Observations of Kelvin-Helmholtz instability at the air-water interface in a circular domain

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We present an analysis of Kelvin-Helmholtz instability in a circular domain in the limit of the azimuthal integer wavenumber, $n \to \infty$ which reproduces the classical results for a rectilinear geometry at the rim, provided that the additional condition that the surface current to surface wind ratio is $(\rho_1/\rho_2)^{1/2}$ where ρ_1 and ρ_2 are respectively the densities of air and water, is satisfied. Experiments were carried out in a circular rig of radius 0.19 m in which a family of unstable waveforms with $n \approx 60$ were observed with properties (including the additional condition) in approximate agreement with theory. The additional condition is consistent with the absence of a surface shear stress in the instability process. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3697487]

INTRODUCTION

Kelvin-Helmholtz instability has been known for over 140 years,[1] however, it is not thought to be important in the open ocean, possibly only occurring when a sudden strong wind gust near the crest of a large wave may give rise to a local transient instability which produces a spray of drops sweeping the surface.[2] Normally a host of other instability mechanisms dominate the air-sea interaction, see, for example, Sullivan and McWilliams.[3] We report here some observations from an experiment in a circular domain where it was the dominant instability mechanism in an air-water system, which are consistent with the inertial coupling hypothesis for momentum exchange across the sea surface.

KELVIN-HELMHOLTZ INSTABILITY AT THE AIR-WATER INTERFACE IN A CIRCULAR DOMAIN

In a circular domain in which the water motion is driven by a rotating air motion, the two-layer coupled interaction can be formulated in terms of a separable solution of the form, $J_n(\gamma r)\sin(n\theta - \omega t)\,F(z)$ where r is the radial co-ordinate, n is the azimuthal integer wavenumber, γ is the radial wavenumber, ω is the frequency, $J_n(\gamma r)$ is a Bessel function of the first kind of order n, and F(z) is the vertical structure of the wave. On matching the pressure at the interface, following the method used in Ref. 4, we obtain a solution for $n \to \infty$ (such that the angle between the wavefronts becomes vanishingly small), with the necessary condition that, $\Omega_2 = \varepsilon\,\Omega_1$ where Ω_1 and Ω_2 are respectively the angular velocities of air and water relative to rest, and $\varepsilon = (\rho_1/\rho_2)^{1/2}$ in which ρ_1 and ρ_2 are respectively the densities of air ($\rho_1 = 1.2$ kg m^{-3}) and water ($\rho_2 = 1000$ kg m^{-3}) and hence $\varepsilon = 0.034$. This relation, which ensures a flat undisturbed interface, see Appendix, cannot be obtained from the classical rectilinear solution. At a radius (R), the critical condition for instability is

$$(u_1 - u_2)^2 > 2(Tg(\rho_2 - \rho_1))^{1/2}(\rho_1 + \rho_2)/(\rho_1\rho_2), \tag{1}$$

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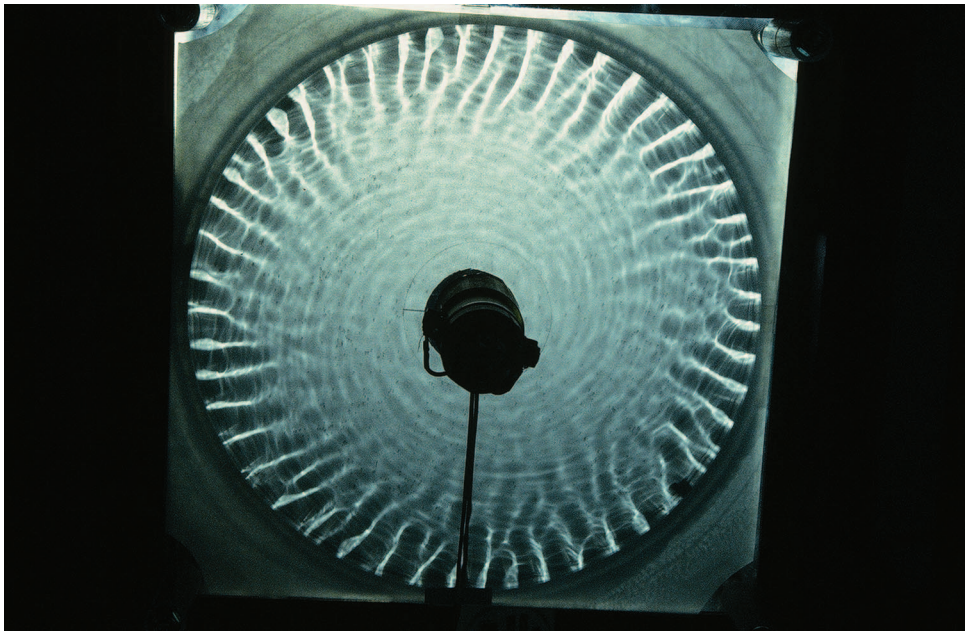

FIG. 1. Kelvin-Helmholtz instability observed in a stationary transparent circular tank of diameter 380 mm in which the air was driven by a rotating clear plastic disk, with a water depth of 60 mm and an air gap of 35 mm. The rotation is clockwise in which the free radial flow takes the form of a trailing logarithmic spiral. The rotation rate of the air was determined by a stroboscope, and the rotation rates of the wave fronts and of small particles introduced into the water were measured by digitally freezing video images. The still and video images were obtained using a light source situated centrally under the tank and directed upwards onto opaque paper under the base of the tank. The central boss is the electric motor which drove the rotating disk. The water used was freshly obtained from the city water supply. In the runs in which Kelvin-Helmholtz instability was observed no wave forms were present at lower rotation rates. The dominance of the Kelvin-Helmholtz instability over other instability mechanisms is possibly due to the radial structure of the wavefronts in which the amplitude of $J_n(\gamma r)$ increases sharply at large n to a maximum just inside the rim where the surface wind is greatest. The instability parameters are the average of two runs in which $n = 68$ and $n = 50$. The image shows the $n = 50$ run. See photograph in Ref. 6. Reprinted with permission from J. C. Vladusic, "Wind wave growth in a circular tank," B.Sc. (Hon.) thesis (The University of Melbourne, 2001).

where g is the acceleration of gravity, T is the surface tension, and $u_1 = R\Omega_1$ and $u_2 = R\Omega_2$ in which $kR = n$ and $k = \gamma$ is the vertical decay constant of the wave, which has respectively the wave speed and wave number,

$$\omega/k = (\rho_1 u_1 + \rho_2 u_2)/(\rho_1 + \rho_2) \quad \text{and} \quad k = (g(\rho_2 - \rho_1)/T)^{1/2} \tag{2}$$

identical with the rectilinear result,[5] but in addition when it is generated the external velocities satisfy the relation

$$u_1 - u_2/\varepsilon = 0. \tag{3}$$

The approach to this situation was investigated in a rig,[6,7] in which air was driven by a transparent circular disk of rotation rate (Ω_0), which in turn generated the water motion (Fig. 1). For a rig of radius, $R = 0.19$ m, it was found, assuming that the rotation rate of the air, $\Omega_1 = 0.3\ \Omega_0$,[8] that instability set in at a rotation speed, $\Omega_1 \approx 38$ rad/s which is similar to the prediction (1), which for $T = 7.4 \times 10^{-2}$ N m^{-1} and $g = 9.8$ ms^{-2} yields $\Omega_1 - \Omega_2 = 35$ rad/s, and also that $n = 59$, which is in very good agreement with the theoretical prediction for k, which satisfies $kR = b_n$, where $b_n = n + 1.86\, n^{1/3}$,[9] and from Eq. (2) yields $n = 62$, which corresponds with an azimuthal wavelength ($2\pi R/n$) of 1.9 cm (Fig. 1).

The progression of the wave fronts was also measured, and the rotation rate ($\Omega_f \approx 1.35$ rad/s) of the surface flow was obtained by timing small particles introduced into the water surface. These observations indicate that $\Omega_f/(\Omega_1 - \Omega_2) \approx 0.039$, which is in approximate agreement with Eq. (3) for which $\Omega_2/(\Omega_1 - \Omega_2) = 0.036$, indicating that Ω_f is a good estimator of Ω_2 (see Section "**The**

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Aerodynamically Rough Coupling Relations For The Interfacial Shearing Stress"), and also that the unstable wave is nearly stationary relative to the surface flow as predicted by Eq. (2).

THE AERODYNAMICALLY ROUGH COUPLING RELATIONS FOR THE INTERFACIAL SHEARING STRESS

How does a relation of the kind (3) arise? The underlying hypothesis is that of aerodynamically rough flow, which is independent of the viscosities in both fluids, and for which the air-sea coupling is represented by a pair of equations for the interfacial (surface) shearing stress in a wave boundary layer bounded below by the surface current (u_2) and above by the surface wind (u_1), which each occur at a distance (z_B) from the interface, $z = 0$,[10,11] where z_B denotes the outer limit of the influence of the wave field. The pair of equations is

$$u_* = K(|z|)^{1/2}(u(z) - u_L) \text{ and } w_* = K(|z|)^{1/2}(-u(-z) + \varepsilon u_L),\ 0 < z < z_B, \tag{4}$$

where $\tau_{sx} = \rho_1 u_*^2 = \rho_2 w_*^2$ in which τ_{sx} is the interfacial shearing stress and u_* and w_* are respectively the friction velocities in air and water, $K(|z|)$ is a drag coefficient, and u_L and εu_L are wave induced velocities, which in the open ocean are due respectively to the phase velocities and the particle velocities of the waves (the surface Stokes velocity). In the Kelvin-Helmholtz problem, the wave motion is controlled by the two velocities u_1 and u_2, and decays exponentially in the vertical with a decay constant, k. Hence the wave induced velocities are simply $u_L = u_1$ and $\varepsilon u_L = u_2$, and $z_B = O(k^{-1})$.

On eliminating u_L and evaluating at z_B we obtain the inertial coupling relation[10]

$$u_* = \rho_1 K_I^{1/2}(u_1 - u_2/\varepsilon), \tag{5}$$

where $K_I = \frac{1}{4}\, K(|z_B|)$, and on matching the pair of Eq. (4) at $z = 0$, we obtain $u_s = (\varepsilon\, u_1 + u_2)/(1 + \varepsilon)$, where $u_s = u(0)$ is the surface drift current. The drift current connects the surface wind and the surface current in a thin unresolved surface layer[11] of thickness much less than z_B.

In the open ocean, the air-sea exchange of momentum occurs through a wind stress (the interfacial shearing stress), and the fully turbulent air and water velocity profiles in Eq. (4) are approximately logarithmic.[11] In Kelvin-Helmholtz instability, however, from Eqs. (3) and (4), $u_* = 0$ and the velocity profiles reduce to $u(z) = u_1$ and $u(-z) = u_2$, and also $u_s = 2\, u_2/(1 + \varepsilon)$, i.e., for a large density contrast system the surface drift velocity is approximately twice the surface current. These conditions are applicable in the inviscid limit of the classical analysis.

For our experimental air-water system, the predicted ratio, $\Omega_s/(\Omega_1 - \Omega_2) = 0.069$, where $\Omega_s = u_s/R$. Hence the ratio ($\Omega_f/(\Omega_1 - \Omega_2) \approx 0.039$) indicates that the observed surface flow is only slightly greater than the surface current and much less than the surface drift velocity.

CONCLUSION

The main conclusion is that the experimental results in the circular rig are consistent with the existence of Kelvin-Helmholtz instability at the air-water interface.

The inertial coupling model for the exchange of momentum across the air-water interface has been shown to be applicable for this special case in which the interfacial shearing stress is zero and the near surface velocity profile in each fluid is uniform, giving rise to a step change in velocity at the interface.

In the ocean, the wind stress (the interfacial shearing stress) is never zero, except in calm conditions, and an approximately constant stress layer exists throughout the wave boundary layer. In addition, the near surface motion occurs relative to the surface geostrophic velocity (u_0), rather than the rest conditions of the experimental rig. On adding this reference velocity, Eq. (5) becomes, $u_* = \rho_1\, K_I^{1/2}((u_1 - u_0) - (u_2 - u_0)/\varepsilon)$. The surface geostrophic velocity reference frame for the wind stress has important consequences for the wind power input to the ocean general circulation by virtue of the temporal and spatial variability of the geostrophic velocity,[12] which are beyond the scope of this note.

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APPENDIX: KEY STEPS IN THE SOLUTION

(1) The matching of the pressure at the interface yields the equality,[13]

$$\gamma^2 = k_1^2(1 - 4\Omega_1^2/(\Omega_1 n - \omega)^2) = k_2^2(1 - 4\Omega_2^2/(\Omega_2 n - \omega)^2), \tag{A1}$$

where in the limit of $n \to \infty$, $\gamma = k$, and the no slip boundary condition at $r = R$ yields $k = n/R$.

(2) On applying the surface equation,

$$\eta = A\sin(n\theta - \omega t)J_n(\gamma r) + \eta_0(r), \tag{A2}$$

where A is a constant, and $\eta_0\,(r) = \frac{1}{2}\, r^2/g\, \{(\rho_2\, \Omega_2{}^2 - \rho_1\, \Omega_1{}^2)/(\rho_2 - \rho_1)\}$. With the necessary condition that $\Omega_2 = \varepsilon\ \Omega_1$, the critical condition for instability (1) is obtained at which the wave speed and the wave number are given by Eq. (2), and the undisturbed surface is flat.

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