

Entanglement, EPR steering, and Bell-nonlocality criteria for multipartite higher-spin systems

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(Received 24 November 2010; published 30 March 2011)

We develop criteria to detect three classes of nonlocality that have been shown by Wiseman *et al.* [*Phys. Rev. Lett.* **98**, 140402 (2007)] to be nonequivalent: entanglement, EPR steering, and the failure of local hidden-variable theories. We use the approach of Cavalcanti *et al.* [*Phys. Rev. Lett.* **99**, 210405 (2007)] for continuous variables to develop the nonlocality criteria for arbitrary spin observables defined on a discrete Hilbert space. The criteria thus apply to multisite qudits, i.e., systems of fixed dimension d , and take the form of inequalities. We find that the spin moment inequalities that test local hidden variables (Bell inequalities) can be violated for arbitrary d by optimized highly correlated nonmaximally entangled states provided the number of sites N is high enough. On the other hand, the spin inequalities for entanglement are violated and thus detect entanglement for such states, for arbitrary d and N , and with a violation that increases with N . We show that one of the moment entanglement inequalities can detect the entanglement of an arbitrary generalized multipartite Greenberger-Horne-Zeilinger state. Because they involve the natural observables for atomic systems, the relevant spin-operator correlations should be readily observable in trapped ultracold atomic gases and ion traps.

DOI: [10.1103/PhysRevA.83.032120](https://doi.org/10.1103/PhysRevA.83.032120)

PACS number(s): 03.65.Ud, 03.67.—a

I. INTRODUCTION

Entanglement and nonlocality has been a central issue in quantum mechanics since the Einstein-Podolsky-Rosen (EPR) paradox [1] which brought into focus the connection between entanglement and nonlocality. The EPR paradox shows that there are correlated quantum states which demonstrate an inconsistency between the completeness of quantum mechanics and the concept of local realism. Schrödinger introduced the term “steering” [2] to describe this apparent nonlocality, and pointed out that these states necessarily involve entanglement, that is, they cannot be separated into factorized terms. Later, Bell’s work [3] served to demonstrate that the situation is even more serious. Bell found quantum states whose correlations cannot be explained by any possible local hidden-variable (LHV) theory. Thus, Einstein’s hope of a local realistic completion of quantum mechanics was not feasible.

Experimental demonstration of EPR and Bell inequality violations to date has supported quantum mechanics. However, almost all work in this direction to date has relied on rather small Hilbert spaces with one or two massless particles. This leaves an important open question, first raised by Schrödinger [2]. Will nonlocal quantum phenomena still persist at a large scale? And do quantum superpositions survive in this limit [4]? Here we note that there are still many difficulties in unifying gravity with quantum mechanics, and the large-scale existence of massive quantum entanglement would directly test such unified theories [5].

Ideally, one would like to generate quantum entanglement of distinct mass distributions. This not only would test quantum theory in a new domain, but also could lead to new types of sensitive gravity detectors. To achieve this, a first step is to obtain macroscopic entanglement of internal degrees of freedom in ultracold atoms, since low temperatures are generally a prerequisite to the observation of quantum superpositions. Already there has been much progress in this direction, with the generation of entangled macroscopic samples of room-temperature atoms [6], entanglement of ionic

degrees of freedom [7], and the observation of spin-squeezing in an ultracold Bose-Einstein condensate [8,9].

While our motivation is to understand something more about Schrödinger’s cat, we focus in particular on the issue of macroscopic nonlocality. This brings into question the usual idea of the classical-quantum correspondence principle, that the system will revert to classical local realism in the “large particle” limit. For many particle systems, macroscopic nonlocality remains to be explored in depth, either experimentally or theoretically. In particular, it is essential to understand the signatures of such effects, in terms of practically accessible observable quantities.

The three forms of nonlocality that we consider are *entanglement* [10]; *steering* [11] (of which the EPR paradox [1] is a special form; hence we follow Ref. [12] in this paper and use the term *EPR steering*); and failure of local hidden-variable (LHV) theories [3], which we call *Bell nonlocality* [12]. For mixed states, these forms of nonlocality are not equivalent [11]. The second type of nonlocality, closely associated with the original EPR paradox [13], has received relatively little attention to date. Recent work of Wiseman and co-workers [11,12] formalizes steering as a nonequivalent nonlocality with its own experimental criteria and has led to EPR steering being the subject of a recent experimental investigation [14].

On the theoretical side, much work has been done for Bell’s nonlocality on multisite qubits [15–17] and bipartite qudits [18–27], though relatively little so far on multipartite systems of higher dimensionality (qudits) [28–30]. Our interest here is to explore these different types of “largeness,” i.e., many sites, many dimensions, and the combination of both. It is conventionally argued that a large system (either multiple sites N or higher dimensional d at each site) must become consistent with classical or LHV theories, but the extent to which this comes about for the three forms of nonlocality is not clear. Whether classical correspondence is achieved by simply increasing d , or N , or whether it occurs through an increasing sensitivity to decoherence, or some other mechanism, is a fundamental question.

Work on N -site qubits gave the surprising result that the deviation of the quantum from the LHV theory increases exponentially with number of qubits [15], for Greenberger-Horne-Zeilinger (GHZ) states [31] and using the Mermin-Ardehali-Belinskii-Klyshko (MABK) Bell inequalities [15–17]. The deviation is relatively robust to noise and loss [32]. However, with multiple sites, it must be noted that the demonstration of entanglement or nonlocality that is necessarily shared between all sites (true multipartite nonlocality) is demonstrably more difficult [33]. We will treat this type of entanglement in a subsequent publication.

One can generalize to qudits or higher-dimensional systems at each site. Violations in the higher-dimensional case for bipartite systems were found possible by Mermin [18], Drummond [19], Peres and co-workers [20,21], and Reid *et al.* [22]. More recent work has confirmed that the violation of LHV in the bipartite case increases or is constant with dimensionality d [23–27] and the CGLMP Bell inequalities (of Collins, Gisin, Linden, Massar, and Popescu) [24] have been shown to be tight. Multisite qudits have been examined by Cabello [28] and Son *et al.* [29], who extended the MABK inequalities, and Chen *et al.* [30], who developed tight inequalities similar to the CGLMP inequalities for bipartite qudits. Results from these authors indicate that the high-dimensional violations are steady, or exponentially increasing, with the number of sites N .

We employ here the idea of studying the multisite higher-dimensionality nonlocality problem using moment inequalities of the type proposed recently by Cavalcanti, Foster, Reid, and Drummond (CFRD) [34]. This originated in the work of Mermin, where the concept was mostly restricted to multipartite qubit measurements, and was applied to Bell violations [15]. The basis of the approach is to construct complex operators $F_k^\pm = X_k \pm iP_k$ from two noncommuting observables X_k and P_k defined for measurements at spatially separated sites $k = 1, \dots, N$. For any separable or local hidden-variable (LHV) model, the inequality

$$\left| \left\langle \prod_{k=1}^N F_k^\pm \right\rangle \right|^2 \leq \int_\lambda d\lambda P(\lambda) \prod_{k=1}^N |\langle F_k^\pm \rangle_\lambda|^2 \quad (1)$$

then holds. Wiseman *et al.* [11] have developed separable models in which *some* of the sites are described by hidden variables that are additionally required to be consistent with localized quantum states. Cavalcanti *et al.* [35] recently pointed out that the nature of the separable model, whether we restrict to local quantum or local hidden-variable states at site k , enables us to deduce a constraint on $|\langle F_k^\pm \rangle_\lambda|^2$, and this leads to a criteria set involving the three types of nonlocality. The case where local states at all sites are constrained to be quantum states leads to the criteria for entanglement developed recently by Hillery and co-workers [36–39].

CFRD [34] used Eq. (1) to develop a multisite continuous variable test of Bell nonlocality. They derived a Bell inequality that is different from previous formulations because it involves moments of the continuous variable outcomes and does not assume bounded outcomes. Defining the outcomes of two *arbitrary* observables to be X_k and P_k , at spatially separated

sites denoted by k ($k = 1, \dots, N$), these inequalities can be written

$$\left| \left\langle \prod_{k=1}^N (X_k + iP_k) \right\rangle \right|^2 \leq \left\langle \prod_{k=1}^N (X_k^2 + P_k^2) \right\rangle. \quad (2)$$

The left side (L) of the inequality is measured by way of the moments involving the Hermitian observables defined at each site: e.g., for $N = 2$, $L = \langle X_1 X_2 - P_1 P_2 \rangle^2 + \langle P_1 X_2 + X_1 P_2 \rangle^2$. Multiobservable and higher-order extensions of these inequalities have been presented by Shchukin and Vogel [40] and Sun *et al.* [41]. CFRD showed that for “position” and “momentum” measurements with a continuous spectrum, there is a violation of Eq. (2) predicted for multipartite qubit GHZ states [31]. Recent work extended the CFRD to examine inequalities involving optimization of functions of these observables that give less strict requirements for loss [42]. The main point is that because X_k and P_k are arbitrary observables, a ready generalization exists to higher-spin observables.

In this paper, we thus develop the CFRD approach further by using arbitrary spin operators, which are generic observables that can be measured in many physical environments for systems with discrete states. We also apply the unified method of Cavalcanti *et al.* [35] to derive CFRD-type criteria for multipartite entanglement, EPR steering, and Bell nonlocality for N -site systems of higher dimensionality d (i.e., specifically, for higher spin J). Cavalcanti *et al.* [35] previously examined only the multipartite qubit ($J = 1/2$) case, but used a method similar to the original approach of Mermin [15] that allows stronger criteria to be derived in this scenario. Direct application of the Bell inequality (2) without optimization shows demonstrations of Bell nonlocality possible for maximally entangled states where $d = 2, 3$, and for all $N \geq 3$. Certain nonmaximally entangled but highly correlated states of the type considered by Acin *et al.* [43] allow violations for all d provided the number of sites is large enough (e.g., $d = 5$ requires $N \geq 9$). While the demonstration of Bell nonlocality is limited in terms of dimensionality d for low N (whereas the CGLMP-type Bell inequalities give constant or increasing violation for bipartite states), the violations that we find for optimized nonmaximally entangled states increase with respect to the number of sites N .

To derive entanglement and EPR steering criteria, we use the lower bound C_J of the quantum uncertainty relation

$$\Delta^2 \hat{j}^x + \Delta^2 \hat{j}^y \geq C_J \quad (3)$$

for two conjugate spins \hat{j}^x and \hat{j}^y [44] with fixed total spin J that has been derived recently for all J (d) [45]. For optimized nonmaximally entangled states, the violation of the “ C_J entanglement inequality” occurs for all spin (dimensionality) J and N , and decreases with J but increases with N . A similar result was obtained for GHZ states by Roy for the spin $J = 1/2$ case [46]. We also derive a second set of more general entanglement and EPR steering criteria based on the original Heisenberg uncertainty relation. This set therefore does not assume the case of fixed total spin J , and for entanglement is a generalization of criteria developed recently by Hillery and co-workers [36–39]. In this paper, we discuss the use of

the two types of criteria for both maximally entangled and nonmaximally entangled highly correlated spin states.

II. THE LOCAL HIDDEN-VARIABLE (LHV) AND LOCAL HIDDEN STATE (LHS) MODELS

We consider measurements \hat{X}_k , with associated outcomes X_k , that can be performed on the k th system ($k = 1, \dots, N$). Following Bell [3], we have a local hidden-variable (LHV) model if the joint probability for outcomes of simultaneous measurements performed on the N spatially separated systems is given by

$$P(X_1, \dots, X_N) = \int_{\lambda} P(\lambda) P(X_1|\lambda) \cdots P(X_N|\lambda) d\lambda. \quad (4)$$

Here λ are Bell's local hidden variables, and $P(X_k|\lambda)$ is the probability of X_k given the values of λ , with $P(\lambda)$ being the probability distribution for λ . The factorization in the integrand is Bell's locality assumption, that $P(X_k|\lambda)$ depends on the parameters λ , and the measurement choice made at k only. The hidden variables λ describe a local state for each site, in that the probability distribution $P(X_k|\lambda)$ for the measurement at k is given as a function of λ . If Eq. (4) fails, then we have proved a failure of LHV theories, which we generically call a *Bell violation* or *Bell nonlocality* [12].

Bell's locality does not exclude that the local hidden state could be a quantum state, in which case there exists a quantum density operator ρ_k for which $P(X_k|\lambda) = \langle X_k | \rho_k | X_k \rangle$. In this case, we write $P(X_k|\lambda) \equiv P_Q(X_k|\lambda)$, where the subscript Q denotes the quantum probability distribution. When all the $P(X_k|\lambda)$ ($k = 1, \dots, N$) are so restricted, we write

$$P(X_1, \dots, X_N) = \int_{\lambda} P(\lambda) P_Q(X_1|\lambda) \cdots P_Q(X_N|\lambda) d\lambda, \quad (5)$$

which is the requirement of a quantum separable (QS) model [11,12]. The model (5) follows from the assumption of a separable density operator ρ , which can be written in the factorized form $\rho = \sum_R P_R \rho_1^R \cdots \rho_N^R$, where R describes a set of local quantum states for each site. Failure of Eq. (5) gives proof of entanglement, following standard definitions [10].

It is clear that all QS models are therefore a subset of the LHV class, that is, quantum separable models are a special case of general local hidden-variable theories, so that one can write

$$\{\text{QS}\} \Rightarrow \{\text{LHV}\}. \quad (6)$$

This means in turn that failure of $\{\text{LHV}\}$ also implies failure of $\{\text{QS}\}$, which is termed entanglement. Another way to state this is that entanglement is a necessary condition for a Bell violation.

Wiseman *et al.* [11] have pointed out that there exists an intermediate case between the local hidden-variable and quantum separable models, in which for the bipartite case $N = 2$, one of the $P(X_k|\lambda)$ is constrained to be a quantum distribution and the other is not. Failure of this asymmetric local hidden-state (LHS) model was shown by them to demonstrate Schrödinger's "steering." The connection between this model and Schrödinger's steering and the EPR paradox for $N = 2$ has been explained in Refs. [11,12] and is summarized in terms of an "elements of reality" approach in Ref. [13]. Where

$N = 2$ and $T = 1$, we arrive at a model which if violated is a demonstration of steering, and also is a demonstration of the EPR paradox as generalized to appropriate observables [13]; hence we follow Ref. [12] and use in this paper the term EPR steering to describe failure of this model.

Recent work of Cavalcanti *et al.* [35] generalizes the LHS model to multiple sites. Following them, when exactly T of the $P(X_k|\lambda)$ ($k = 1, \dots, N$) of the separable model (4) are quantum probabilities, one can write (we label these T sites by $k = 1, \dots, T$)

$$P(X_1, \dots, X_N) = \int_{\lambda} P(\lambda) \prod_{k=1}^T P_Q(X_T|\lambda) \cdots \prod_{k=T+1}^N P(X_N|\lambda) d\lambda. \quad (7)$$

This condition implies that we assume normal quantum uncertainties for X_1, \dots, X_T , while for the remaining observables we can have a complete classical knowledge, i.e., they are predetermined elements of local reality in Einstein's language. In this paper, we refer to the multipartite separability model (7) as a T th order EPR model (EPR_T) and follow [35] to denote this local hidden-state model by $\text{LHS}(T, N)$. With $T = N$, one has a simple case of proving entanglement, while with $T = 0$ one has a Bell violation. Importantly for this paper, Cavalcanti *et al.* [35] show that the case of violation of $\text{LHS}(1, N)$ ($T = 1$) implies EPR-steering nonlocality across at least one bipartition.

For larger N , we have a range of separable models as has been shown by Ref. [35]: violations of these models provide a step-by-step transition in increasing degrees of *quantum nonlocality*. It is clear that all $\{\text{EPR}_T\}$, [i.e., $\text{LHS}(T, N)$], models are included in the LHV class, so that one can write

$$\{\text{QSM}\} \Rightarrow \{\text{EPR}_{T+1}\} \Rightarrow \{\text{EPR}_T\} \Rightarrow \{\text{LHV}\}, \quad (8)$$

and hence when violations are observed, one similarly obtains the negation of these relations

$$\{\text{Bell}\} \Rightarrow \{\mathcal{S}_T\} \Rightarrow \{\mathcal{S}_{T+1}\} \Rightarrow \{\text{entanglement}\}, \quad (9)$$

where $\{\mathcal{S}_T\}$ symbolizes the nonlocality associated with the failure of the $\{\text{EPR}_T\}$ [i.e., $\text{LHS}(T, N)$] model. This is shown in a Venn diagram in Fig. 1 for the $N = 2$ or bipartite case. More generally there is a nested sequence of nonlocality inequalities. For the bipartite case, Werner [47] proved that the entangled states are a strict superset of Bell states (those that show Bell nonlocality). Wiseman *et al.* [11] proved that those states able to show steering are a strict superset of Bell states and a strict subset of entangled states, and hence showed that there are three distinct classes as illustrated in the Venn diagram in Fig. 1.

III. DERIVATION OF A SPIN NONLOCALITY CRITERIA SET

We are considering N sites that are in principle causally separated. From previous references summarized in the last section, we know that depending on the assumption of what type of local hidden state is present at each site, whether "quantum" or "hidden variable", one can derive criteria for Bell nonlocality, EPR steering, or entanglement [12,35]. There are many possible observable signatures for these types of

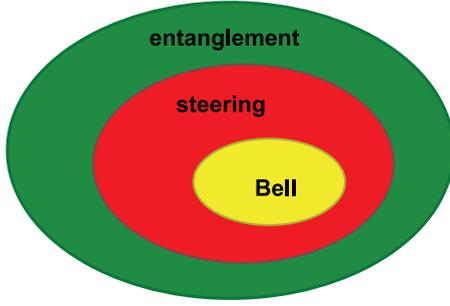


FIG. 1. (Color online) Three famous types of quantum nonlocality: Bell nonlocality is a stronger result than EPR steering, which is stronger than entanglement.

nonlocality. Generally speaking it is simplest to construct conditions *sufficient* to deduce the nonlocality, by deriving inequalities for observed correlations that follow *necessarily* from the LHV, LHS(1, N), and QS models, respectively. This is the route we take here. Our criteria will thus take the form of inequalities that if violated prove the nonlocality, but will not necessarily be violated by all such nonlocal states.

We will use the general results of Cavalcanti *et al.* [35] to derive nonlocality inequalities for higher-spin measurements. We follow their derivation, using their notation as much as possible, but applying to our special case of arbitrary spin observables. First, one follows Mermin [15] and Cavalcanti *et al.* [34] who consider $F_k^\pm = (X_k \pm iP_k)$ for each site k , where X_k and P_k denote the outcomes of two observables \hat{X}_k and \hat{P}_k . Now we turn to the specific case of interest in this paper, and consider spin measurements, at each site. Here, we make the simplest choice:

$$F_k^\pm = J_k^\theta \pm iJ_k^{\theta'}, \quad (10)$$

where $J_k^\theta = J_k^x \cos \theta + J_k^y \sin \theta$, and $J_k^{x/y}$ are the outcomes for the measurements represented by spin operators \hat{J}_k^x and \hat{J}_k^y . In fact we will focus on the case where $\theta' = \theta + \pi/2$ for which $F_k^\pm = J^\pm$ corresponds to the spin raising and lowering operators \hat{J}^\pm , i.e.,

$$F_k^\pm \equiv J_k^\pm = J_k^x \pm iJ_k^y, \quad (11)$$

which is the choice $X_k = J_k^x$ and $P_k = J_k^y$. For notational convenience, we thus denote the raising operator \hat{J}^+ by \hat{J}^+ and the lowering operator \hat{J}^- by \hat{J}^- , and their outcomes by J^+ and J^- , respectively. Note the distinction between the operator \hat{J}_k^\pm and the *measurable complex number* $\langle J_k^\pm \rangle = \langle J_k^x \rangle \pm i \langle J_k^y \rangle$.

Following Ref. [35], for any LHV or LHS model (7), we can write

$$\left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle = \int_\lambda d\lambda P(\lambda) \prod_{k=1}^N \langle J_k^{s_k} \rangle_\lambda, \quad (12)$$

where $s_k = +1$ or -1 . Here $\langle J_k^\pm \rangle_\lambda = \langle J_k^x \rangle_\lambda \pm i \langle J_k^y \rangle_\lambda$ where the subscript λ denotes the complex number average, for a given hidden-variable specification λ . Then one follows the

Holder inequality techniques of Refs. [34,36] and uses the inequality (1), which holds since

$$\begin{aligned} & \left| \left\langle \prod_{k=1}^N F_k^\pm \right\rangle \right| \\ & \leq \int d\lambda P(\lambda) |\langle F_1^\pm \rangle_\lambda| |\langle F_2^\pm \rangle_\lambda| \cdots \\ & = \int d\lambda P(\lambda) [|\langle F_1^\pm \rangle_\lambda|^2 |\langle F_2^\pm \rangle_\lambda|^2 \cdots]^{1/2} \\ & \leq \left[\int d\lambda P(\lambda) \right]^{1/2} \left[\int d\lambda P(\lambda) |\langle F_1^\pm \rangle_\lambda|^2 |\langle F_2^\pm \rangle_\lambda|^2 \cdots \right]^{1/2} \\ & = \left[\int d\lambda P(\lambda) |\langle F_1^\pm \rangle_\lambda|^2 |\langle F_2^\pm \rangle_\lambda|^2 \cdots \right]^{1/2}. \end{aligned} \quad (13)$$

Here, the Cauchy-Schwarz inequality $\langle xy \rangle^2 \leq \langle x^2 \rangle \langle y^2 \rangle$, where $x = \sqrt{P(\lambda)}$ and $y = \sqrt{P(\lambda)} |\langle F_1^\pm \rangle_\lambda| |\langle F_2^\pm \rangle_\lambda| \cdots$, has been used to justify the third line.

By definition $|\langle J_k^\pm \rangle_\lambda|^2 = \langle J_k^x \rangle_\lambda^2 + \langle J_k^y \rangle_\lambda^2$, and since variances are nonnegative it follows that for any *local hidden-variable theory* λ ,

$$\langle (J_k^x)^2 \rangle_\lambda - \langle J_k^x \rangle_\lambda^2 \geq 0, \quad \langle (J_k^y)^2 \rangle_\lambda - \langle J_k^y \rangle_\lambda^2 \geq 0, \quad (14)$$

and hence

$$|\langle J_k^\pm \rangle_\lambda|^2 \leq \langle (J_k^x)^2 \rangle_\lambda + \langle (J_k^y)^2 \rangle_\lambda. \quad (15)$$

The case where the separable model specifies N local *quantum* states, as in the assumption of the separable density operator (5), has been employed, for example, by Roy [46]. Here there are further restrictions due to the Heisenberg uncertainty principle and its generalizations. For spins, $\Delta J_k^x \Delta J_k^y \geq |\langle J_k^z \rangle|/2$ and hence

$$(\Delta J_k^x)^2 + (\Delta J_k^y)^2 \geq |\langle J_k^z \rangle|. \quad (16)$$

Quantum uncertainty relations of the form

$$(\Delta J_k^x)^2 + (\Delta J_k^y)^2 \geq C_k, \quad (17)$$

where C_k is a constant, can also be derived that give useful entanglement and steering criteria, as will be introduced in the following parts. This leads to the inequalities

$$|\langle J_k^\pm \rangle_\lambda|^2 \leq \langle (J_k^x)^2 \rangle_\lambda + \langle (J_k^y)^2 \rangle_\lambda - C_k, \quad (18)$$

$$|\langle J_k^\pm \rangle_\lambda|^2 \leq \langle (J_k^x)^2 \rangle_\lambda + \langle (J_k^y)^2 \rangle_\lambda - |\langle J_k^z \rangle_\lambda|. \quad (19)$$

The last inequality in fact implies

$$|\langle J_k^+ \rangle_\lambda|^2 \leq \langle (J_k^x)^2 \rangle_\lambda + \langle (J_k^y)^2 \rangle_\lambda \pm \langle J_k^z \rangle_\lambda, \quad (20)$$

$$|\langle J_k^- \rangle_\lambda|^2 \leq \langle (J_k^x)^2 \rangle_\lambda + \langle (J_k^y)^2 \rangle_\lambda \pm \langle J_k^z \rangle_\lambda. \quad (21)$$

We now use the results of Ref. [35]: we assume the model LHS(T, N) where sites $k = 1, \dots, T$ are quantum, and the remainder local hidden variable, so we have a hybrid case as



FIG. 2. (Color online) Hybrid model of Ref. [11] involves different local hidden states, either quantum (LQS) or local hidden variable (LHV), at each spatially separated site 1 and 2. The asymmetric use of quantum uncertainty relations that results because of this gives rise to criteria for steering and the EPR paradox (EPR steering) [12].

studied for $N = 2$ in Ref. [11] (Fig. 2). Using the relations then the following holds:

$$\begin{aligned} & \left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 \\ & \leq \int d\lambda P(\lambda) \prod_{k=1}^N |\langle J_k^{s_k} \rangle_\lambda|^2 \\ & \leq \left\langle \prod_{k=1}^T [(J_k^x)^2 + (J_k^y)^2 - C_k] \prod_{k=T+1}^N [(J_k^x)^2 + (J_k^y)^2] \right\rangle. \end{aligned} \quad (22)$$

Here s_k can be selected + or – at both sides, respectively. If $T = 0$ one recovers a Bell inequality whose violation will prove failure of LHV, while for $T = N$, one recovers an inequality which if violated will simply prove entanglement. The intermediate case of Ref. [35], where $T = 1$, recovers an inequality which if violated will prove an EPR steering. It is clearly necessary to have entanglement as a starting point toward observation of stronger forms of nonlocality. Similarly, using Eqs. (20), (21), and (15), the result of Ref. [35] becomes

$$\begin{aligned} & \left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 \\ & \leq \int d\lambda P(\lambda) \prod_{k=1}^N |\langle J_k^{s_k} \rangle_\lambda|^2 \\ & \leq \left\langle \prod_{k=1}^T [(J_k^x)^2 + (J_k^y)^2 \pm J_k^z] \prod_{k=T+1}^N [(J_k^x)^2 + (J_k^y)^2] \right\rangle, \end{aligned} \quad (23)$$

where the \pm appearing in the first T factors of the right-side product can be chosen independently for each factor.

IV. SPIN NONLOCALITY CRITERIA

In general, the total spin may itself be an observable, so that J at each site is not known in advance. In this general case, we note that for all quantum states, we must have $|\langle \hat{J}^+ \rangle| |\langle \hat{J}^- \rangle| \leq \langle \hat{J}^+ \hat{J}^- \rangle$ and $|\langle \hat{J}^+ \rangle| |\langle \hat{J}^- \rangle| \leq \langle \hat{J}^- \hat{J}^+ \rangle$. This implies

$$|\langle \hat{J}^+ \rangle|^2 \leq \langle \hat{J}^\pm \hat{J}^\mp \rangle = \langle (\hat{J}^x)^2 + (\hat{J}^y)^2 \pm i[\hat{J}^x, \hat{J}^y] \rangle, \quad (24)$$

which is another way to arrive at the conditions (20) and (21). Using (23) we now obtain three nonlocality inequalities that

apply to all systems, with no assumptions being placed on the total spin.

A. Entanglement inequalities: Generalized Hillery-Zubairy (HZ) entanglement criterion

Entanglement is verified if

$$\begin{aligned} & \left| \left\langle \prod_{k=1}^N \hat{J}_k^{s_k} \right\rangle \right|^2 > \left\langle \prod_{k=1}^N [(\hat{J}_k)^2 - (\hat{J}_k^z)^2 \pm \hat{J}_k^z] \right\rangle \\ & \equiv \left\langle \prod_{k=1}^N [(\hat{J}_k)^2 - (\hat{J}_k^z)^2 + l_k \hat{J}_k^z] \right\rangle \\ & = \left\langle \prod_{k=1}^N \hat{J}_k^\pm \hat{J}_k^\mp \right\rangle \equiv \left\langle \prod_{k=1}^N \hat{J}_k^{l_k} \hat{J}_k^{-l_k} \right\rangle, \end{aligned} \quad (25)$$

where $l_{k,s_k} = \pm$ and we note that $(\hat{J}_1)^2 - (\hat{J}_1^z)^2 \pm \hat{J}_1^z = \hat{J}_1^\pm \hat{J}_1^\mp$, so the final line has been rewritten in terms of the lowering and raising operators. The \pm value of l_k in each factor on the right side (R) can be selected *independently* for each factor, and independently of the choice s_k , in order to minimize R . Also, the choice of + or – for s_k on the left side (L) can be selected independently to optimize the criterion for the state used.

We call this a *generalized* HZ criterion, since a similar but not identical criterion (25) has been derived recently by Hillery and co-workers [36–39]. This HZ multipartite criterion is an extension of the criteria developed previously by Hillery and Zubairy [36], and Hillery, Dung, and Niset [37]. There have been recent applications of this criterion to spins systems [38,39]. We recall that \hat{J}^2 is defined as

$$(\hat{J})^2 = (\hat{J}^x)^2 + (\hat{J}^y)^2 + (\hat{J}^z)^2, \quad (26)$$

so we can use $(\hat{J})^2 - (\hat{J}^z)^2 = (\hat{J}^x)^2 + (\hat{J}^y)^2$ to reexpress the generalized HZ entanglement criterion (25) as

$$\left| \left\langle \prod_{k=1}^N \hat{J}_k^{s_k} \right\rangle \right|^2 > \left\langle \prod_{k=1}^N [(\hat{J}_k^x)^2 + (\hat{J}_k^y)^2 \pm \hat{J}_k^z] \right\rangle. \quad (27)$$

B. EPR steering inequalities

An EPR paradox or steering nonlocality is demonstrated if the EPR steering inequality is violated, i.e., if

$$\left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 > \left\langle [(J_1^2)^2 - (J_1^z)^2 \pm J_1^z] \prod_{k=T+1}^N [(J_k^x)^2 + (J_k^y)^2] \right\rangle. \quad (28)$$

We note in this case $T = 1$, so there is only one quantum site (which we select to be $k = 1$). The criterion can also be expressed as

$$\left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 > \left\langle [(J_1^x)^2 + (J_1^y)^2 \pm J_1^z] \prod_{k=T+1}^N [(J_k^x)^2 + (J_k^y)^2] \right\rangle \quad (29)$$

for direct comparison with Eq. (27). The criterion to detect failure of the LHS(T, N) model (7) is

$$\left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 > \left\langle \prod_{k=1}^T [(J_k^x)^2 + (J_k^y)^2 \pm J_k^z] \right\rangle \times \prod_{k=T+1}^N [(J_k^x)^2 + (J_k^y)^2]. \quad (30)$$

C. Bell-nonlocality inequalities

A Bell inequality is violated, proving a failure of all possible LHV, if

$$\left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 > \left\langle \prod_{k=1}^N [(J_k^x)^2 + (J_k^y)^2] \right\rangle. \quad (31)$$

We note the criterion has been expressed in terms of the moments of outcomes corresponding to the observables. This is done because the test of nonlocality is a test of local hidden-variable theories and hence does not assume quantum mechanics. The EPR steering criterion is expressed similarly in terms of outcome moments, for similar reasons.

This Bell inequality differs from the MABK Bell inequalities [15–17] because the right side is not a fixed bound, but varies as a moment. We will note that for spin-1/2 observables this Bell inequality does not give as strong a violation as the MABK Bell inequalities for maximally entangled states. The different nature of the right-hand side may, however, make the violations stronger in other scenarios.

These results [Eqs. (25)–(31)] apply for all spin measurements and systems, even when the spin quantum number itself is a quantum observable.

V. FIXED-DIMENSIONALITY J ENTANGLEMENT AND EPR STEERING CRITERIA

We now consider states of fixed spin dimensionality J . The most general pure quantum state of this type at a single site is simply a general qudit state of dimension $d = 2J + 1$:

$$|\psi\rangle = \frac{1}{\sqrt{n}} [r_{-J} e^{-i\phi_{-J}} |J, -J\rangle + r_{-J+1} e^{-i\phi_{-J+1}} |J, -J+1\rangle + \dots + r_J e^{-i\phi_J} |J, +J\rangle]. \quad (32)$$

Here $n = \sum_{m=-J}^J r_m^2$, r_m, ϕ_m ($m = -J, \dots, J$) are real numbers indicating amplitude and phase, respectively. Where we have fixed dimensionality, i.e., a spin- J system, the quantum uncertainty relation (17) can be used to derive different entanglement and steering inequalities beyond those of Eqs. (25)–(29). In particular there will be a quantum uncertainty relation of the form [45]

$$\Delta^2 \hat{J}^x + \Delta^2 \hat{J}^y \geq C_J, \quad (33)$$

where $C_J \neq 0$ because there exist no simultaneous eigenstates of \hat{J}^x and \hat{J}^y [44]. The results for selected values of J are tabulated in Table I [45].

These fixed- J uncertainty relations can be used to derive additional entanglement and EPR steering criteria based on Eq. (22):

TABLE I. Lower bound C_J of quantum uncertainty relation with spin J , where $C_J = 7/16$ for spin 1 agrees with the result in [44].

J	1/2	1	3/2	2	5/2	3	7/2	4	...
C_J	1/4	7/16	0.6009	0.7496	0.8877	1.0178	1.1416	1.26	...

(1) Entanglement is verified if

$$\left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 > \left\langle \prod_{k=1}^N [(J_k)^2 - (J_k^z)^2 - C_J] \right\rangle. \quad (34)$$

(2) An EPR steering nonlocality is verified if

$$\left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 > \left\langle [(J_1)^2 - (J_1^z)^2 - C_J] \times \prod_{k=2}^N [(J_k^x)^2 + (J_k^y)^2] \right\rangle, \quad (35)$$

where again we note that $(\hat{J}^y)^2 - (\hat{J}^z)^2 = (\hat{J}^x)^2 + (\hat{J}^y)^2$. The criterion to detect failure of the more general LHS(T, N) separable model (7) of [35] is

$$\left| \left\langle \prod_{k=1}^N J_k^{s_k} \right\rangle \right|^2 > \left\langle \prod_{k=1}^T [(J_k)^2 - (J_k^z)^2 - C_J] \times \prod_{k=T+1}^N [(J_k^x)^2 + (J_k^y)^2] \right\rangle. \quad (36)$$

We will refer to these criteria throughout the paper as the C_J nonlocality criteria.

The Bell inequality (31), which is not dependent on C_J , also applies. The criteria we have derived here are all *sufficient* to detect entanglement, but may not *necessarily* detect entanglement. In the following, we analyze these spin nonlocality sets in greater detail, and examine how the sensitivity to entanglement changes as J changes. We note here that either form of entanglement and EPR steering nonlocality criterion is valid for systems of fixed dimensionality (i.e., J), and the optimal choice that is made will depend on the value of J and the states that are selected.

VI. FIXED- J ENTANGLED STATES

There are many possible entangled states. In particular, in the following sections we choose to analyze the following highly correlated spin states:

(1) The maximally entangled and highly correlated states of form

$$|\Psi\rangle_{\max} = \frac{1}{\sqrt{d}} \sum_{m=-J}^J |J, m\rangle_1 |J, m\rangle_2 |J, m\rangle_3 \dots, \quad (37)$$

where $|J, m\rangle$ is an eigenstate of \hat{J}^2 and \hat{J}^z . In particular the bipartite ($N = 2$) maximally entangled state is

$$|\Psi\rangle_{\max} = \frac{1}{\sqrt{d}} \sum_{m=-J}^J |J, m\rangle_1 |J, m\rangle_2. \quad (38)$$

This state can also be written in terms of the boson operators as [18]

$$|\Psi\rangle_{\max} = \frac{1}{(2J)!(2J+1)^{1/2}} (\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{b}_1^\dagger \hat{b}_2^\dagger)^{2J} |0\rangle, \quad (39)$$

where \hat{a}_1, \hat{b}_1 are the two modes at site 1, $\hat{J}_1^z = (\hat{b}_1^\dagger \hat{b}_1 - \hat{a}_1^\dagger \hat{a}_1)/2$, with the same definition at site 2. This equivalence is due to the Schwinger representation, which is used to map bosonic states into spin states. We note that due to the Schwinger representation [48], one can write

$$\begin{aligned} |J, m\rangle_1 &= \frac{(a_1^\dagger)^{J+m} (b_1^\dagger)^{J-m}}{\sqrt{(J+m)!(J-m)!}} |0_a, 0_b\rangle_1 \\ &= |(J+m)_a, (J-m)_b\rangle_1. \end{aligned} \quad (40)$$

Here $|(J+m)_a, (J-m)_b\rangle_1$ represents a general harmonic oscillator state of two bosonic or harmonic oscillator modes localized at position 1.

(2) More general, nonmaximally entangled but highly correlated spin states of form

$$\begin{aligned} |\psi\rangle_{\text{non}} &= \frac{1}{\sqrt{n}} [r_{-J} |J, -J\rangle^{\otimes N} + r_{-J+1} |J, -J+1\rangle^{\otimes N} \\ &\quad + \dots + r_J |J, +J\rangle^{\otimes N}], \end{aligned} \quad (41)$$

where $|J, m\rangle^{\otimes N} = \prod_{k=1}^N |J, m\rangle_k$, $n = \sum_{m=-J}^J r_m^2$. Here we will be restricted to the case of real parameters symmetrically distributed around $m=0$. The amplitude r_m can be selected to optimize the nonlocality result. For example, with N sites and a spin-1 system, the state has the form

$$|\psi\rangle = \frac{1}{\sqrt{r^2+2}} (|1, -1\rangle^{\otimes N} + r|1, 0\rangle^{\otimes N} + |1, +1\rangle^{\otimes N}), \quad (42)$$

which has been shown by Acin *et al.* [43] to give improved violation over the maximally entangled state for some Bell inequalities.

VII. SPIN-1/2 CASE

General nonlocality criteria. The general entanglement, EPR steering, and Bell inequalities (25)–(31) apply in this case, in addition to the C_J criteria (34)–(35) specific to spin $J=1/2$.

For spin-1/2, it is convenient to use the Pauli spin operators $\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z$ so that a connection can be made with previous criteria. The Bell inequality of Eq. (31) becomes

$$\left\langle \left\langle \prod_{k=1}^N \sigma_k^{s_k} \right\rangle \right\rangle^2 \leq \left\langle \left\langle \prod_{k=1}^N [(\sigma_k^x)^2 + (\sigma_k^y)^2] \right\rangle \right\rangle = 2^N, \quad (43)$$

where $\hat{\sigma}_k^\pm = \hat{\sigma}_k^x \pm i\hat{\sigma}_k^y$. Bell nonlocality is implied by the violation of this inequality. Similarly, the generalized HZ entanglement inequality of (25) and (27) becomes

$$\begin{aligned} \left\langle \left\langle \prod_{k=1}^N \sigma_k^{s_k} \right\rangle \right\rangle^2 &\leq \left\langle \left\langle \prod_{k=1}^N [(\sigma_k^x)^2 + (\sigma_k^y)^2 + 2l_k \sigma_k^z] \right\rangle \right\rangle \\ &= 2^N \left\langle \left\langle \prod_{k=1}^N [1 + l_k \sigma_k^z] \right\rangle \right\rangle, \end{aligned} \quad (44)$$

where l_k and s_k can be independently selected as + or –, and the EPR steering inequality (28) becomes

$$\begin{aligned} \left| \left\langle \left\langle \prod_{k=1}^N \sigma_k^{s_k} \right\rangle \right\rangle \right|^2 &\leq 2^{N-1} \langle (\sigma_k^x)^2 + (\sigma_k^y)^2 + 2l_k \sigma_k^z \rangle \\ &= 2^N \langle 1 + l_k \sigma_k^z \rangle. \end{aligned} \quad (45)$$

C_J nonlocality criteria. The quantum uncertainty relation

$$\Delta^2 \hat{\sigma}^x + \Delta^2 \hat{\sigma}^y \geq 1 \quad (46)$$

follows from $\Delta^2 \hat{\sigma}^x + \Delta^2 \hat{\sigma}^y + \Delta^2 \hat{\sigma}^z \geq 2$ [44]. Hence the C_J entanglement inequality of Eq. (34) becomes

$$\left| \left\langle \left\langle \prod_{k=1}^N \sigma_k^{s_k} \right\rangle \right\rangle \right|^2 \leq \left\langle \left\langle \prod_{k=1}^N [(\sigma_k^x)^2 + (\sigma_k^y)^2 - 1] \right\rangle \right\rangle = 1. \quad (47)$$

The violation of this inequality thus implies entanglement. EPR steering is implied by violation of the EPR steering inequality

$$\begin{aligned} \left| \left\langle \left\langle \prod_{k=1}^N \sigma_k^{s_k} \right\rangle \right\rangle \right|^2 &\leq \left\langle \left\langle [(\sigma_1^x)^2 + (\sigma_1^y)^2 - 1] \right. \right. \\ &\quad \left. \left. \times \prod_{k=2}^N [(\sigma_k^x)^2 + (\sigma_k^y)^2] \right\rangle \right\rangle = 2^{N-1}. \end{aligned} \quad (48)$$

The entanglement criterion (47) has been derived by Roy [46], while the Bell inequality (43) (when expressed in terms of the real or imaginary parts of the left side) becomes that of Mermin [15] (for N even) and Ardehali [16] (for N odd). This Bell inequality is known to be weaker than the full MABK Bell inequalities (that of Mermin's for N odd, and Ardehali's for N even) [17]. For EPR steering, Eq. (48) reduces to one of the EPR steering inequalities derived for the qubit case by Cavalcanti *et al.* [35].

Quantum prediction for multisite qubits. We consider the N -partite GHZ states, denoting $|\frac{1}{2}, -\frac{1}{2}\rangle$ and $|\frac{1}{2}, +\frac{1}{2}\rangle$ symbolically by $|0\rangle$ and $|1\rangle$, respectively:

$$|\psi\rangle_{\max} = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \quad (49)$$

We define B_{Bell} , B_{EPR} , and B_{ent} for the Bell-nonlocality, steering, and entanglement inequalities, respectively, as the square root of the ratio of the left side (L) and right side (R) of the inequalities. The generalized HZ entanglement inequality (25), i.e., Eq. (44), allows the strongest violation B_{ent} possible for this state. In fact we can consider the generalized GHZ state

$$|\psi\rangle_{\max} = \cos\theta |0\rangle^{\otimes N} + \sin\theta |1\rangle^{\otimes N}. \quad (50)$$

The l_k can be chosen so that the right side (R) is zero, and the s_k so that the left side (L) is nonzero. In the spin notation, we choose specifically

$$\begin{aligned} L &= \left| \left\langle \left\langle \prod_{k=1}^N J_k^- \right\rangle \right\rangle \right|^2 = \left| \sum_{m=-J+1}^J r_{m-1}^* r_m \right. \\ &\quad \left. \times [(J+m)(J-m+1)]^{N/2} \right|^2 = (\cos\theta \sin\theta)^2, \end{aligned} \quad (51)$$

and

$$R = \left\langle J_1^+ J_1^- \prod_{k=2}^N J_k^- J_k^+ \right\rangle = \sum_{m=-J}^J |r_m|^2 (J-m)(J+m+1) \times [(J-m)(J+m+1)]^{N-1} = 0, \quad (52)$$

where $J = 1/2$, and $m = \pm 1/2$. The ratio $B_{\text{ent}} = \sqrt{L/R} \rightarrow \infty$ for choices of θ other than $0, \pi/2$ and the criterion can detect all entanglement for this state. This is a stronger result than that of [39] who considered the generalized GHZ state where the coefficients are asymmetric, but they did not consider the generalized entanglement criterion which involves independent choices of l_k . We note the EPR steering inequality (45) reduces to the Bell inequality for these states because the correlation $\langle \sigma_k^z \rangle = 0$, and hence it is better to use the C_J EPR steering criterion in this case.

The C_J entanglement criterion (34), i.e., Eq. (47), can also be studied for the generalized GHZ state, since L is also given by Eq. (51). The R in this case (converting to the spin-1/2 operators), however, is always $1/2^{2N}$, meaning that entanglement is only detected when $\sin^2 2\theta > 2^2/2^{2N} = 1/2^{2(N-1)}$, i.e., when $\sin 2\theta > 1/2^{(N-1)}$. Considering the symmetric case for $\theta = \pi/4$, we note the C_J criterion is satisfied for the GHZ states for all $N \geq 2$ [46].

The Bell criterion is satisfied only for $N \geq 3$ [15]. The EPR steering inequality (48) for $T = 1$ is also violated for $N \geq 2$. We note the amount of violation for these inequalities increases exponentially with N , as shown in Fig. 3 and as reported by Mermin [15], Roy [46], and Cavalcanti *et al.* [35]. Defining B_T to correspond to the ratio of L to R for the general case of T quantum sites, we note that $B_T = 2^{(N+T-2)/2}$, so that $B_{\text{ent}} = 2^{N-1}$, $B_{\text{EPR}} = 2^{(N-1)/2}$, and $B_{\text{Bell}} = 2^{(N-2)/2}$. For Bell nonlocality, the MABK Bell inequality [15–17] gives stronger violations for these GHZ states ($B_{\text{Bell}} = 2^{(N-1)/2}$), and it is violated for all $N \geq 2$. The MABK inequalities involve a different derivation, and these alternative nonlocality inequalities derived for the qubit (spin-1/2) case have been studied in Ref. [35].

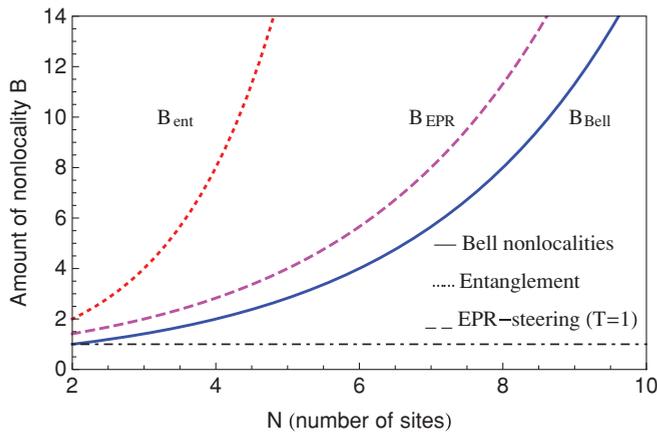


FIG. 3. (Color online) Spin 1/2: Entanglement (dotted), EPR steering ($T = 1$) (dashed), and Bell nonlocalities (solid) can be detected for GHZ states of Eq. (49) using C_J criteria in Eqs. (47) and (48) and Bell inequalities (43) when $B_{\text{ent}}, B_{\text{EPR}}, B_{\text{Bell}} > 1$. B_{ent} for the generalized HZ entanglement criterion (25), i.e., Eq. (44), becomes infinite in this case.

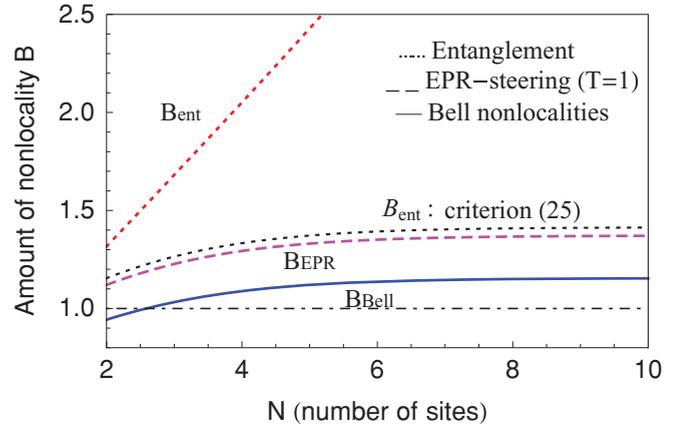


FIG. 4. (Color online) Spin-1 case for the maximally entangled state (37). Nonlocality detected by the Bell inequality (31) and the C_J entanglement and EPR steering criteria (34) and (35). The nonlocality is detected when the appropriate $B > 1$. The entanglement measured by the generalized HZ entanglement criterion (25) is plotted for comparison.

VIII. SPIN-1 CASE

A Bell inequality for multiple spin-1 systems is given by Eq. (31). The C_J entanglement and EPR steering criteria (34) and (35) can be used to test for the entanglement and EPR steering nonlocalities, which for spin-1 are based on the value $C_J = 7/16$ derived by [44]. In this case, we find the C_J criterion (34) becomes more useful than the generalized HZ entanglement criterion (25) in the sense that the right-hand side of the relevant inequality becomes smaller.

Maximally entangled state. In this case, the inequalities can be investigated for the bipartite case of two sites ($N = 2$). First the *maximally entangled* state (38) can be written more explicitly for spin-1 ($d = 3$) as

$$|\psi\rangle_{\text{max}} = \frac{1}{\sqrt{3}}(|1, -1\rangle_1 |1, -1\rangle_2 + |1, 0\rangle_1 |1, 0\rangle_2 + |1, +1\rangle_1 |1, +1\rangle_2). \quad (53)$$

This state can also be written in terms of the boson operators as [18]

$$|\Psi\rangle_{\text{max}} = \frac{1}{2\sqrt{3}}(a_1^\dagger a_2^\dagger + b_1^\dagger b_2^\dagger)^2 |0\rangle. \quad (54)$$

No violation of the Bell inequality (31) is observed here, but the C_J entanglement criterion (34) is satisfied.

Allowing for more sites ($N > 2$), we consider the state (37) that can be written equivalently as

$$|\Psi\rangle_{\text{max}} = \text{norm}(a_1^\dagger a_2^\dagger \cdots a_N^\dagger + b_1^\dagger b_2^\dagger \cdots b_N^\dagger)^{2J} |0\rangle, \quad (55)$$

with $J = 1$. Violation of Bell's inequality (31) is possible for this state with $N > 2$, as shown in Fig. 4. The limit of the square root of the ratio of the left side (L) to the right side (R) of the Bell criterion (31) as $N \rightarrow \infty$ is $2/\sqrt{3}$. In fact generally the ratio is given as

$$B_{\text{Bell}} = \frac{2(2^{N-1})^{1/2}}{\sqrt{3}(2^{N-1} + 1)^{1/2}}. \quad (56)$$

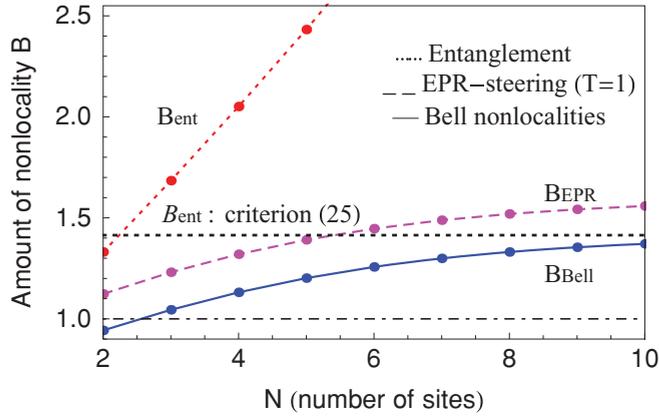


FIG. 5. (Color online) Spin-1 case for the nonmaximally entangled state (42) with optimal r . The amount of nonlocality B_{Bell} (solid), B_{EPR} ($T = 1$) (dashed), and B_{ent} (dotted) detected by the Bell inequality (31) and C_J criteria (35) and (34) is larger than for state (37) shown in Fig. 4. Here, $B_{\text{Bell}}, B_{\text{EPR}}, B_{\text{ent}} > 1$ implies the detection of the respective nonlocality. The entanglement detected by the generalized HZ entanglement criterion (25) with optimal r is shown in here for comparison.

This ratio may be compared with the C_J criterion (34) for entanglement:

$$B_{\text{ent}} = \frac{2^{3N}}{(3^{2N}2^{N-1} + 5^{2N})^{1/2}(2^{N-1} + 1)^{1/2}}. \quad (57)$$

Entanglement can be proved for all $N \geq 2$. The ratio B_{ent} increases with N , and is favorable compared to that obtained from the generalized HZ entanglement criterion (25) (Fig. 4). The amount of nonlocality for the asymmetric EPR steering case ($T = 1$) is also plotted. EPR steering can be detected via the C_J criterion (35) when $B_{\text{EPR}} = 2^{(N+4)/2} / [\sqrt{17}(2^{N-1} + 1)^{1/2}] > 1$ for $T = 1$, $N \geq 2$.

Nonmaximally entangled state. We can also consider the more general state of Eq. (42). In this case, the left (L) and right (R) sides of the Bell inequality (31) become

$$L = \frac{2^{N+2}r^2}{(r^2 + 2)^2}, \quad (58)$$

$$R = \frac{2^N r^2 + 2}{r^2 + 2}, \quad (59)$$

then

$$B_{\text{Bell}} = \frac{2^{(N+2)/2}r}{(r^2 + 2)^{1/2}[2^N r^2 + 2]^{1/2}}. \quad (60)$$

Optimizing r for each value of N , we can get a violation from $N = 3$ sites for spin 1, as for the maximally entangled state, but the violations are greater. The amount of violation is $B_{\text{Bell}} \rightarrow \sqrt{2}$ as the number of sites N increases (Fig. 5).

The result for the C_J entanglement inequality (34) is

$$B_{\text{ent}} = \frac{2^{(N+2)/2}r}{(r^2 + 2)^{1/2}[(25/16)^N r^2 + 2(9/16)^N]^{1/2}}, \quad (61)$$

which increases with the number of sites N . The results are shown in Fig. 5.

The value of B for the C_J criterion for EPR steering (35) can be derived as

$$B_{\text{EPR}} = \frac{2^{(N+2)/2}r}{(r^2 + 2)^{1/2}[(9/8)^N + r^2 2^{N-5} 25]^{1/2}}, \quad (62)$$

as shown in Fig. 5. More generally, using C_J criterion (36), the result is

$$B_T = \frac{2^{(N+2)/2}r}{(r^2 + 2)^{1/2}[2(9/16)^T + r^2 2^{N-T} (25/16)^T]^{1/2}}. \quad (63)$$

IX. SPIN- J CASE

A. Bell nonlocality

The maximally entangled state (55) gives violation of the Bell inequalities (31) only if $d = 2, 3$ and $N \geq 3$. However, the Bell inequalities can be violated for larger d by the optimally selected symmetric nonmaximally entangled states (41), provided the number of sites N is high enough. Figure 6 shows the results for $d = 2, \dots, 7$. In this section we treat cases of arbitrary spin J .

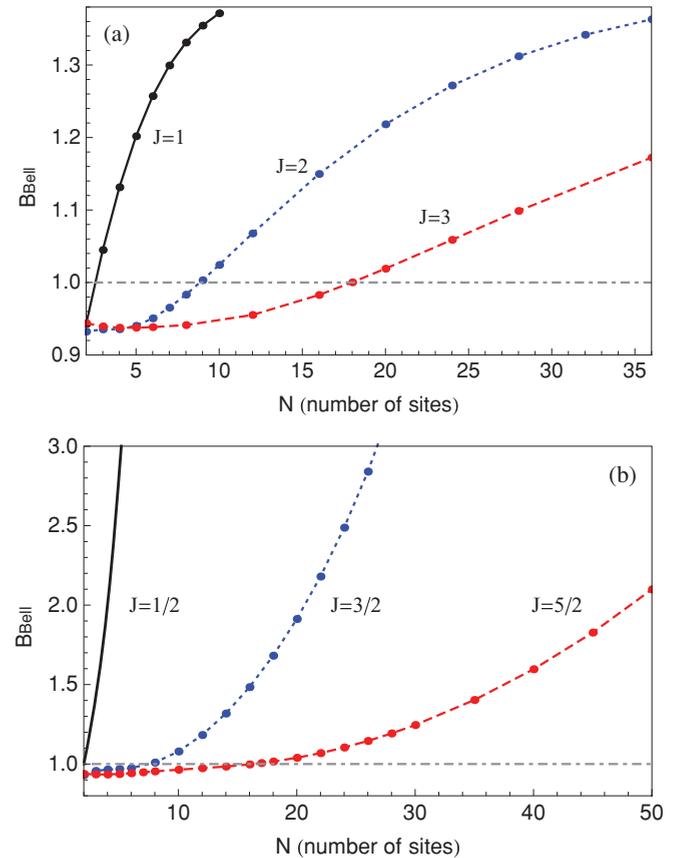


FIG. 6. (Color online) Violation of the Bell inequality (31) as measured by B_{Bell} vs N for the symmetric nonmaximally entangled state (41): (a) $J = 1, 2, 3$ and (b) $J = 1/2, 3/2, 5/2$. Bell nonlocality is confirmed when $B_{\text{Bell}} > 1$.

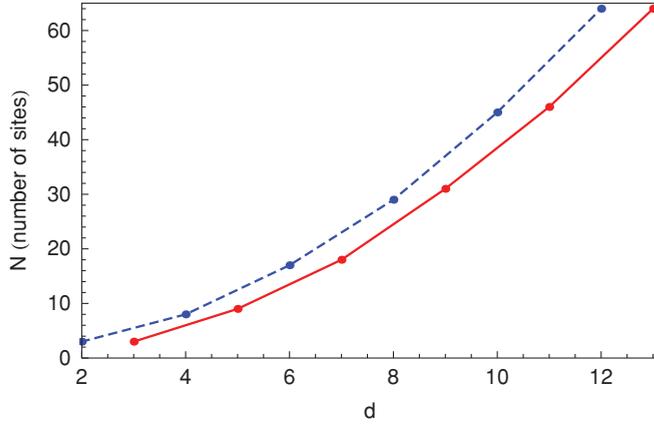


FIG. 7. (Color online) Minimum number of sites N needed for a given d ($d = 2J + 1$) in order to violate the Bell inequality (31) for a nonmaximally entangled state (41): even d (dashed curve), odd d (solid curve).

For nonmaximally entangled states (41), first we need to calculate the values

$$\begin{aligned} \langle (J_1^z)^2 \rangle &= \frac{1}{n} \sum_{m=-J}^J m^2 r_m^2, \\ \langle (J_1^z)^2 (J_2^z)^2 \rangle &= \frac{1}{n} \sum_{m=-J}^J m^4 r_m^2, \end{aligned} \quad (64)$$

where $n = \sum_{m=-J}^J r_m^2$, $m = -J, \dots, +J$. Then we obtain

$$\begin{aligned} L &= \left| \left\langle \prod_{k=1}^N J_k^- \right\rangle \right|^2 \\ &= \frac{1}{n^2} \left[\sum_{m=-J}^J r_m r_{m+1} (\sqrt{J-m} \sqrt{J+m+1})^N \right]^2, \end{aligned} \quad (65)$$

$$\begin{aligned} R &= \langle [(J_1)^2 - (J_1^z)^2] \cdots [(J_N)^2 - (J_N^z)^2] \rangle \\ &= \frac{1}{n} \sum_{m=-J}^J r_m^2 [J(J+1) - m^2]^N, \end{aligned} \quad (66)$$

$$B_{\text{Bell}} = \frac{\sum_{m=-J}^J r_m r_{m+1} (\sqrt{J-m} \sqrt{J+m+1})^N}{\left\{ n \sum_{m=-J}^J r_m^2 [J(J+1) - m^2]^N \right\}^{1/2}}. \quad (67)$$

Optimizing the value of r_m for each spin value J and number of sites N , we see that the Bell inequalities can be violated ($B_{\text{Bell}} > 1$) for larger d provided N is sufficiently large (Fig. 7).

B. Entanglement

The left side (L) of the inequality for the generalized HZ entanglement criterion (25) and the C_J entanglement criterion (34) is the same as that for the Bell inequality; the right side (R), however, changes. Entanglement can be proved for the

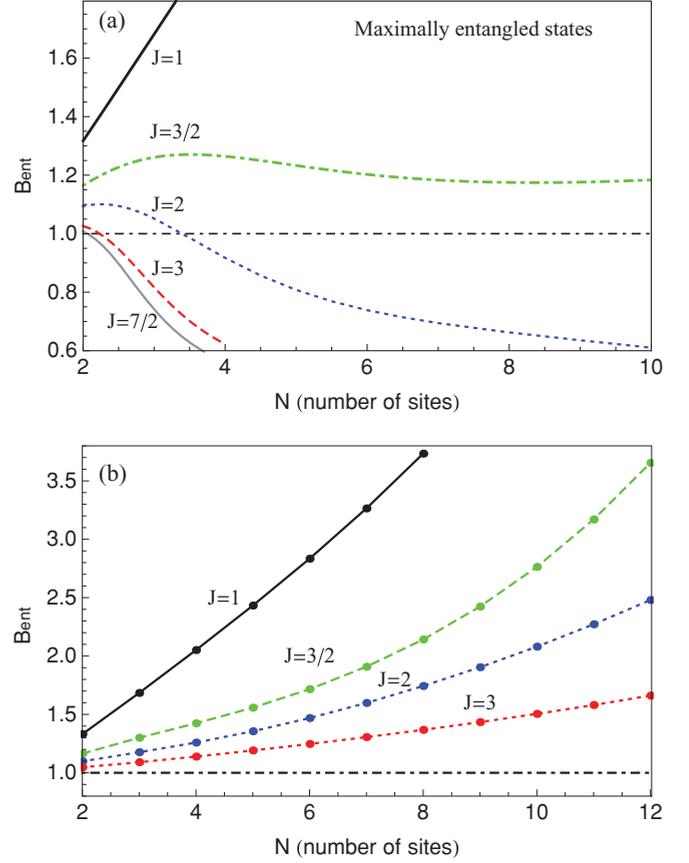


FIG. 8. (Color online) Entanglement as measured by the C_J criterion (34): B_{ent} vs N for (a) the maximally entangled state (37) and (b) the optimal nonmaximally entangled state (41). Entanglement is confirmed when $B_{\text{ent}} > 1$.

state (41) via the C_J entanglement criterion (34) for spin J when

$$B_{\text{ent}} = \frac{\sum_{m=-J}^J r_m r_{m+1} (\sqrt{J-m} \sqrt{J+m+1})^N}{\left\{ n \sum_{m=-J}^J r_m^2 [J(J+1) - m^2 - C_J]^N \right\}^{1/2}} > 1. \quad (68)$$

For maximally entangled states, r_m is fixed as $r_m = [\sqrt{(J-m)!(J+m)!}]^{N-2}$, and then the criterion is only satisfied for lower $J < 4$ and increases or is steady with N only for $J < 2$ [Fig. 8(a)]. However, for the symmetric nonmaximally entangled states (41), $B_{\text{ent}} > 1$ can occur for all spin J and N , provided the amplitudes r_m are optimally chosen. For fixed J , the value of B_{ent} increases with N , while for fixed N the violation decreases with increasing J [Fig. 8(b)].

The generalized HZ entanglement criterion (25), however, has a different R (we choose appropriate s_k, l_k to get larger L and smaller R):

$$\begin{aligned} L &= \left| \left\langle \prod_{k=1}^N J_k^- \right\rangle \right|^2 \\ &= \left| \sum_{m=-J+1}^J \frac{r_{m-1} r_m}{n} [(J+m)(J-m+1)]^{N/2} \right|^2, \end{aligned} \quad (69)$$

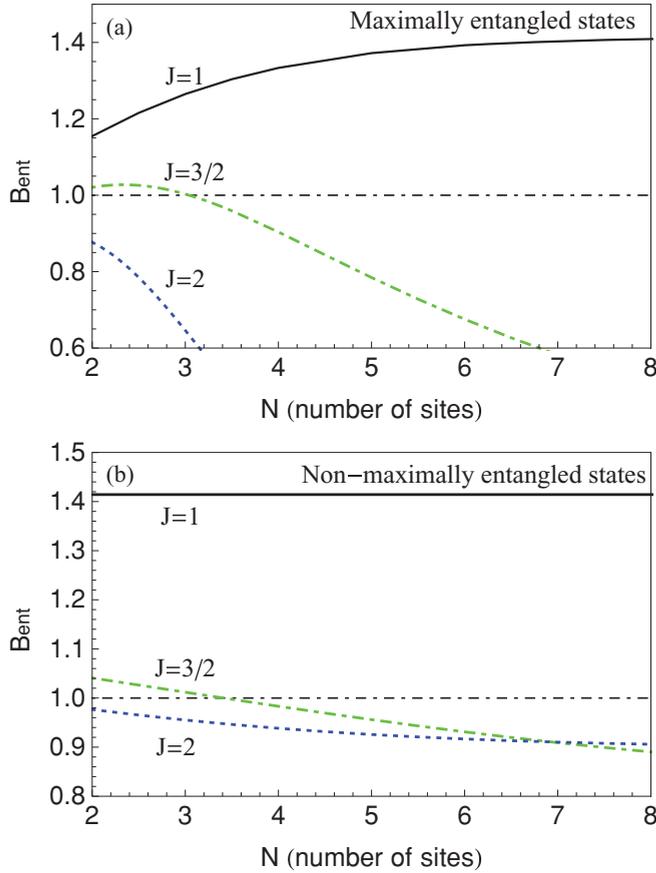


FIG. 9. (Color online) Entanglement as measured by the generalized HZ entanglement criterion (25): B_{ent} vs N for (a) the maximally entangled state (37) and (b) the optimal nonmaximally entangled state (41). Entanglement is confirmed when $B_{\text{ent}} > 1$.

and

$$R = \left\langle J_1^+ J_1^- \prod_{k=2}^N J_k^- J_k^+ \right\rangle = \sum_{m=-J}^J \frac{r_m^2}{n} \{(J-m)(J+m+1) \times [(J-m)(J+m+1)]^{N-1}\}, \quad (70)$$

so that entanglement is detected when this $B_{\text{ent}} = \sqrt{L/R} > 1$. For the maximally entangled state, entanglement is only detected for lower J , as shown in Fig. 9(a), while the result for symmetric but otherwise optimized r_m is shown in Fig. 9(b). The generalized HZ entanglement criterion becomes less effective than the C_J entanglement criterion for $J > 1/2$.

Entanglement for the bipartite spin- J case. The bipartite ($N = 2$) case for arbitrary spin J has been considered recently by Zheng *et al.* [38] using criteria similar to Eq. (25) (but with restricted choices of l_k). Here we use the generalized HZ entanglement criterion (25): entanglement is detected when

$$|\langle J_1^{s_1} J_2^{s_2} \rangle|^2 > \langle J_1^{l_1} J_1^{-l_1} J_2^{l_2} J_2^{-l_2} \rangle, \quad (71)$$

where s_k and l_k are independently chosen to be + or -. We consider the highly correlated state (41)

$$|\psi\rangle = \frac{1}{n} \sum_{m=-J}^J r_m |J, m\rangle_1 |J, m\rangle_2, \quad (72)$$

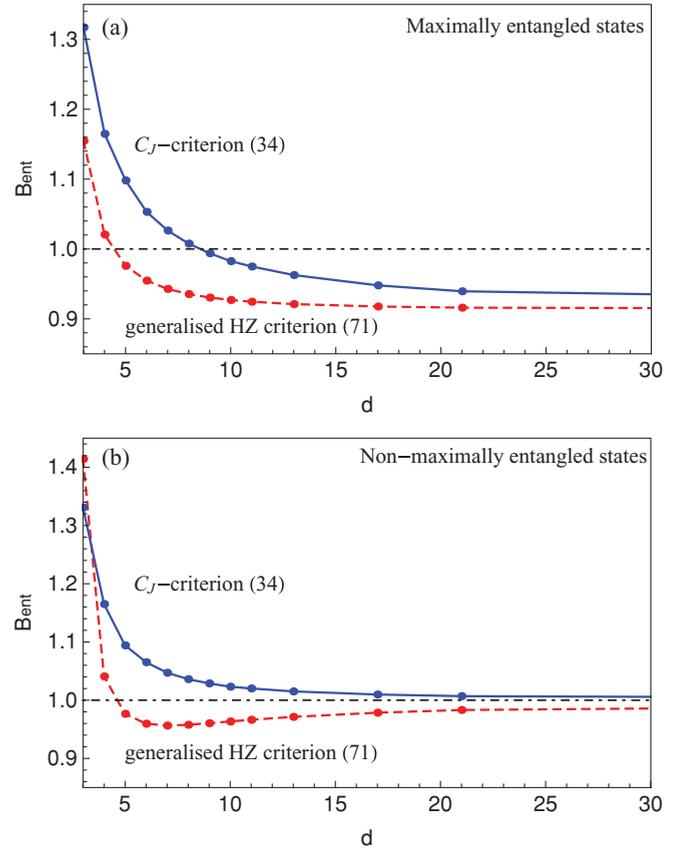


FIG. 10. (Color online) Bipartite case: Entanglement as measured by the generalized HZ criterion (71) (dashed curve) and the C_J criterion (34) (solid curve): B_{ent} vs $d = 2J + 1$ for (a) the bipartite maximally entangled state (37) and (b) the nonmaximally entangled state (41) with optimal r_m . Entanglement is confirmed when $B_{\text{ent}} > 1$.

for which

$$L = |\langle J_1^- J_2^- \rangle|^2 = \frac{1}{n^2} \left| \sum_{m=-J+1}^J r_{m-1} r_m (J+m)(J-m+1) \right|^2, \quad (73)$$

and

$$R = \langle J_1^+ J_1^- J_2^- J_2^+ \rangle = \frac{1}{n} \sum_{m=-J}^J r_m^2 (J^2 - m^2) [(J+1)^2 - m^2]. \quad (74)$$

Immediately, for $J = 1/2$, we see that R is zero for all choices of r_m , i.e., for the generalized Bell state

$$|\psi\rangle_{\text{max}} = \cos \theta |0\rangle^{\otimes 2} + \sin \theta |1\rangle^{\otimes 2}, \quad (75)$$

and hence the criterion detects *all* entanglement for this state. This contrasts with the criterion considered by [39], which does not detect entanglement for the symmetric case $\cos \theta = \sin \theta$.

For spin $J = 1$, detection is still possible though less ideal, as shown for the choice of constant and real r_m (a maximally entangled state) in Fig. 4 and for an optimally chosen but real and symmetric r_m in Fig. 5.

Figure 10 presents results for detection of entanglement for the bipartite case with increasing J for the generalized

HZ entanglement criterion (71) and the C_J criterion (34). Neither criterion can detect entanglement of the maximally entangled state (37) for high J . The C_J criterion can be used to detect entanglement for all J that we have calculated using the optimized states (41). These criteria may be compared with the variance local uncertainty relation (LUR) criteria of [44,45] which detect entanglement for all the highly correlated states

(37) and (41) of arbitrary J . We note the earlier spin squeezing criteria of [49] are not sensitive to entanglement in cases where $\langle J_z \rangle = 0$.

C. EPR steering

For the nonmaximally entangled state (41), the C_J criterion for EPR steering (35) is satisfied when

$$B_{\text{EPR}} = \frac{\sum_{m=-J}^J r_m r_{m+1} (\sqrt{J-m} \sqrt{J+m+1})^N}{\left[n \sum_{m=-J}^J r_m^2 \{ [J(J+1) - m^2]^{N-1} [J(J+1) - m^2 - C_J] \} \right]^{1/2}} > 1. \quad (76)$$

This is predicted for all dimensions d with optimal r_m , provided the number of sites N is high enough. For the more general model (7) of T quantum sites, the C_J criterion (36) for the nonlocality becomes, for spin J ,

$$B_T = \frac{\sum_{m=-J}^J r_m r_{m+1} (\sqrt{J-m} \sqrt{J+m+1})^N}{\left[n \sum_{m=-J}^J r_m^2 \{ [J(J+1) - m^2]^{N-T} [J(J+1) - m^2 - C_J]^T \} \right]^{1/2}} > 1. \quad (77)$$

This more general criterion is also verified for all dimensions d with optimal r_m , provided N is high enough.

D. Summary

Plots of the violation of the relevant inequalities for the three types of nonlocality for fixed d and increasing N are shown in Fig. 11. The strength of the violation as measured by B for these particular inequalities increases with N , but this occurs for all d only for the optimized nonmaximally entangled states. This effect is similar to that reported for the MABK-type Bell inequalities of Cabello [28], though an increase in violation with N was not reported for the multipartite qudit Bell inequalities of Chen and Deng [30].

For fixed N and increasing d (J), the strength of the violation reduces (Fig. 12). This result differs from that of Collins *et al.* [24] for $N = 2$, who obtained steady violation for increasing d for the maximally entangled states. Cabello [28]

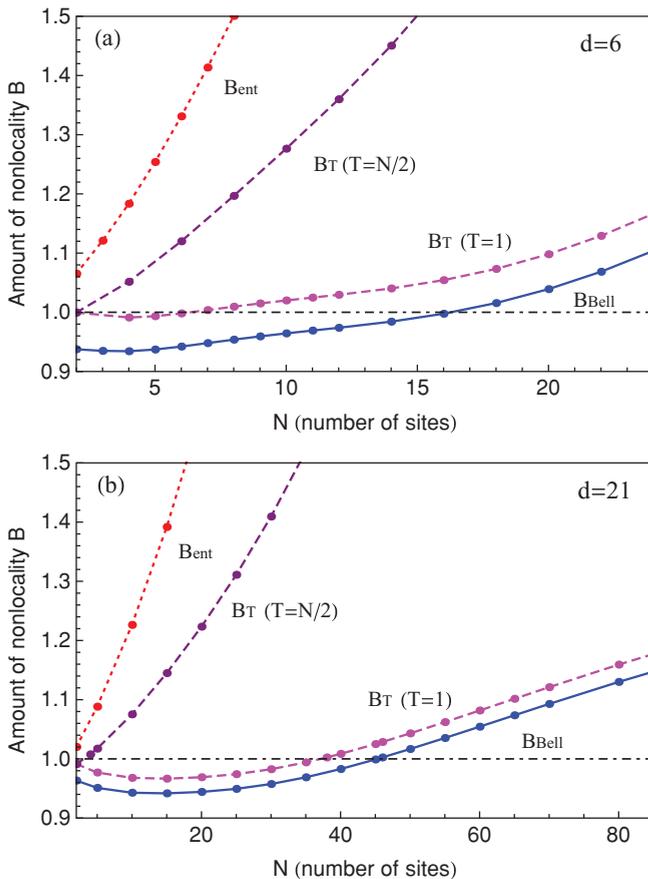


FIG. 11. (Color online) Effect of increasing the number of sites N for fixed d on the strength B of the nonlocality as measured by the Bell inequality (31) and the C_J criteria (34), (35), and (36): (a) $d = 6$ ($J = 5/2$) and (b) $d = 21$ ($J = 10$). Results are for the nonmaximally entangled states (41): entanglement (dotted) can be detected when $B_{\text{ent}} > 1$; Bell nonlocalities (solid) can be detected when $B_{\text{Bell}} > 1$. The C_J criterion (36) for the more general model of T quantum sites can be verified when $B_T > 1$.

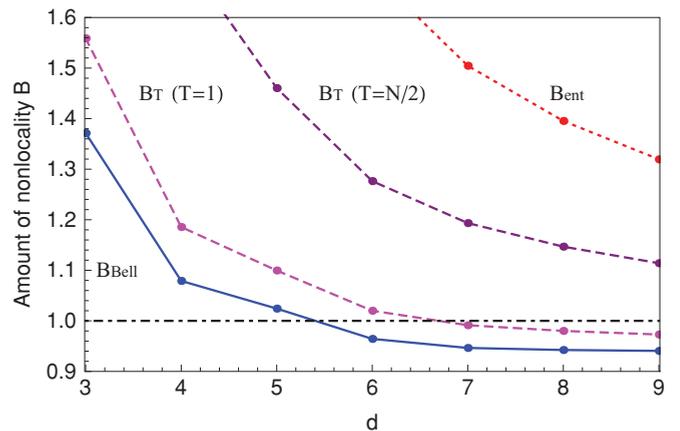


FIG. 12. (Color online) Entanglement, EPR steering, and Bell nonlocalities vs d (i.e., $d = 2J + 1$) as measured by the C_J criteria (34), (35) and the Bell inequality (31) for the nonmaximally entangled states (41): $N = 10$. The nonlocality is confirmed when $B > 1$.

also reported a steady violation with increased d with any fixed N , but this effect was not observed by the violations of the inequalities of Son *et al.* [29].

X. CONCLUSION

We have derived a unified set of measurement-based criteria for multipartite entanglement, steering, and Bell nonlocality for N -site systems of higher dimensionality d . Direct application of the Bell inequality (31) shows that demonstrations of Bell nonlocality are possible for maximally entangled highly correlated states $|\psi_m\rangle$ where $d = 2, 3$, and all $N \geq 3$. Symmetric nonmaximally entangled but highly correlated states of the type considered by Acin *et al.* [43] show violations for all higher d , provided the number of sites N is large enough and the state is optimized.

Our work also includes the derivation of entanglement and steering criteria that take a very similar form to the Bell inequality. We have introduced two types of such criteria for entanglement. One is valid for all spin states, and for entanglement is similar to criteria that have been presented by Hillery and co-workers [36,37,39]. We therefore call this a generalized HZ criterion. The other (a “ C_J criterion”) is valid for states with a fixed total spin J , and reduces to the entanglement criterion of Roy [46] for $J = 1/2$. For maximally entangled states $|\psi_m\rangle$ the C_J entanglement criterion can only detect entanglement for low spin J ($J \leq 3$). The violation of the C_J inequality increases with N for $J = 1/2$, and $J = 1$, but for higher spin it decreases with N , so that the bipartite case is optimal. The generalized HZ entanglement criterion, in a form different from that considered originally in [39], is remarkably sensitive in the spin-1/2 case, being

able to detect all entanglement of a generalized multipartite GHZ state. This criterion, however, becomes generally less sensitive than the C_J criterion for higher J . Both entanglement criteria can detect entanglement for some optimized symmetric nonmaximally entangled states, but the first criterion is only sensitive for low J . Violation of the appropriate C_J entanglement inequality is possible in this case for all J and N : the violation decreases for increasing J , but will increase with N for fixed J .

The degree of violation obtained from these Bell inequalities shows the MABK-type growth of violation with N , but the violation decreases with increasing J . However, for fixed J , one can achieve a violation by increasing N sufficiently. Our approach has the advantage that it readily gives entanglement and steering EPR paradox criteria and gives analytical predictions for simple quantum states. It might be noted that the different form of the right-hand side of the Bell and nonlocality CFRD-type inequalities of this paper may mean a more advantageous result for other scenarios, such as where loss is included, as studied in [42].

As a last point, while the nonlocality criteria of this paper refer to multipartite scenarios, they do not necessarily detect a genuine multipartite entanglement or Bell’s nonlocality that is shared between all N parties. Such entanglement is crucial in addressing the real existence of macroscopic entanglement and will be treated elsewhere.

ACKNOWLEDGMENTS

We are grateful for support from the Australian Research Council via ACQAO COE and Discovery grants. One of us (P.D.D.) thanks the Humboldt Society for financial support.

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