Swinburne Research Bank

http://researchbank.swinburne.edu.au

Abramov, V. M. (2008). Large closed queueing networks in semi-Markov environment and their application.

Originally published in *Acta Applicandae Mathematicae, 100*(3), 201–226.
Available from: http://dx.doi.org/10.1007/s10440-007-9180-4

Copyright © 2007 Springer Science+Business Media, LLC.

This is the author's version of the work, posted here with the permission of the publisher for your personal use. No further distribution is permitted. You may also be able to access the published version from your library. The definitive version is available at http://www.springerlink.com/.

Swinburne University of Technology | CRICOS Provider 00111D | swinburne.edu.au

arXiv:math/0612224v6 [math.PR] 13 Nov 2007

LARGE CLOSED QUEUEING NETWORKS IN SEMI-MARKOV ENVIRONMENT AND THEIR APPLICATION

VYACHESLAV M. ABRAMOV

Abstract. The paper studies closed queueing networks containing a server station and k client stations. The server station is an infinite server queueing system, and client stations are single-server queueing systems with autonomous service, i.e. every client station serves customers (units) only at random instants generated by a strictly stationary and ergodic sequence of random variables. The total number of units in the network is N. The expected times between departures in client stations are $(N\mu_j)^{-1}$. After a service completion in the server station, a unit is transmitted to the jth client station with probability p_j $(j = 1, 2, \ldots, k)$, and being processed in the jth client station, the unit returns to the server station. The network is assumed to be in a semi-Markov environment. A semi-Markov environment is defined by a finite or countable infinite Markov chain and by sequences of independent and identically distributed random variables. Then the routing probabilities p_j $(j = 1, 2, \ldots, k)$ and transmission rates (which are expressed via parameters of the network) depend on a Markov state of the environment. The paper studies the queue-length processes in client stations of this network and is aimed to the analysis of performance measures associated with this network. The questions risen in this paper have immediate relation to quality control of complex telecommunication networks, and the obtained results are expected to lead to the solutions to many practical problems of this area of research.

Contents

1. Introduction 3
2. Motivation 7
3. Definition of semi-Markov environment and classification of bottleneck stations 8
3.1. Semi-Markov environments 8

1991 *Mathematics Subject Classification.* 60K25, 60K30, 60H30, 60H35.
Key words and phrases. Closed queueing network; Random environment; Martingales and Semimartingales; Skorokhod reflection principle.

3.2. Classification of bottleneck stations 9
4. Queue-length processes in the client stations of networks with semi-Markov environment 12
5. Bottleneck analysis in the case of a piece-wise constant environment 15
6. The main theorem on queue-length processes in the client stations of networks with semi-Markov environment 25
7. Application: Performance analysis of symmetric large closed client/server computer networks with unreliable client stations 27
7.1. Formulation of the problem 27
7.2. Solution of the problem 28
7.3. Numerical calculation 29
8. Discussion of new problems and associated monotonicity conditions for the networks in Markov environment 30
9. Example of numerical study 31
10. Concluding remarks 33
Acknowledgement 33
References 34

1. Introduction

We consider closed queueing networks containing a server station and k client stations. The server station is an infinite server queueing system with identical servers. Client stations are single-server queueing systems with an autonomous service mechanism, where customers (units) are served only at random instants generated by a strictly stationary and ergodic sequence of random variables.

Queueing systems with an autonomous service mechanism were introduced and originally studied by Borovkov [11], [12]. The formal definition of these systems in the simplest case of single arrivals and departures is as follows. Let $A(t)$ denote an arrival point process, let $S(t)$ denote a departure point process, and let $Q(t)$ be a queue-length process, and all of these processes are started at zero ($A(0) = S(0) = Q(0) = 0$). Then the autonomous service mechanism is defined by the equation:

$$Q(t) = A(t) - \int_0^\infty \mathbb{I}\{Q(s-) > 0\}\mathrm{d}S(s).$$

The queueing systems with an autonomous service mechanism have been studied in many papers (e.g. Abramov [1], [3], [5], Fricker [16], [17], Gelenbe and Iasnogorodski [18]). The structure of queueing systems or networks with autonomous service and their analysis is much easier than those structure and analysis of "usual" systems with generally distributed service times. Queueing systems and networks with autonomous service, because of their simple construction, are studied under general settings on dependent inter-arrival and inter-departure times, and their analysis is often based on methods of stochastic calculus and the theory of martingales. The corresponding results for usual Markovian queueing systems or networks follow as a particular case of the corresponding results for queueing systems or networks with Poisson input and autonomous service. For different applications of queueing systems (networks) with autonomous service see e.g. [1], [3], [4] and [5].

The assumption that the queueing mechanism is autonomous substantially simplify the analysis. However, according to sample-path results of [4] this assumption can be removed. So, all of the main results remain valid for quite general client/server networks without the special assumption that the service mechanism is autonomous.

In the present paper we study client/server networks in a semi-Markov environment. There has been an increasing attention to queueing systems in a random environment in the literature (e.g. [9], [13],

[15], [19], [26], [29], [31] and others). However, most of these papers mainly develop the theory and remain far from real-world applications.

The aim of the present paper is twofold. First, we establish new theoretical results for client/server networks in semi-Markov environment describing the behavior of queue-length processes of this network. Second, we show how these theoretical results can be applied to solve real-world problems. Some of these problems are solved in the present paper. Other ones will be solved in the future.

The model of the network, which is considered in this paper, is very close to the models considered in [22] and [1] (see Figure 1). For a more general construction of network with two types of node and multiple customer classes see [3].

The description of the present model is based on that of the model of [1]. For other papers studying the models of client/server networks see also [10], [21], [23], [24], [25], [28], [30], [35] and other papers.

The previous assumptions of [1], that are also used in the present paper, are repeated below.

The departure instants in the jth client station ($j = 1, 2, \ldots, k$) are denoted by $\xi_{j,N,1}$, $\xi_{j,N,1} + \xi_{j,N,2}$, $\xi_{j,N,1} + \xi_{j,N,2} + \xi_{j,N,3}, \ldots$, and

- each sequence $\{\xi_{j,N,1}, \xi_{j,N,2}, \ldots\}$ forms a strictly stationary and ergodic sequence of random variables.

The corresponding point process associated with departures from the client station j is denoted

$$S_{j,N}(t) = \sum_{i=1}^{\infty} \mathbb{I}\left\{\sum_{l=1}^{i} \xi_{j,N,l} \leq t\right\}.$$

The total number of units in the network is N. The number N is a large parameter, and we assume that N increases indefinitely. This means that we assume that N is a series parameter, and the series of models (with different N) are considered on the same probability space.

It was assumed in [1] that the service time of each unit in the server station is an exponentially distributed random variable with a *given* parameter λ. In the present paper, the assumption is another. Under the assumption that an environment is random (Markov or semi-Markov), the parameter λ is not longer a *constant value*. It is a random variable, taking values in dependence of environment states. (By environment states we mean the states of the corresponding Markov chain, which is described later.) The same is noted regarding the routing probability matrix.

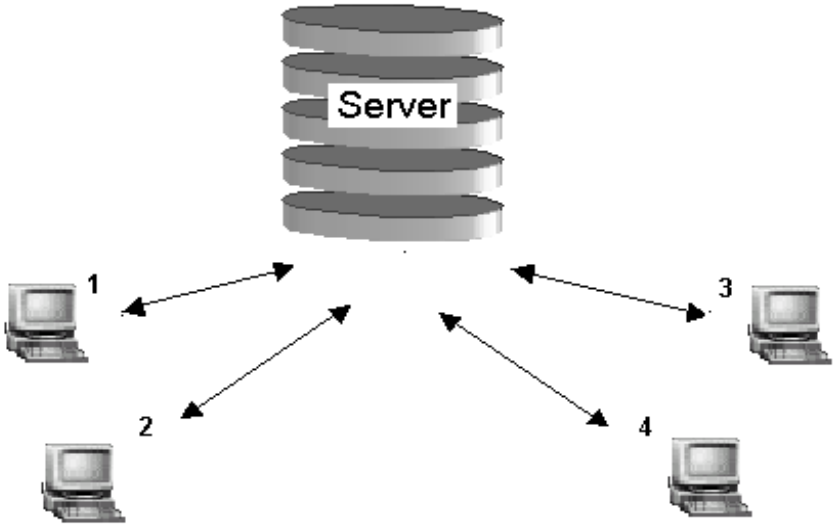

FIGURE 1. An example of client/server network topology

There were the standard assumptions in [1] that after a service completion at the server station, a unit was transmitted to the client station j with probability p_j, $p_j \geq 0$, and $\sum_{j=1}^{k} p_j$=1. These assumptions are not longer valid in the case of the system considered in this paper. The routing probabilities are assumed to be random, taking the values in dependence of the environment states as well.

Let us first describe *Markov states* of semi-Markov environment, and then latter we describe the notion of semi-Markov environment as well. The network is assumed to be complete in the following sense.

The server station and k client stations communicate by links. The number of links is k. Therefore, the Markov chain of the environment states is defined as follows. The number of possible environment states is assumed to be finite or countable infinite. These states are denoted by $\mathcal{E}_i$, $i = 1, 2, \ldots$. For the Markov or semi-Markov environment considered later we shall also use the notation $\mathcal{E}(t)$. The meaning of this notation is the state of Markov chain in time t. The initial state is denoted by $\mathcal{E}_0$ or $\mathcal{E}(0)$.

The completeness of the network means that

- for any j=1,2,..., k there exists $i_j \geq 1$, such that $p_j(\mathcal{E}_{i_j}) > 0$.

This assumption is not used in our proofs explicitly. Nevertheless, it must be mentioned. If $p_j(\mathcal{E}_{i_j}) = 0$ for all $i_j \geq 1$, then the station j becomes isolated and therefore not representative. In the above assumption by $p_j(\mathcal{E}_i)$ we mean the value of probability p_j when the network is in state $\mathcal{E}_i$. It is assumed additionally that for any $i = 1, 2, \ldots$, the sum of probabilities $\sum_{j=1}^{k} p_j(\mathcal{E}_i)$ =1. The notation with similar meaning is used for parameter λ. Namely, $\lambda(\mathcal{E}_i)$ is assumed to be strictly positive for any state $\mathcal{E}_i$, however $\lambda_j(\mathcal{E}_i) = \lambda(\mathcal{E}_i)p_j(\mathcal{E}_i)$ can be equal to 0 (because $p_j(\mathcal{E}_i)$ need not be strictly positive in general).

The results of [1] are associated with the asymptotic analysis of a closed client/server network with a bottleneck station as N increases indefinitely. One of the main results of [1] was then developed in [3] for networks containing two types of node and multiple customer classes, where one of client stations was bottleneck. The results obtained in [1], [2], [3] as well as in preceding paper [22] all are a theoretical contribution to the theory of client/server computer networks with bottlenecks.

The aim of the present paper is another. We follow towards performance analysis, and are aimed to use the known theoretical results of [22] and [1]. However, the known theoretical results obtained in these papers are scanty for their immediate application to real telecommunication systems, whose parameters can change in time. Therefore, for the purpose of performance analysis, a substantial development of

the earlier results related to this model is required. Therefore, before studying the behaviour of queueing networks in semi-Markov environment, we first study the behaviour of this network in its special case of piece-wise deterministic environment. Then the results are extended to the more general situation of semi-Markov environment.

The paper is organized as follows. In Section 2, the paper is motivated by formulating the concrete real-world problems, where the results of the present paper can be applied. In section 3, necessary notions of piece-wise deterministic, Markov and semi-Markov environments as well as classification of bottleneck stations are defined. In Section 4, the semi-martingale decomposition for the queue-length process in semi-Markov environment is derived. Then in Section 5, the queue-length processes in piece-wise deterministic environment are studied. In fact we study "usual" closed client/server networks under general assumptions. In Section 6, the results are extended to the case of semi-Markov environment. In Section 7, an example of an application of the main theorems of the present paper to client/server computer network with failing client stations is considered. In Section 8, new problems and monotonicity conditions associated with these problems for the case of Markov environment are discussed. In Section 9, the example of numerical study for the simplest network is considered. The results of the paper are concluded in Section 10.

2. Motivation

In this section we motivate our study by real-world problems associated with performance analysis of the network, its separate stations or a subset of those stations. The main results of the present paper are the subject for many concrete practical problems having significant value. Two of them are considered in [6], [7]. Article [6] assumes that client stations all are identical and subject to breakdowns. Article [7] discusses the similar problem for not identical client stations. As a client station is breakdown the parameters of the network are changed. One or other breakdown leads to bottleneck and risk of a destruction of the network performance. The aforementioned articles [6], [7] establish confidence intervals helping to avoid with a given high probability that performance destruction. Some results of these articles are outlined in Section 7 of this paper.

Another circle of problems is as follows.

If a computer network operates for a given fixed time interval $(0, T)$, then its performance characteristic (for example, payment for the increasing a given level of queue in a given client station, or in a subset

of client stations) depends on the characteristics of this network at the initial time $t = 0$. Using the terminology of the present paper, these characteristics at the initial time $t = 0$ specify the initial condition of the environment. For one initial condition, the measure of time that the queue-length is greater than a given fixed level L is x, and we must pay, say Cx. For other initial condition, this measure of time is y and the corresponding payment is Cy.

On the other hand, the cost for initialization the first initial condition related to the above client station of the network is X, and that cost for initialization the second initial condition is Y.

So, the total expenses in the first case are $X + Cx$, and in the second case $Y + Cy$. If $X + Cx < Y + Cy$, then we say that the first *strategy* is more profitable than the second one. In practical context, the first strategy can mean *the first type repair* of the network, or a failing client station, or a subset of failing client stations of the network, and the second strategy - *the second type repair* of that network, or a failing client station, or a subset of failing client stations correspondingly. For example, the first type repair can contain an additional prophylactical service and therefore to be more expensive that the second type repair (i.e. $X > Y$).

3. Definition of semi-Markov environment and classification of bottleneck stations

3.1. **Semi-Markov environments.** Let us now define a semi-Markov environment in Mathematical terms. For each state $\mathcal{E}_i$ of the Markov chain let us define the sequence of independent and identically distributed random variables:

$$\zeta_{i,1}, \zeta_{i,2}, \ldots \tag{3.1}$$

Definition 3.1. Let $\{Z_n\}$ be a Markov chain with finite or countable infinite states space $\{\mathcal{E}_i\}$. Let $Z(t)$ be a random process, and let $\{\sigma_n\}$ be an increasing sequence of random points, i.e. $0 = \sigma_0 < \sigma_1 < \ldots$. Suppose that the process $Z(t)$ is defined as follows: $Z(\sigma_n) = Z_n$, and $Z(t) \equiv Z(\sigma_n)$ for all $\sigma_n \leq t < \sigma_{n+1}$. Suppose also that the increments $\sigma_{n+1} - \sigma_n$, n=0,1,2,..., coincide in distribution with $\zeta_{i,n}$, where the equality $\mathcal{E}(\sigma_n) = \mathcal{E}_i$ defines the index i. Then the process $Z(t)$ is called *semi-Markov environment*.

The above semi-Markov environment $Z(t)$ is assumed to be given on special probability space $\{\Omega_Z, \mathcal{F}_Z, \mathbb{P}_Z\}$, which in turn is contained in the common filtered probability space $\{\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), \mathbb{P}\}$.

The two significant special cases of semi-Markov environment are as follows.

Definition 3.2. A semi-Markov environment is called piece-wise deterministic, if

• the sequences of (3.1) all are deterministic, i.e. $\zeta_{i,v} = z_i$, where z_i is the same constant for all v.

• the parameters of networks are not randomly dependent of the environment states, i.e. $\lambda_j(\mathcal{E}_0) = \lambda_j^{(0)}$, $\lambda_j(\mathcal{E}(\sigma_1)) = \lambda_j^{(1)}, \ldots$, $\lambda_j(\mathcal{E}(\sigma_l)) = \lambda_j^{(l)}, \ldots$, for all j=1,2,..., k, where $\lambda_j^{(0)}$, $\lambda_j^{(1)}$, ...all are non-random constants.

In other words, a piece-wise deterministic environment defines time dependent closed client/server network with fixed piece-wise constant parameters depending on time.

Definition 3.3. In the case where the sequences (3.1) consist of independent and exponentially distributed random variables with rates z_i, i=1,2,..., then the semi-Markov environment $Z(t)$ is called *Markov environment*.

Another equivalent definition of Markov environment and independent of Definition 3.1 is as follows. Let $Z(t)$ be a homogeneous Markov process with initial state $\mathcal{E}(0)$ and transition probabilities $z_{l,m}\triangle t + o(\triangle t)$ from the state $\mathcal{E}_l$ to the state $\mathcal{E}_m$ ($l \neq m$) during a small time interval $(t,\, t+\triangle t)$, and there is probability $1 - \sum_{m\neq l} z_{l,m}\triangle t + o(\triangle t)$ to stay in the same state $\mathcal{E}_l$ during the same time interval $(t,\, t+\triangle t)$.

In many papers on queueing theory, telecommunication systems, inference of stochastic processes, statistics and other areas, the above Markov environment is often associated with *Markov Modulated Poisson Process*. The phrases *Markov environment*, *piece-wise deterministic environment* and *semi-Markov environment* are more appropriate in the context of the present paper.

3.2. **Classification of bottleneck stations.** There was mentioned that $\mathcal{E}(t)$ denotes the state of the semi-Markov environment in time t. For example, the equality $\mathcal{E}(4) = \mathcal{E}_5$ means that in time $t = 4$ the associated Markov chain is in state $\mathcal{E}_5$. We also use the following notation: $\lambda_j(\mathcal{E}(t)) = \lambda(\mathcal{E}(t))p_j(\mathcal{E}(t))$.

In the sequel we shall also use the notation $\lambda(t)$, $p_j(t)$ and correspondingly, $\lambda_j(t)$. It is worth noting, that the definition of $\lambda(t)$ and $\lambda(\mathcal{E}(t))$ and consequently $\lambda_j(t)$ and $\lambda_j(\mathcal{E}(t))$ have the different meaning. In general $\lambda(\mathcal{E}(t)) \neq \lambda(t)$ and $\lambda_j(\mathcal{E}(t)) \neq \lambda_j(t)$. $\lambda_j(t)$ and $\lambda_j(\mathcal{E}(t))$ both are random parameters, and $\lambda_j(t) = \lambda_j(t,\omega)$ where $\omega \in \Omega$, while

$\lambda_j(\mathcal{E}(t)) = \lambda_j(t, \omega_Z)$ where $\omega_Z \in \Omega_Z$. (In the sequel this dependence upon ω_Z will be always implied, but shown explicitly only in the cases where it is necessary.)

More specific explanation of the above difference is as follows. If at the initial time moment $t = 0$ all of units are in the server station, then the input rate to the jth client station is $\lambda_j(0)N$. ($\lambda_j(0)$ is the individual rate of each unit arriving to the client station j, and therefore the rate between arrivals is $\lambda_j(0)N$.) The time parameter 0 in parentheses is associated with the state of Markov environment in time 0. Specifically, if at the initial time moment $t = 0$ all of units are in the server station, then $\lambda_j(0) = \lambda_j(\mathcal{E}(0))$. If the network is considered without Markov or semi-Markov environment, then that initial arrival rate is $\lambda_j N$ which is associated with the individual rate λ_j of each unit of the server station. If at the initial time moment there are αN units in the server station, $\alpha < 1$, then for arrival process to any client station j of a standard network (without random environment) we also use the notation $\lambda_j(0)$. However, the meaning of $\lambda_j(0)$ is not longer the individual rate of each unit at time $t = 0$ arriving to the station j. More specifically, $\lambda_j(0) = \lambda_j\alpha$, where λ_j is the individual rate of each unit at time $t = 0$ arriving to the station j. The meaning of $\lambda_j(t)$ is similar. The only difference that it is said about an arbitrary time t. For example, if there are $\alpha(t)N$ units in the server station in time t, $\alpha(t) < 1$, then $\lambda_j(t) = \lambda_j\alpha(t)$. Resuming the above, $\lambda_j(\mathcal{E}(t))$ is associated with individual service rate of each unit, while $\lambda_j(t)$ is a recalculated (relative) rate depending on the state of the queue-length processes in time t, as explained above.

Regarding the departure (service) rates our assumption in the paper is as follows. The departure rate of the jth client station is assumed to be independent of semi-Markov environment as well as of input rates $\lambda_j(t)$. Specifically,

- it is assumed that the the expectation of service (inter-departure) time in the jth client station is $\mathbb{E}\xi_{j,N,l} = \frac{1}{\mu_j N}$ for all l=1,2,...

Therefore, if at the initial time moment $t = 0$ all of units are in the server station, then the load parameter of the jth client station is $\rho_j(0) = \frac{\lambda_j(0)}{\mu_j}$, and in the case where there is no semi-Markov environment, $\rho_j(0) = \rho_j = \frac{\lambda_j}{\mu_j}$.

In the case of network in Markov environment, the meaning of the notation $\lambda_j(t)$ is the same as well. If there are $N\alpha(t)$ units in the server station in time t, $\alpha(t) < 1$, and the rate of arrival of a unit from the

server to the client station j is $\lambda_j(\mathcal{E}(t))$, then $\lambda_j(t) = \lambda_j(\mathcal{E}(t))\alpha(t)$. Then the load of the jth client station in time t is $\rho_j(t) = \frac{\lambda_j(t)}{\mu_j}$.

Now introduce necessary definitions. The first two definitions are related to both standard client/server networks and client/server networks in semi-Markov environment.

Definition 3.4. The client station j is called *locally non-bottleneck* in time t if $\rho_j(t) < 1$. Otherwise, the jth client station is called *locally bottleneck* in time t. A client station locally (non-) bottleneck in time 0 will be also called *initially (non-) bottleneck*.

Definition 3.5. The client station j is called *non-bottleneck* in time interval $[t_1, t_2]$ if it is locally non-bottleneck in all points of this interval. Otherwise, if there is a point $t^* \in [t_1, t_2]$ such that $\rho_j(t^*) \geq 1$, then the client station is called *bottleneck* in time interval $[t_1, t_2]$. A client station is called *(non-) bottleneck* if it is (non-) bottleneck for all t.

The special definition for standard client/server networks (without semi-Markov environment) is as follows.

Definition 3.6. A client station is called *absolutely non-bottleneck* if it is a locally non-bottleneck station at the moment when all of units are in the server station. Otherwise, a client station is called *absolutely bottleneck*.

Clearly, that absolutely non-bottleneck client station j is a non-bottleneck client station, because then for all t we have $\lambda_j(t) < \mu_j$. In the next section we prove that absolutely bottleneck client station is a bottleneck client station as well. That is, if a client station is currently locally bottleneck station, then it never can become a locally non-bottleneck. This means that the client station is forever bottleneck.

Definition 3.6 can be extended to network stations in semi-Markov environment for an arbitrary time t. Specifically, we have the following definition.

Definition 3.7. The client station j of a network in random environment is called absolutely non-bottleneck in time t if $\lambda_j(\mathcal{E}(t)) < \mu_j$. In other words, the client station j is absolutely non-bottleneck in time t if in that time t the network belongs to some state $\mathcal{E}_i$ of the environment, i.e. $\mathcal{E}(t) = \mathcal{E}_i$, and in this state $\lambda_j(\mathcal{E}_i) < \mu_j$. Otherwise, this client station is called absolutely bottleneck in time t.

The last notion enables us to judge on the behavior of client stations in random intervals $[\sigma_i, \sigma_{i+1})$, where the network is in given state $\mathcal{E}_l$.

Recall that σ_i is a time instant when the state of semi-Markov environment is changed. So, during the random interval $[\sigma_i, \sigma_{i+1})$ the network is in a fixed state of the semi-Markov environment.

4. Queue-length processes in the client stations of networks with semi-Markov environment

Consider a client station j ($j = 1, 2, \ldots, k$). Let $Q_{j,N}(t)$ denote a queue-length there in time t. Assume that at the initial time instant $t = 0$, all of units are in the server station, i.e. $Q_{j,N}(0) = 0$ for all $j = 1, 2, \ldots, k$. This is the simplest case, and we start from its study. For $t > 0$,

$$Q_{j,N}(t) = A_{j,N}(t) - D_{j,N}(t), \tag{4.1}$$

where $A_{j,N}(t)$ is the arrival process to client station j, and $D_{j,N}(t)$ is the departure process from that client station j. The equation for departure process is as follows. Let

$$S_{j,N}(t) = \sum_{i=1}^{\infty} \mathbb{I}\left\{\sum_{l=1}^{i} \xi_{j,N,l} \le t\right\}, \quad j = 1, 2, \ldots, k$$

be a point process associated with consecutive departures from the jth client station. Then,

$$\begin{aligned} D_{j,N}(t) &= \int_0^t \mathbb{I}\{Q_{j,N}(s-) > 0\} \mathrm{d}S_{j,N}(s) \\ &= S_{j,N}(t) - \int_0^t \mathbb{I}\{Q_{j,N}(s-) = 0\} \mathrm{d}S_{j,N}(s), \\ &\quad j = 1, 2, \ldots, k. \end{aligned} \tag{4.2}$$

The definition of the departure process given by (4.2) is as in [1]. However, the construction of arrival process is more difficult.

Specifically,

$$A_{j,N}(t) = \int_0^t \sum_{i=1}^{N} \mathbb{I}\left\{N - \sum_{l=1}^{k} Q_{l,N}(s-) \ge i\right\} \mathrm{d}\pi_{j,i}(s, \omega_Z). \tag{4.3}$$

The processes $\{\pi_{j,i}(s, \omega_Z)\}$, $i = 1, 2, \ldots, N$, appearing in relation (4.3) are a collection of conditionally independent Poisson processes with parameters depending on ω_Z. This means the following. Assume that for a given realization ω_Z we have a sequence: $0 < \sigma_1 < \sigma_2 < \ldots$, and for some n, $\sigma_n \le s < \sigma_{n+1}$. Assume that $\mathcal{E}(\sigma_n) = \mathcal{E}_l$. Then $\{\pi_{j,i}(s, \omega_Z)\}$, $i = 1, 2, \ldots, N$, is the sequence of Poisson processes with the parameter $\lambda_j(\mathcal{E}_l)$. That is the rate of Poisson process depends on the state of semi-Markov environment in time s. This is just the main

difference between the consideration of [1], where λ_j, $j = 1, 2, \ldots, k$, were non-random constants.

Relation (4.3) can be then rewritten

$$A_{j,N}(t) = \sum_{v=1}^{\infty} \int_{t\wedge\sigma_{v-1}}^{t\wedge\sigma_v} \sum_{i=1}^{N} \mathbb{I}\left\{N - \sum_{l=1}^{k} Q_{l,N}(s-) \geq i\right\} \mathrm{d}\pi_{j,i,v}(s), \tag{4.4}$$

where $\pi_{j,i,v}(s)$ is an associated sequence of (conditionally) independent Poisson process with parameter depending on the state of the semi-Markov environment $\mathcal{E}(\sigma_{v-1})$. (Here in (4.4) and later we use the standard notation for a minimum of two numbers: $a \wedge b \equiv \min(a, b)$.)

Relations (4.1) and (4.2) enable us to write the following representation for the queue-length process $Q_{j,N}(t)$:

$$Q_{j,N}(t) = A_{j,N}(t) - S_{j,N}(t) + \int_0^t \mathbb{I}\{Q_{j,N}(s-) = 0\}\mathrm{d}S_{j,N}(s). \tag{4.5}$$

This implies that $Q_{j,N}(t)$ is the normal reflection of the process

$$X_{j,N}(t) = A_{j,N}(t) - S_{j,N}(t), \quad X_{j,N}(0) = 0 \tag{4.6}$$

at zero. More accurately, $Q_{j,N}(t)$ is a nonnegative solution of the Skorokhod problem (see [33] as well as [8], [32], [34] and others) on the normal reflection of the process $X_{j,N}(t)$ at zero (for the detailed arguments see [22]). Recall that according to the Skorokhod problem,

$$\int_0^t \mathbb{I}\{Q_{j,N}(s-) = 0\}\mathrm{d}S_{j,N}(s) = -\inf_{s\leq t} X_{j,N}(s), \tag{4.7}$$

and $Q_{j,N}(t)$ has the representation

$$Q_{j,N}(t) = X_{j,N}(t) - \inf_{s\leq t} X_{j,N}(s). \tag{4.8}$$

In the sequel, it is convenient to use the notation:

$$\Phi_t(X) = X(t) - \inf_{s\leq t} X(s)$$

for any cádlág function $X(t)$ satisfying $X(0) = 0$ (see e.g. [22]). According to this notation, (4.8) can be rewritten $Q_{j,N}(t) = \Phi_t(X_{j,N})$.

Next, we take into account that the process $A_{j,N}(t)$ is a semimartingale adapted with respect to the filtration $\mathcal{F}_t$. Let $\widehat{A}_{j,N}(t)$ denote the compensator of $A_{j,N}(t)$ and $M_{A_{j,N}}(t)$ denote the square integrable martingale of $A_{j,N}(t)$ in the Doob-Meyer semimartingale decomposition: $A_{j,N}(t) = \widehat{A}_{j,N}(t) + M_{A_{j,N}}(t)$, $j = 1, 2, \ldots, k$. Then, the process $X_{j,N}(t)$ given by (4.6) can be represented

$$X_{j,N}(t) = \widehat{A}_{j,N}(t) - S_{j,N}(t) + M_{A_{j,N}}(t), \tag{4.9}$$

where

$$\widehat{A}_{j,N}(t) = \sum_{l=1}^{\infty} \int_{\sigma_{l-1}\wedge t}^{\sigma_l \wedge t} \lambda_j(\mathcal{E}(\sigma_{l-1})) \left\{ N - \sum_{i=1}^{k} Q_{i,N}(s) \right\} \mathrm{d}s. \tag{4.10}$$

The details for last formula (4.10) can be obtained from [14] or [27], Theorem 1.6.1.

Let us now study asymptotic properties of the normalized queue-lengths $q_{j,N}(t) = \frac{Q_{j,N}(t)}{N}$ as $N \to \infty$. For normalized processes we will use small Latin letters. For example, $x_{j,N}(t) = \frac{X_{j,N}(t)}{N}$, $\widehat{a}_{j,N}(t) = \frac{\widehat{A}_{j,N}(t)}{N}$, and so on.

Then, from (4.9) we have

$$x_{j,N}(t) = \widehat{a}_{j,N}(t) - s_{j,N}(t) + m_{A_{j,N}}(t), \ j = 1, 2, \ldots, k. \tag{4.11}$$

Let us derive a relation for $\mathbb{P}\text{-}\lim_{N\to\infty} x_{j,N}(t)$. ($\mathbb{P}\text{-}\lim$ denotes the limit in probability.)

From Lenglart-Rebolledo inequality (e.g. Liptser and Shiryayev [27]), we have:

(4.12)

$$\begin{aligned} \mathbb{P}\left\{ \sup_{0\le s\le t} |m_{A_{j,N}}(t)| > \delta \right\} &= \mathbb{P}\left\{ \sup_{0\le s\le t} |A_{j,N}(s) - \widehat{A}_{j,N}(s)| > \delta N \right\} \\ &\le \frac{\epsilon}{\delta^2} + \mathbb{P}\{\widehat{A}_{j,N}(t) > \epsilon N^2\}, \end{aligned}$$

where $\epsilon = \epsilon(N)$ vanishes such that $\epsilon N \to \infty$. Then, by virtue of (4.10), the term $\mathbb{P}\{\widehat{A}_{j,N}(t) > \epsilon N^2\} = \mathbb{P}\{\widehat{a}_{j,N}(t) > \epsilon N\}$ vanishes as well, and for any small $\delta > 0$ the fraction $\frac{\epsilon}{\delta^2}$ vanishes. Therefore,

$$\mathbb{P}\text{-}\lim_{N\to\infty} m_{A_{j,N}}(t) = 0 \tag{4.13}$$

for all $j = 1, 2, \ldots, k$ and $t \ge 0$.

Next, according to the assumption above, $\mathbb{P}\text{-}\lim_{N\to\infty} s_{j,N}(t) = \mu_j t$.

Therefore,

$$\begin{aligned} x_j(t) &= \mathbb{P}\text{-}\lim_{N\to\infty} x_{j,N}(t) \\ &= \mathbb{P}\text{-}\lim_{N\to\infty} \widehat{a}_{j,N}(t) - \mu_j t \\ &= \sum_{l=1}^{\infty} \int_{\sigma_{l-1}\wedge t}^{\sigma_l \wedge t} \left[\lambda_j(\mathcal{E}(\sigma_{l-1})) \left\{ 1 - \sum_{i=1}^{k} \Phi_s(x_i) \right\} - \mu_j \right] \mathrm{d}s. \end{aligned} \tag{4.14}$$

Representation (4.14) is the extension of the similar result of [1] for queue-length processes in client stations of standard client/server networks. Recall that the representation obtained in [1] and [22] is

$$x_j(t) = \int_0^t \left[\lambda_j \left\{ 1 - \sum_{i=1}^{k} \Phi_s(x_i) \right\} - \mu_j \right] \mathrm{d}s. \tag{4.15}$$

As we can see, the representations given by (4.14) and (4.15) are similar. The only difference is in the presence of infinite sum containing the integrals with random upper and lower bounds, and (4.15) is a particular case of (4.14).

5. Bottleneck analysis in the case of a piece-wise constant environment

In this section we discuss the behavior of the queue-length processes in a large closed client/server network in the case of piece-wise constant environment. For this case relation (4.14) reduces to

$$\begin{aligned} x_j(t) &= \sum_{l=1}^{r} \int_{\sigma_{l-1}}^{\sigma_l} \left[\lambda_j \left\{ 1 - \sum_{i=1}^{k} \Phi_s(x_i) \right\} - \mu_j \right] \mathrm{d}s \\ &\quad + \int_{\sigma_r}^{t} \left[\lambda_j \left\{ 1 - \sum_{i=1}^{k} \Phi_s(x_i) \right\} - \mu_j \right] \mathrm{d}s, \end{aligned} \tag{5.1}$$

where r is the number of the state changes before time t.

Therefore, the bottleneck analysis of the network in piece-wise constant environment reduces to the analysis of the network in traditional formulation (without random environment) in some given time intervals such as $[\sigma_{l-1}, \sigma_l)$, l=1,2,...,r, or $[\sigma_r, t)$.

Notice, that the bottleneck analysis of the Markovian client/server model has been originally studied by Kogan and Liptser [22]. Then these results were extended for the case of autonomous service mechanism in client stations in [1]. However, the results obtained in both of these papers are related to a single special case and are not enough for the purpose of our performance analysis. Therefore we will study all possible cases including the behaviour of the network under different initial lengths of queues in client stations and several absolutely bottleneck and absolutely non-bottleneck client stations.

In [22] and [1] there has only been considered the case where in the initial time moment $t = 0$ all of units are in the server station (i.e. the client stations all are empty) and only one (the kth) client station

is a bottleneck station. (In this particular case the notions of (non-)bottleneck and absolutely (non-)bottleneck client station coincide.) Specifically, there has been proved the following result in [1].

Proposition 5.1. *Let* $S^*_{j,N}(t) = \inf\{s > 0 : S_{j,N}(s) = S_{j,N}(t)\}$, *and* $Q_{j,N}(t)$ *denotes the queue-length in the* j*th client station in time* t. *Under the assumption that the* k*th client section is bottleneck, for* $j = 1, 2, \ldots, k-1$ *and for any* $t > 0$ *we have:*

$$\lim_{N\to\infty} \mathbb{P}\{Q_{j,N}[S^*_{j,N}(t)] = 0\} = 1 - \rho_j(t), \tag{5.2}$$

$$\begin{aligned} &\lim_{N\to\infty} \int_0^t \rho_j(s)\mathbb{P}\{Q_{j,N}(s) = l\}\,ds \\ &= \lim_{N\to\infty} \int_0^t \mathbb{P}\{Q_{j,N}[S^*_{j,N}(s)] = l+1\}\,ds, \\ &\quad l = 0, 1, \ldots, \end{aligned} \tag{5.3}$$

where

$$\rho_j(t) = \rho_j(0)[1 - q(t)], \tag{5.4}$$

$$q(t) = \left(1 - \frac{1}{\rho_k(0)}\right)(1 - e^{-\rho_k(0)\mu_k t}). \tag{5.5}$$

In the case of Markovian network, relations (5.2), (5.3) can be resolved explicitly, and queue-length distribution in the non-bottleneck stations is the time-dependent geometric distribution [1], [22].

The meaning of $1 - q(t)$ in (5.4) is the limiting fraction of units remaining at the server station in time t as $N \to \infty$. For example, in the case where $\rho_k(0) = 1$ this fraction remains the same at any time t as initially, that is, as N large, the number of units in the server station remains asymptotically equivalent to N. However, if $\rho_k(0) > 1$, then the number of units in the server station in time t is asymptotically equivalent to $N[1 - q(t)]$. Then, the number of units remaining in bottleneck station in time t is asymptotically equivalent to $Nq(t)$.

Let us study various cases of the client/server network with bottlenecks. These cases will be studied in order of increasing complexity. We possibly shorten the proofs.

Assuming that the initial condition of the network is the same as in Proposition 5.1 (i.e. at the initial time $t = 0$ all of units are at the server station), let us study the case where the client stations 1,2,...,k_0 are non-bottleneck, while the rest client stations $k_0 + 1, \ldots, k$ are bottleneck. In this case we have the following result.

Proposition 5.2. *Under the assumption that the client sections $k_0+1,\dots,k$ are bottleneck, for $j=1,2,\dots,k_0$ and for any $t>0$ we have:*

$$\lim_{N\to\infty}\mathbb{P}\{Q_{j,N}[S^*_{j,N}(t)]=0\}=1-\rho_j(t), \tag{5.6}$$

$$\begin{aligned}&\lim_{N\to\infty}\int_0^t \rho_j(s)\mathbb{P}\{Q_{j,N}(s)=l\}\,ds\\ &=\lim_{N\to\infty}\int_0^t \mathbb{P}\{Q_{j,N}[S^*_{j,N}(s)]=l+1\}\,ds,\\ &\quad l=0,1,\dots,\end{aligned} \tag{5.7}$$

where

$$\rho_j(t) = \rho_j(0)[1-q(t)], \tag{5.8}$$

$$q(t) = \left(1-\frac{\sum_{v=k_0+1}^k \mu_v}{\sum_{v=k_0+1}^k \lambda_v}\right)\left(1-\exp\left[-t\sum_{v=k_0+1}^k \lambda_v\right]\right). \tag{5.9}$$

Note, that equations (5.6) and (5.7) are the same as (5.2) and (5.3), and the main difference between Proposition 5.1 and Proposition 5.2 is only in the expression for $q(t)$. The difference between expressions (5.4), (5.5) and (5.8), (5.9) can be easily explained in the framework of the proof of Proposition 5.2.

Considering all of the bottleneck client stations as a separate subsystem, one can notice that the arrival rate to this subsystem is $\sum_{v=k_0+1}^k \lambda_v$, and the service rate (the sum of reciprocals of the expected inter-departure times) is $\sum_{v=k_0+1}^k \mu_v$. This subsystem can be thought as a bottleneck station with the load

$$\frac{\sum_{v=k_0+1}^k \lambda_v}{\sum_{v=k_0+1}^k \mu_v}.$$

To be specific, note the method of Sections 2, 3, and 4 of [1] leads to the same equations, and the equations for normalized queue-lengths in the bottleneck stations all are an elementary extension of the case considered in [1].

Therefore Proposition 5.2 is an elementary extension of Proposition 5.1 and its proof is the same as in [1] or [22]. Notice, that the number of units remaining at the bottleneck station v, $v=k_0+1,\dots,k$ in time t is asymptotically equal to

$$Nq_v(t)=N\left([\lambda_v-\mu_v]t-\lambda_v\int_0^t q(s)\mathrm{d}s\right). \tag{5.10}$$

Recall the main elements of the known proof for the representation $q(t)$ given by (5.5), and consequently explain the proof of (5.10). We use the notation similar to that of the earlier papers [1] and [22].

The difference between arrival and departure processes in the jth client station is denoted $X_{j,N}(t) = A_{j,N}(t) - S_{j,N}(t)$ and its normalization $x_{j,N}(t) = \frac{X_{j,N}(t)}{N}$. Let $x_j(t)$, $j = 1, 2, \ldots, k$, denote the limit in probability of $x_{j,N}(t)$, as $N \to \infty$. The queue-length in the jth client station in time t is denoted $Q_{j,N}(t)$, its normalization is denoted $q_{j,N}(t) = \frac{Q_{j,N}(t)}{N}$, and the limit of $q_{j,N}(t)$ in probability, as $N \to \infty$, is denoted $q_j(t)$.

Next, let $\Phi_t(X) = X(t) - \inf_{s \le t} X(s)$ for any cádlág function X satisfying the condition $X(0) = 0$. The functional $\Phi_t(X)$ has been introduced in [22]. It characterizes a solution of the Skorokhod problem on normal reflection at zero. Therefore $\Phi_t(x_j)$, being the functional Φ applied to the function $x_j(t)$, describes the dynamic of normalized queue-length in the jth client station under the "usual" initial conditions given in Propositions 5.1 and 5.2. Under these "usual" conditions, the functions $x_j(t)$, $j = 1, 2, \ldots, k$ satisfy the system of equations:

$$x_j(t) = \int_0^t \left\{ \lambda_j \left[1 - \sum_{l=1}^{k} \Phi_s(x_l) \right] - \mu_j \right\} \mathrm{d}s, \quad j = 1, 2, \ldots, k. \tag{5.11}$$

Note, that the normalized functions $x_j(t)$, $j = 1, 2, \ldots, k$ are usual (non-random) continuous functions, and (5.11) characterizes a usual system of linear differential equations.

The statement of Proposition 5.1 is based on the solution of the system of these equations. (It is proved in [22] that there is a unique solution of the system of equations (5.11).) More specifically, in the case where the node k is bottleneck, $\inf_{s \le t} x_k(s) = x_k(0) = 0$, and we therefore have $\Phi_s(x_k) = x_k(s)$. In other words, $\Phi_s(x_k)$ can be replaced by $x_k(s)$ in these equations of (5.11).

The solution of the system of equations (5.11) is $x_j(t)$=0 for the non-bottleneck stations $j = 1, 2, \ldots, k-1$, and $x_k(t) = q(t)$ for the bottleneck station $j = k$, where $q(t)$ is given by (5.5). In the case of several bottleneck stations, we write the similar equation for the dynamic of the normalized cumulated queue-length process in all of the bottleneck stations. Specifically,

$$\sum_{v=k_0+1}^{k} x_v(t) = \int_0^t \left\{ \sum_{v=k_0+1}^{k} \lambda_v \left[1 - \sum_{l=k_0+1}^{k} x_l(s) \right] - \sum_{v=k_0+1}^{k} \mu_v \right\} \mathrm{d}s,$$

gives solution (5.9) for $q(t)$ in the similar statement of Proposition 5.2. Then, the solution of system (5.11) for the bottleneck client stations v=k_0+1,$k_0+2\ldots$, k is given by

$$x_v(t) = q_v(t) = (\lambda_v - \mu_v)t - \lambda_v \int_0^t q(s)\mathrm{d}s,$$

where $q(t)$ is defined by (5.9), and the queue-lengths in the bottleneck stations are asymptotically evaluated by relation (5.10).

Now, we discuss the behavior of the network, in which the client stations are not initially empty. This is the next step of the extension of the original result of Proposition 5.1. The analysis of cases related to initially not empty queues is much more difficult. Therefore, we start from the simplest case of the network containing only one client station, i.e. $k = 1$.

Let $\beta_1 \leq 1$ be a positive real number, and let us assume that the initial number of units in this client station is asymptotically equivalent to $N\beta_1$. Consider the following two cases: (i) the client station is initially bottleneck, i.e. $\lambda_1(0) \geq \mu_1$, and (ii) the client station is initially non-bottleneck, i.e. $\lambda_1(0) < \mu_1$.

Case (i) is relatively simple. It is a simple extension of the cases considered above. Specifically, we have the following system of equations:

$$\begin{aligned} x_1(t) &= \beta_1 + (1-\beta_1)z_1(t), \\ z_1(t) &= \int_0^t \left\{\lambda_1(0)\left[1 - z_1(s)\right] - \mu_1\right\}\mathrm{d}s. \end{aligned} \tag{5.12}$$

From (5.12) we have the following solution:

$$x_1(t) = \beta_1 + (1-\beta_1)\left(\frac{\lambda_1(0)-\mu_1}{\lambda_1(0)}\right)\left(1 - \mathrm{e}^{-\lambda_1(0)t}\right). \tag{5.13}$$

The normalized queue-length $q_1(t)$ in this client station is $q_1(t) = x_1(t)$.

Case (i) can be easily extended to a more general case of k initially bottleneck client stations. Let β_j, $j = 1, 2, \ldots, k$, denote nonnegative real numbers, and $\beta_1 + \beta_2 + \ldots + \beta_k \leq 1$. Assume then that the initial number of units in the jth client station is asymptotically equivalent to $N\beta_j$. Let $q(t)$ denote the cumulated normalized queue-length in all of client stations.

Proposition 5.3. *Assume that all client stations are initially bottleneck, and the initial queue-lengths in client stations are asymptotically equivalent to $N\beta_1$, $N\beta_2$,..., $N\beta_k$ correspondingly ($\beta_1+\beta_2+\ldots+\beta_k \leq$*

1), as $N \to \infty$. Then,

$$
\begin{aligned}
q(t) &= \sum_{j=1}^{k} \beta_j + \left(1 - \sum_{j=1}^{k} \beta_j\right) r(t), \\
r(t) &= \left(\frac{\sum_{j=1}^{k}(\lambda_j(0) - \mu_j)}{\sum_{j=1}^{k} \lambda_j(0)}\right) \left(1 - \exp\left[-t \sum_{j=1}^{k} \lambda_j(0)\right]\right),
\end{aligned}
\tag{5.14}
$$

and the normalized queue-length in the jth client station is defined as

$$
q_j(t) = \beta_j + \left(1 - \sum_{j=1}^{k} \beta_j\right) \left([\lambda_j(0) - \mu_j]t - \lambda_j(0) \int_0^t r(s)\,ds\right). \tag{5.15}
$$

Let us now discuss case (ii). This case is also described by system of equations (5.12), and the dynamic of the process $x_1(t)$ is therefore similar to the case considered above. However, this case is more delicate. The client station is initially non-bottleneck, i.e. $\lambda_1(0) < \mu_1$, and the function $x_1(t)$ is therefore decreasing in the right side of 0. According to the convention, the initial value of queue is asymptotically equivalent to $\beta_1 N$, and therefore $\lambda_1(0) = (1-\beta_1)\lambda_1^*$. Then the meaning of λ_1^* is a maximally possible rate of units arriving from the server station to client station, when all of units are in the server station and the client station is empty. Then the client station is absolutely non-bottleneck if $\lambda_1^* < \mu_1$, and it is absolutely bottleneck if $\lambda_1^* \geq \mu_1$.

Consider first the case of an absolutely bottleneck station, i.e. $\lambda_1^* \geq \mu_1$. In this case, from the solution given by (5.13) we have

$$
\begin{aligned}
\lim_{t\to\infty} q_1(t) &= \beta_1 + (1-\beta_1)\frac{\lambda_1(0) - \mu_1}{\lambda_1(0)} \\
&= \frac{\lambda_1(0) - \mu_1(1-\beta_1)}{\lambda_1(0)} \\
&= \frac{\lambda_1^* - \mu_1}{\lambda_1^*}.
\end{aligned}
\tag{5.16}
$$

The meaning of the last result is the following. Let $N\overline{\beta_1}$ be an asymptotic value of the queue-length in the client station, when this queue station (at the first time) becomes a locally bottleneck station. Then, according to (5.16) the normalized queue-length in the client station approaches to this level $\overline{\beta_1} = \frac{\lambda_1^* - \mu_1}{\lambda_1^*}$ as $t \to \infty$. Notice, that the same level for normalized queue-length is achieved for an initially bottleneck station in case (i). The result of (5.16) is also supported by results (5.4) and (5.5) of Proposition 5.1. Thus, for a bottleneck station, the same level is asymptotically achieved independently of an initial

queue-length. For this reason for any absolutely bottleneck station in which $\lambda_1^* \geq \mu_1$ we do not distinguish between two cases $\lambda_j(0) < \mu_j$ and $\lambda_j(0) \geq \mu_j$, and absolutely bottleneck client station is always a bottleneck station.

Consider now the case of an absolutely non-bottleneck client station (and therefore non-bottleneck client station) where $\lambda_1^* < \mu_1$. Then, according to the same calculation as in (5.16) we have

$$\lim_{t\to\infty} x_1(t) = \frac{\lambda_1^* - \mu_1}{\lambda_1^*} < 0. \tag{5.17}$$

Therefore, there exists the time instant τ_1 when the normalized queue-length becomes at the first time empty. For this time instant we have the equation

$$\tau_1 = -\frac{1}{\lambda_1(0)} \log\left(1 + \frac{\beta_1\lambda_1(0)}{(1-\beta_1)(\lambda_1(0)-\mu_1)}\right). \tag{5.18}$$

Let us extend the result of case (ii) for a network with k initially non-bottleneck client stations, all satisfying the condition $\lambda_j(0) < \mu_j$, $j = 1, 2, \ldots, k$. It is assumed that the initial number of units in the jth client station is asymptotically equivalent to $N\beta_j$ ($\beta_1 + \beta_2 + \ldots + \beta_k \leq 1$) as $N \to \infty$. Assume also that the first k_0 client stations are (absolutely) non-bottleneck, i.e. $\lambda_j^* = \lambda_j(0)(1-\beta_1-\beta_2-\ldots-\beta_k) < \mu_j$, $j = 1, 2, \ldots, k_0$, while the rest $k - k_0$ client stations are (absolutely) bottleneck, i.e. $\lambda_v^* = \lambda_v(0)(1-\beta_1-\beta_2-\ldots-\beta_k) \geq \mu_v$, $v = k_0+1, k_0+2, \ldots, k$.

Similarly to (5.12) we have the following system of equations

$$\begin{aligned} x(t) &= \sum_{j=1}^{k} \beta_j + \left(1 - \sum_{j=1}^{k} \beta_j\right) z(t), \\ z(t) &= \int_0^t \left\{ (1 - z(s)) \sum_{j=1}^{k} \lambda_j(0) - \sum_{j=1}^{k} \mu_j \right\} \mathrm{d}s, \end{aligned} \tag{5.19}$$

and for $z(t)$ we have the solution

$$z(t) = \left(\frac{\sum_{j=1}^{k}(\lambda_j(0) - \mu_j)}{\sum_{j=1}^{k} \lambda_j(0)}\right)\left(1 - \exp\left[-t\sum_{j=1}^{k} \lambda_j(0)\right]\right). \tag{5.20}$$

Then for $x_j(t)$, $j = 1, 2, \ldots, k$, we have the following solutions:

$$x_j(t) = \beta_j + \left(1 - \sum_{j=1}^{k} \beta_j\right)\left((\lambda_j(0) - \mu_j)t - \lambda_j(0)\int_0^t z(s)\mathrm{d}s\right). \tag{5.21}$$

(Recall that $x_j(t) = \mathbb{P}\text{-}\lim_{N\to\infty} \frac{A_{j,N}(t)-S_{j,N}(t)}{N}$, $j = 1, 2, \ldots, k$.)

However, since the first k_0 client stations are non-bottleneck, then the equality $q_j(t) = x_j(t)$ for the normalized queue-lengths in client stations is valid only for the values t of the interval $0 \le t \le \tau_1$, where the value τ_1 can be found from (5.21) as

$$\tau_1 = \min_{1\le j\le k_0} \inf\{t : x_j(t) \le 0\}. \tag{5.22}$$

Let $j_0 = \arg\min_{1\le j\le k_0} \inf\{t : x_j(t) \le 0\}$. Then the normalized queue-length process $q_{j_0}(t)$ is as follows. For $0 \le t \le \tau_1$, $q_{j_0}(t) = x_{j_0}(t)$, and for $t \ge \tau_1$ it satisfies the equation

$$x_{j_0}(t) = \int_0^{t-\tau_1} \left[\lambda(\tau_1)\left(1 - \sum_{j\ne j_0} x_j(s) - \Phi_s(x_{j_0})\right) - \mu_{j_0}\right] \mathrm{d}s, \tag{5.23}$$

where $\Phi_s(x_{j_0}) = x_{j_0}(s) - \inf_{0\le u\le s} x_{j_0}(u)$.

Together with (5.23) for all remaining j =1,2,..., $j_0 - 1$, $j_0 + 1$,..., k and $t \ge \tau_1$ we have:

$$\begin{aligned} x(t) &= \sum_{j\ne j_0} x_j(\tau_1) + \left(1 - \sum_{j\ne j_0} x_j(\tau_1)\right) z(t), \\ z(t) &= \int_0^{t-\tau_1} \left\{(1 - z(s)) \sum_{j\ne j_0} \lambda_j(\tau_1) - \sum_{j\ne j_0} \mu_j\right\} \mathrm{d}s, \end{aligned} \tag{5.24}$$

and similarly to (5.20) for $z(t)$ we have the solution

$$z(t) = \left(\frac{\sum_{j\ne j_0}(\lambda_j(\tau_1) - \mu_j)}{\sum_{j\ne j_0} \lambda_j(\tau_1)}\right)\left(1 - \exp\left[-(t - \tau_1) \sum_{j\ne j_0} \lambda_j(\tau_1)\right]\right). \tag{5.25}$$

Therefore for j =1,2,..., j_0-1, j_0+1,..., k and $t \ge \tau_1$ we obtain

$$\begin{aligned} x_j(t) =& x_j(\tau_1) + \left(1 - \sum_{j\ne j_0} x_j(\tau_1)\right) \times \\ & \times \left((\lambda_j(\tau_1) - \mu_j)(t - \tau_1) - \lambda_j(\tau_1) \int_0^{t-\tau_1} z(s)\mathrm{d}s\right), \end{aligned} \tag{5.26}$$

and since $x_{j_0}(t)$ is nonnegative, for $j = j_0$ and $t \ge \tau_1$ we obtain

$$x_{j_0}(t) = 0. \tag{5.27}$$

Thus the dimension of the system is decreased by 1, and the procedure can be repeated similarly. Specifically, again since the client

stations j =1,2,..., $j_0 - 1$, $j_0 + 1$,..., k_0 all are non-bottleneck stations, then the equality $q_j(t) = x_j(t)$ for the normalized queue-length processes is valid for all $t \le \tau_2$, where τ_2 is defined as

$$\tau_2 = \min_{1 \le j \ne j_0 \le k_0} \inf\{t : x_j(t) \le 0\}. \tag{5.28}$$

Setting now $j_1 = \arg\min_{1 \le j \ne j_0 \le k_0} \inf\{t : x_j(t) \le 0\}$ one can continue this procedure to find τ_3,..., τ_{k_0} and then to know the behaviour of the queue-length processes in all of (non-bottleneck and absolutely bottleneck) client stations for all t.

The considered extension of case (ii) is in fact the general case in which amongst k client stations there are k_0 non-bottleneck client stations, and the rest $k - k_0$ stations are absolutely bottleneck (i.e. part of them can be initially non-bottleneck), and all k client stations are with arbitrarily large initial queue-lengths.

The result can be formulated as follows.

Theorem 5.4. *Assume that there are k client stations, where k_0 client stations (non necessarily the first ones) are absolutely non-bottleneck, and the initial normalized queue-length in all of these k client stations are β_j in limit as $N \to \infty$ correspondingly ($j = 1, 2, \ldots, k$). Then there are time instants $\tau_1 \le \tau_2 \le \ldots \le \tau_{k_0}$ when the normalized queue-lengths in these k_0 absolutely non-bottleneck stations correspondingly achieves zero and remains then to stay at zero. These points as well as the normalized queue-lengths in time t in the rest $k - k_0$ stations are defined according to Scheme 5.5 below.*

Scheme 5.5. Consider the system

$$x_j(t) = x_j(0) + \left(1 - \sum_{j=1}^{k} x_j(0)\right)\left((\lambda_j(0) - \mu_j)t - \lambda_j(0)\int_0^t z(s)\mathrm{d}s\right), \quad j = 1, 2, \ldots, k, \tag{5.29}$$

where $x_j(0) = \beta_j$, and

$$z(t) = \left(\frac{\sum_{j=1}^{k}(\lambda_j(0) - \mu_j)}{\sum_{j=1}^{k}\lambda_j(0)}\right)\left(1 - \exp\left[-t\sum_{j=1}^{k}\lambda_j(0)\right]\right). \tag{5.30}$$

Then,

$$\tau_1 = \min_{1 \le j \le k} \inf\{t : x_j(t) \le 0\},$$

and the argument t in (5.29) and (5.30) belongs to the interval $[0,\tau_1]$. In this case the normalized queue-length at the jth client station, $q_j(t) =$

$x_j(t)$ for all j=1,2,...,k. Let $j_0 = \arg\min_{1\le j\le k} \inf\{t : x_j(t) \le 0\}$. Then $q_{j_0}(t)$=0 for all $t \ge \tau_1$.

In the next step, we consider the system of $k-1$ equations, and $t \ge \tau_1$ (the equation for $x_{j_0}(t)$, $t \ge \tau_1$, is excluded):

$$\begin{aligned} x_j(t) =& x_j(\tau_1) + \left(1 - \sum_{j\ne j_0} x_j(\tau_1)\right) \times \\ & \times \left((\lambda_j(\tau_1) - \mu_j)(t-\tau_1) - \lambda_j(\tau_1) \int_0^{t-\tau_1} z(s)\mathrm{d}s\right), \end{aligned} \tag{5.31}$$

where

$$z(t) = \left(\frac{\sum_{j\ne j_0}(\lambda_j(\tau_1) - \mu_j)}{\sum_{j\ne j_0} \lambda_j(\tau_1)}\right)\left(1 - \exp\left[-t \sum_{j\ne j_0} \lambda_j(\tau_1)\right]\right). \tag{5.32}$$

Then,

$$\begin{aligned} \tau_2 &= \min_{\substack{1\le j\le k\\ j\ne j_0}} \inf\{t \ge \tau_1 : x_j(t) \le 0\}, \\ j_1 &= \arg\min_{\substack{1\le j\le k\\ j\ne j_0}} \inf\{t \ge \tau_1 : x_j(t) \le 0\}, \end{aligned}$$

and the argument t in (5.31) and (5.32) belongs to the interval $[\tau_1,\tau_2]$, and subscript j in these equations is $j = 1, 2, \ldots, j_0 - 1, j_0 + 1, \ldots, k$. In this case $q_j(t) = x_j(t)$, and $q_{j_1}(t) = 0$ for all $t \ge \tau_2$.

Consequently excluding equations one-by-one, then the lth instant τ_l, $2 \le l \le k_0$, is defined from the system

$$\begin{aligned} x_j(t) =& x_j(\tau_{l-1}) + \left(1 - \sum_{j\ne j_0, j_1, \ldots, j_{l-2}} x_j(\tau_{l-1})\right) \times \\ & \times \left((\lambda_j(\tau_{l-1}) - \mu_j)(t-\tau_{l-1}) - \lambda_j(\tau_{l-1}) \int_0^{t-\tau_{l-1}} z(s)\mathrm{d}s\right), \end{aligned} \tag{5.33}$$

where

$$\begin{aligned} z(t) =& \left(\frac{\sum_{j\ne j_0, j_1, \ldots, j_{l-2}}(\lambda_j(\tau_{l-1}) - \mu_j)}{\sum_{j\ne j_0, j_1, \ldots, j_{l-2}} \lambda_j(\tau_{l-1})}\right) \times \\ & \times \left(1 - \exp\left[-t \sum_{j\ne j_0, j_1, \ldots, j_{l-2}} \lambda_j(\tau_{l-1})\right]\right). \end{aligned} \tag{5.34}$$

Then,

$$\tau_l = \min_{\substack{1\le j\le k\\ j\ne j_0,j_1,\dots,j_{l-2}}} \inf\{t \ge \tau_{l-1} : x_j(t) \le 0\},$$

$$j_{l-1} = \arg \min_{\substack{1\le j\le k\\ j\ne j_0,j_1,\dots,j_{l-2}}} \inf\{t \ge \tau_{l-1} : x_j(t) \le 0\},$$

and the argument t in (5.33) and (5.34) belongs to the interval $[\tau_{l-1},\tau_l]$, and subscript j in these equations takes the values from 1 to k but j_0, $j_1,\dots,$ j_{l-2}. In this case the normalized queue-lengths $q_j(t) = x_j(t)$, and $q_{j_{l-1}}(t)$=0 for all $t \ge \tau_l$.

Theorem 5.4 containing Scheme 5.5 is easily applied to networks with piece-wise deterministic environment. Specifically, the theorem is applied to each of the intervals $[\sigma_{l-1}, \sigma_l)$, $l = 1, 2, \dots, r$ and $[\sigma_r, t)$.

6. The main theorem on queue-length processes in the client stations of networks with semi-Markov environment

The statement of Theorem 5.4 is easily adapted to client/server networks in semi-Markov environment. The difference between the approach to the piece-wise deterministic environment and semi-Markov environment is only that the points σ_l are random, and general relation (4.14) rather than (5.1) must be used.

For example, considering the first term of (4.14), we have

$$\begin{aligned} &x_j(\sigma_1 \wedge t)\\ &= \sum_{l=1}^{\infty} \int_{\sigma_{l-1}\wedge(\sigma_1\wedge t)}^{\sigma_l\wedge(\sigma_1\wedge t)} \left[\lambda_j(\mathcal{E}(\sigma_{l-1}))\left\{1-\sum_{i=1}^{k}\Phi_s(x_j)\right\}-\mu_j\right]\mathrm{d}s \\ &= \int_0^{\sigma_1\wedge t}\left[\lambda_j(\mathcal{E}(0))\left\{1-\sum_{i=1}^{k}\Phi_s(x_j)\right\}-\mu_j\right]\mathrm{d}s, \end{aligned} \tag{6.1}$$

and the extension the above theory of Section 5 is elementary. For example one can make the assumption about arbitrary initial conditions in client stations and arbitrary number of bottleneck stations. The same theory of Section 5 can be developed for any interval with random lower and upper bounds as well, such that

$$\int_{\sigma_{l-1}\wedge t}^{\sigma_l\wedge t}\left[\lambda_j(\mathcal{E}(\sigma_{l-1}))\left\{1-\sum_{i=1}^{k}\Phi_s(x_j)\right\}-\mu_j\right]\mathrm{d}s,$$

resulting in adaptation of the theory of Section 5 to client/server networks in semi-Markov environment.

The theorem below is the adaptation of Theorem 5.4 to the case of a random interval $[0,\, \sigma_1 \wedge t)$. (The extension of Theorem 5.4 to the other intervals $[\sigma_{l-1} \wedge t,\, \sigma_l \wedge t)$, l=1,2,..., is similar.)

Theorem 6.1. *Assume that amongst k client stations, there are k_0 absolutely non-bottleneck in time 0, and the rest $k - k_0$ client stations are absolutely bottleneck in time 0. Assume that the initial normalized queue-lengths in all of these k client stations are β_j in limit as $N \to \infty$ correspondingly ($j = 1, 2, \ldots, k$). Then there are time instants $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_{k_0}$ which are defined recurrently by Scheme 5.5 of Theorem 5.4. We only take into account the values τ_i satisfying the inequality $\tau_i \leq \sigma_1 \wedge t$, i.e. we set $\ell = \ell(\sigma_1) = \max\{i : \tau_i \leq \sigma_1 \wedge t\}$. Then, the only instants $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_\ell$ are taken into account in this theorem. The main relation* (5.29) *of Scheme 5.5 now looks*

$$
\begin{aligned}
x_j(\sigma_1 \wedge t) =& x_j(0) + \left(1 - \sum_{j=1}^{k} x_j(0)\right) \times \\
& \times \left((\lambda_j(0) - \mu_j)(\sigma_1 \wedge t) - \lambda_j(0) \int_0^{\sigma_1 \wedge t} z(s)\, ds\right), \\
& j = 1, 2, \ldots, k,
\end{aligned} \tag{6.2}
$$

where $x_j(0) = \beta_j$, and

$$
z(t) = \left(\frac{\sum_{j=1}^{k} (\lambda_j(0) - \mu_j)}{\sum_{j=1}^{k} \lambda_j(0)}\right) \left(1 - \exp\left[-t \sum_{j=1}^{k} \lambda_j(0)\right]\right). \tag{6.3}
$$

The other relations of this theorem are defined similarly to the corresponding relations of Scheme 5.5 where only argument t is replaced by $\sigma_1 \wedge t$ in the corresponding places. Specifically, (5.31) *now looks*

$$
\begin{aligned}
x_j(\sigma_1 \wedge t) = x_j(\tau_1) + \left(1 - \sum_{j \neq j_0} x_j(\tau_1)\right) \times \\
\times \left((\lambda_j(\tau_1) - \mu_j)((\sigma_1 \wedge t) - \tau_1) - \lambda_j(\tau_1) \int_0^{(\sigma_1 \wedge t) - \tau_1} z(s)\, ds\right),
\end{aligned}
$$

where $z(t)$ *is defined by* (5.32), *and* (5.33) *now looks*

(6.4)

$$x_j(\sigma_1 \wedge t) = x_j(\tau_{l-1}) + \left(1 - \sum_{j \neq j_0, j_1, \ldots, j_{l-2}} x_j(\tau_{l-1})\right) \times$$
$$\times \left((\lambda_j(\tau_{l-1}) - \mu_j)((\sigma_1 \wedge t) - \tau_{l-1}) - \lambda_j(\tau_{l-1}) \int_0^{(\sigma_1 \wedge t) - \tau_{l-1}} z(s)\, ds\right),$$

where $z(t)$ *is defined by* (5.34), *and* $l = 1, 2, \ldots, \ell$.

7. Application: Performance analysis of symmetric large closed client/server computer networks with unreliable client stations

In this section we discuss application of the above main theorem for symmetric large closed client server computer networks.

7.1. **Formulation of the problem.** We consider a network with k identical servers. We assume that all of the processes started at zero, i.e. $A_{j,N}(0) = S_{j,N}(0) = Q_{j,N}(0) = 0$, and the following other assumptions and notation are used. The service time of each unit in the server station is exponentially distributed with parameter λ. There are k client stations in total, and each of client stations is a subject to breakdown. A lifetime of each client station has the probability distribution $G(x)$. In this case the moments of breakdown are associated with change states of semi-Markov environment, and this example is one of applications of the above theory.

The process $S_{j,N}(t)$ satisfies the condition

$$\mathbb{P}\left\{\lim_{N\to\infty} \frac{S_{j,N}(t)}{Nt} = \mu\right\} = 1. \tag{7.1}$$

(The equivalent form of condition (7.1) is considered in Section 3. It is assumed here that $\mu_1 = \mu_2 = \ldots = \mu_k \equiv \mu$.)

The relations between parameters λ, μ and k are assumed to be

$$\frac{\lambda}{k\mu} < 1, \tag{7.2}$$

and

$$\frac{\lambda}{\mu} > 1. \tag{7.3}$$

Condition (7.2) means that all of the client stations are initially non-bottleneck. Condition (7.3) means that after one of other breakdown

all of the client stations become bottleneck. Denote

$$l_0 = \max\left\{ l : \frac{\lambda}{l\mu} > 1 \right\}$$

the maximum number of available client stations when the client stations all are bottleneck. Then for all $l \le l_0$ the rest l client stations will be bottleneck as well.

Let $\alpha < 1$ be a given positive number. We say that the network is *at risk* if the total number of units in client stations exceeds the value αN. Assuming that at the initial time moment all of the units are in the server stations, the aim of this section is to find a confidence interval $[0, \theta)$ such that with a given high probability P (say $P = 0.95$) the network will not be at risk during that time interval $[0, \theta)$. The parameter N is assumed to be large. Therefore we study asymptotic solution of this problem as $N \to \infty$, that is we study limiting (as $N \to \infty$) normalized cumulative queue-length process in client stations. This problem is hard in general. Here we consider a particular case of $l_0 = 1$.

7.2. **Solution of the problem.** The limiting normalized queue-length process is denoted $q(t)$. At the initial time moment $t = 0$ there are k available client stations. Let τ_1, τ_2,...,τ_k be the moments of their breakdown, $0 \le \tau_1 \le \tau_2 \le \ldots \le \tau_k$. The above moments of breakdown are associated with the behavior of the time dependent network, which can be considered as a network in semi-Markov environment. Let us then apply Theorems 5.4 and 6.1. (Last Theorem 6.1 is used consequently from one random interval to another. Here we do not explain the details of this application assuming that they are clear.)

The random time interval $[0, \tau_k]$ is the lifetime of the entire system. Therefore $q(t)$ is to be considered during the aforementioned random interval $[0, \tau_k]$. We assume that $l_0 = 1$. Therefore, according to Theorem 5.4 and Theorem 6.1 we obtain that in the random interval $[0, \tau_{k-1})$, $q(t) = 0$, while in the random interval $[\tau_{k-1}, \tau_k)$ the equation for $q(t)$ is

$$q(t) = (\lambda - \mu)(t - \tau_{k-1}) - \lambda \int_0^{t-\tau_{k-1}} r(s)\mathrm{d}s, \tag{7.4}$$

where $r(t)$ is given by

$$r(t) = \left(1 - \frac{\mu}{\lambda}\right)(1 - \mathrm{e}^{-\lambda t}). \tag{7.5}$$

In the last endpoint τ_k we set $q(\tau_k) = 1$.

We have the following relationships:

$$\mathbb{P}\{q(t)=0\}=\sum_{i=2}^{k}\binom{k}{i}[1-G(t)]^i[G(t)]^{k-i}, \tag{7.6}$$

$$\begin{aligned}&\mathbb{P}\{q(t)\le\gamma<1\}\\&=[1-G(t)]\sum_{i=1}^{k-1}\binom{k-1}{i}[1-G(t-t_\gamma)]^i[G(t-t_\gamma)]^{k-i-1},\end{aligned} \tag{7.7}$$

where t_γ is such the value of t under which

$$(\lambda-\mu)t-\lambda\int_0^t r(s)\mathrm{d}s=\gamma. \tag{7.8}$$

The value t_γ is found from the relation

$$\frac{\int_0^\infty[1-G(t)]\sum_{i=1}^{k-1}\binom{k-1}{i}[1-G(t-t_\gamma)]^i[G(t-t_\gamma)]^{k-i-1}\mathrm{d}t}{\int_0^\infty\sum_{i=2}^{k}\binom{k}{i}[1-G(t-t_\gamma)]^i[G(t-t_\gamma)]^{k-i}\mathrm{d}t}=P, \tag{7.9}$$

and then from (7.8) one can find the corresponding value γ.

If the value of γ is not greater than α, then the value θ of the interval $[\tau_{k-1},\,\theta)$ is to be taken $\theta=\tau_{k-1}+t_\gamma$. Otherwise, if $\gamma>\alpha$, then the value θ is to be taken $\theta=\tau_{k-1}+t_\alpha$.

In the particular case of $k=2$ we have the following results. Relations (7.6), (7.7) and (7.9) reduces correspondingly to

$$\mathbb{P}\{q(t)=0\}=[1-G(t)]^2, \tag{7.10}$$

$$\mathbb{P}\{q(t)\le\gamma<1\}=[1-G(t)][1-G(t-t_\gamma)], \tag{7.11}$$

and

$$\frac{\int_0^\infty[1-G(t)][1-G(t-t_\gamma)]\mathrm{d}t}{\int_0^\infty[1-G(t-t_\gamma)]^2\mathrm{d}t}=P. \tag{7.12}$$

7.3. **Numerical calculation.** We consider the following example for $k=2$: $\lambda=4$, $\mu=3$, $\alpha=0.2$, $P=0.95$, $G(x)=1-\mathrm{e}^{-2x}$. From (7.12) we have:

$$\frac{\int_0^\infty \mathrm{e}^{-2(t-t_\gamma)}\mathrm{e}^{-2t}\mathrm{d}t}{\int_0^\infty \mathrm{e}^{-4(t-t_\gamma)}\mathrm{d}t}=\mathrm{e}^{-2t_\gamma}=0.95.$$

Solution of the equation $\mathrm{e}^{-2t_\gamma}=0.95$ yields $t_\gamma=0.025647$. From (7.8) we obtain:

$$\gamma=\int_0^{t_\gamma}\mathrm{e}^{-4t}\mathrm{d}t=\int_0^{0.025647}\mathrm{e}^{-4t}\mathrm{d}t=0.25-0.25\mathrm{e}^{-0.102588}\approx 0.024375.$$

This value of γ is less than $\alpha = 0.2$, and therefore this value $\gamma = 0.024375$ is a required value for parameter, which defines a desired confidence interval.

8. Discussion of new problems and associated monotonicity conditions for the networks in Markov environment

Theorem 6.1 looks very complicated, and its further analysis is very difficult to make a conclusion on the behavior of queue-length processes. For example, it seems very difficult to obtain any numerical characteristics of normalized queue-length processes analytically, $\mathbb{E}x_j(\sigma_1 \wedge t)$ for example. Therefore, the numerical work should be based on simulation of semi-Markov environment in order to obtain required performance characteristics of the process. (By simulation of semi-Markov environment we mean a multiple realizations for a Markov process in order to calculate required numerical performance characteristics of the process.)

For the purpose of performance analysis we also should restrict the class of networks and processes describing the behavior of queue-length processes in client stations. This restriction is related to application of the results rather than development of the theory. In many practical examples the quality characteristics of networks are changed monotonically resulting in one or other strategy of repair mentioned in Section 2.

The aforementioned comparing of two different strategies for fixed interval $(0, T)$ requires application of Theorem 6.1, and the problem can be solved without any additional assumption requiring monotonicity. However, under general settings we cannot answer to many significant questions. One of them is *How behave this criteria when the considered time interval is changed?* For example, we have two strategies corresponding two different initial conditions of semi-Markov environment, and suppose we concluded that the first strategy is more profitable than the second one for specific time interval $(0, T)$. *Is this conclusion remains correct (or becomes not correct) for another time interval $(0, T^*)$?* Another typical question is as follows. Again, we have two strategies corresponding two different initial conditions of semi-Markov environment. Suppose we established that for an interval $(0, T)$ the both strategies are equivalent. Let T^* be a new time instant, and $T^* > T$. *Which one of the strategies is now more profitable in the new time interval $(0, T^*)$, the first or second one?* The same question can be asked under the opposite inequality $T^* < T$.

These questions can be answered in the case when the class of the processes studied numerically has a monotone stricture and is described by the properties listed below. Then in certain cases the behavior of queue-length processes in client stations and consequently a conclusion about better strategy for other time intervals can be established as well.

Assuming for simplicity that the environment is *Markov*, then aforementioned properties are as follows.

(1) For any two positive integers $l \leq m$ assume that $z_{l,m} \geq z_{m,l}$.

Recall that $z_{l,m}\triangle t + o(\triangle t)$ $(l \neq m)$ are the transition probabilities from the state $\mathcal{E}_l$ to the state $\mathcal{E}_m$ of a homogeneous Markov process for a small time interval $(t,\, t + \triangle t)$.

(2) $\lambda_j(\mathcal{E}_l) \leq \lambda_j(\mathcal{E}_m)$ for all $j = 1, 2, \ldots, k$, and $l \leq m$.

Property (1) means that the Markov process $Z(t)$ is an increasing process in the following sense: for two time moments σ_{l-1} and σ_l we have $Z(\sigma_{l-1}) \leq_{st} Z(\sigma_l)$, which means that the state of a Markov process in time σ_{l-1} is not greater (in stochastic sense) than that state in time σ_l for any integer positive l. The above property remains correct for any $t_1 \leq t_2$, i.e. $Z(t_1) \leq_{st} Z(t_2)$.

Property (1) also means that for two Markov processes $Z_1(t)$ and $Z_2(t)$ having the same transition probabilities, but different initial conditions satisfying $Z_1(0) \leq_{st} Z_2(0)$, we also have $Z_1(t) \leq_{st} Z_2(t)$, $t \geq 0$. (For details of the proof of these properties see e.g. Kalmykov [20].)

Consequently, from property (2) we have $\lambda_j(\mathcal{E}(\sigma_{l-1})) \leq_{st} \lambda_j(\mathcal{E}(\sigma_l))$ for all $j = 1, 2, \ldots, k$ and any integer positive l. Moreover, for all $j = 1, 2, \ldots, k$ and any $t_1 \leq t_2$ we have $\lambda_j(\mathcal{E}(t_1)) \leq_{st} \lambda_j(\mathcal{E}(t_2))$.

Thus the rates $\lambda_j(\mathcal{E}(t))$ are increasing in time. As a result, the queue-length processes in client stations increase sharper than in the case of fixed λ_j of "usual" network, and more extended problems mentioned in this section seem can be solved as well. We however do not provide their solutions in the present paper.

In the next section, numerical investigation for concrete client server networks in a given Markov environment, satisfying the above two properties is provided.

9. Example of numerical study

In this section we do not intend to challenge a problem of comparing two different strategies or finding an optimal strategy. We only show (step-by-step) how to study the behavior of queue-lengths in client stations numerically. However, the detailed explanations of the given example can help to solve some of the aforementioned problems of

Section 2. The example models a Markov environment, i.e. in our example we simulate exponentially distributed random variables describing the state changes in the Markov environment. (In the given case by simulation we mean one realization of the process in order to study numerically a specific sample path of normalized queue-length process.)

We consider the simplest case of Markov transition matrix of the order 4

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

associated with the continuous Markov process $Z(t)$. In this matrix $P_{l,l+1} = 1$, $l = 1, 2, 3$, and $P_{4,4}$=1. This means that if the initial state of the process is $\mathcal{E}(0) = \mathcal{E}_1$, then the next state is $\mathcal{E}(\sigma_1) = \mathcal{E}_2$. Consequently, $\mathcal{E}(\sigma_2) = \mathcal{E}_3$, and $\mathcal{E}(\sigma_3) = \mathcal{E}_4$. Then $\mathcal{E}(\sigma_l) = \mathcal{E}_4$ for all $l \geq 3$. Assume also that $z_{l,l+1} = 1$, $l = 1, 2, 3$, so that $\mathbb{E}(\sigma_l - \sigma_{l-1})$ =1, l=1,2,3.

For simplicity, the network contains only 2 client stations. Assume that $\lambda_1(\mathcal{E}_1)$=1, $\lambda_1(\mathcal{E}_2)$=2, and for l=3,4, $\lambda_1(\mathcal{E}_l)$=3. We also assume that $\lambda_1(\mathcal{E}_l) = \lambda_2(\mathcal{E}_l)$ for $l = 1, 2, 3, 4$. The values $\mu_1 = \mu_2 = 2$. Next, $\beta_1 = \beta_2 = 0.1$, i.e. at the initial time moment each client station contains 10% of all units in the queue.

We set T=3, and study behavior of queue-length processes in client station in the time interval (0, 3). By simulation we obtained the following exponentially distributed random variables: 0.5488, 1.0892 and 1.8734. The sum of these 3 random variables is greater than 3, so this quantity of exponentially distributed random variables is enough for our experiment.

Note, that $\lambda_1(0) = \lambda_2(0) = 0.8$. There three time intervals: [0, 0.5488), [0.5488, 1.6380), [1.6380, 3).

For the time interval [0, 0.5488) the two client stations are absolutely non-bottleneck because $\frac{\lambda_j(\mathcal{E}_1)}{\mu_j}$=0.5, and we have the following equations:

$$\begin{aligned} x_1(t) = x_2(t) &= 0.1 - 0.96t - 0.8 \int_0^t z(s)\mathrm{d}s, \\ z(t) &= -1.5\left(1 - \mathrm{e}^{-1.6t}\right). \end{aligned} \tag{9.1}$$

Therefore, from (9.1) we obtain:

$$x_1(t) = x_2(t) = -0.65 + 0.24t + 0.75\mathrm{e}^{-1.6t}. \tag{9.2}$$

Substituting 0.5488 for t in (9.2) one can see that

$$x_1(0.5488) = x_2(0.5488) \approx -0.2066.$$

The endpoints are negative, therefore we are to find such the values τ_1 and τ_2 such that $x_1(\tau_1)$=0 and $x_2(\tau_2)$=0. In our case $\tau_1 = \tau_2 \approx 0.117$. This means that $q_1(t) = q_2(t) = 0$ for all $t \geq 0.117$ of the given interval [0, 0.5488). Therefore, in the endpoint of this interval $q_1(0.5488) = q_2(0.5488) = 0$.

Consider now the time interval [0.5488, 1.6380). In point 0.5488 we now set $x_1(0.5488) = x_2(0.5488)$=0. Therefore, $\lambda_1(0.5488) = \lambda_2(0.5488)$ =2, and the both client stations are bottleneck in [0.5488, 1.6380). Since $\frac{\lambda_j(0.5488)}{\mu_j}$=1, $j = 1, 2$, then $x_1(t)$ and $x_2(t)$ are equal to zero in this interval, and $q_1(t) = q_2(t)$ =0 in this interval as well.

We arrive at the last time interval [1.6380, 3). Similarly to the above, we have $x_1(1.6380) = x_2(1.6380)$=0, and $\lambda_1(1.6380) = \lambda_2(1.6380)$=3, and the both client stations are bottleneck in [1.6380, 3). However, in the both client stations we have $\frac{\lambda_j(1.6380)}{\mu_j}$=1.5, $j = 1, 2$. Therefore, after a little algebra we have the following equations:

$$q_j(t) = \frac{1}{6}\left(1 - \mathrm{e}^{-6(t-1.6380)}\right), \ j = 1, 2,$$

for all t from the interval [1.6380, 3).

10. Concluding remarks

In the present paper we introduced a class of client/server networks in order to study performance measures of real client/server networks. Our analysis was based on the results of earlier papers related to closed queueing networks with bottleneck. However, for purpose of real applications, we developed the earlier results and provided complete analysis of standard bottleneck client/server networks. We then extended our results for client/server networks in semi-Markov environment. The results obtained in this paper are then used for analysis of confidence intervals of client/server networks with failing client stations. Numerical study given in this paper will help to clearly understand solution for many related problems. The future work can be related to application of the theoretical results of this paper to concrete technological problems similar to those formulated in Section 2.

Acknowledgement

The research was supported by the Australian Research Council, grant #DP0771338.

References

[1] Abramov, V.M. (2000). A large closed queueing network with autonomous service and bottleneck. *Queueing Systems*, 35, 23-54.

[2] Abramov, V.M. (2001). Some results for large closed queueing networks with and without bottleneck: Up- and down-crossings approach. *Queueing Systems*, 38, 149-184.

[3] Abramov, V.M. (2004). A large closed queueing network containing two types of node and multiple customers classes: One bottleneck station. *Queueing Systems*, 48, 45-73.

[4] Abramov, V.M. (2005). The stability of join-the-shortest-queue models with general input and output processes. arXiv: math/PR 0505040.

[5] Abramov, V.M. (2006). The effective bandwidth problem revisited. arXiv: math/PR 0604182.

[6] Abramov, V.M. (2007). Confidence intervals associated with performance analysis of symmetric large closed client/server computer networks. *Reliability: Theory and Applications*, 2 (2), 35-42.

[7] Abramov, V.M. (2007). Further analysis of confidence intervals for large client/server computer networks, submitted.

[8] Anulova, S.V. and Liptser, R. Sh. (1990). Diffusion approximation for processes with normal reflection. *Theory of Probability and its Applications*, 35, 413-423.

[9] Baccelli, F. and Makovsky, A.M. (1986). Stability and bounds for single-server queue in a random enviromnent. *Stochastic Models*, 2, 281-292.

[10] Berger, A., Bregman, L. and Kogan, Ya. (1999) Bottleneck analysis in multiclass closed queueing networks and its application. *Queueing Systems*, 31, 217-237.

[11] Borovkov, A.A. (1976). *Stochastic Processes in Queueing Theory.* Springer, Berlin.

[12] Borovkov, A.A. (1984). *Asymptotic Methods in Queueing Theory.* John Wiley, New York.

[13] Boxma, O.J. and Kurkova, I.A. (2000). The M/M/1 queue in heavy-tailed random environment. *Statistica Neerlandica*, 54, 221-236.

[14] Dellacherie, C. (1972). *Capacités et Processus Stochastiques.* Springer-Verlag, Berlin.

[15] D'Auria, B. M/M/∞ queues in quasi-Markovian random environment. arXiv: math/PR 0701842.

[16] Fricker, C. (1986). Etude d'une file GI/G/1 á service autonome (avec vacances du serveur). *Advances in Applied Probability*, 18, 283-286.

[17] Fricker, C. (1987). Note sur un modele de file GI/G/1 á service autonomé (avec vacances du serveur). *Advances in Applied Probability*, 19, 289-291.

[18] Gelenbe, E. and Iasnogorodski, R. (1980). A queue with server of walking type (autonomous service). *Ann. Inst. H. Poincare*, 16, 63-73.

[19] Helm, W.E. and Waldmann, K.-H. (1984). Optimal control of arrivals to multiserver queues in a random environment. *Journal of Applied Probability*, 21, 602-615.

[20] Kalmykov, G.I. (1962). On the partial ordering of one-dimensional Markov processes. *Theory of Probability and its Applications*, 7, 456-459.

[21] Kogan, Ya. (1992). Another approach to asymptotic expansions for large closed queueing networks. *Operations Research Letters*, 11, 317-321.

[22] Kogan, Ya. and Liptser, R.Sh. (1993). Limit non-stationary behavior of large closed queueing networks with bottlenecks. *Queueing Systems*, 14, 33-55.

[23] Kogan, Ya., Liptser, R.Sh. and Smorodinskii, A.V. (1986). Gaussian diffusion approximation of a closed Markov model of computer networks. *Problems of Information Transmission*, 22, 38-51.

[24] Krichagina, E.V., Liptser, R.Sh. and Puhalskii, A.A. (1988). Diffusion approximation for a system that an arrival stream depends on queue and with arbitrary service. *Theory of Probability and its Applications*, 33, 114-124.

[25] Krichagina, E.V. and Puhalskii, A.A. (1997). A heavy-traffic analysis of closed queueing system with GI/∞ server. *Queueing Systems*, 25, 235-280.

[26] Krieger, U., Klimenok, V.I., Kazmirsky, A.V., Breuer, L. and Dudin, A.N. (2005). A BMAP/PH/1 queue with feedback operating in a random environment. *Mathematical and Computer Modelling*, 41, 867-882.

[27] Liptser, R.Sh. and Shiryayev, A.N. (1989). *Theory of Martingales.* Kluwer, Dordrecht.

[28] McKenna, J. and Mitra, D. (1982). Integral representation and asymptotic expansions for closed Markovian queueing networks. Normal usage. *Bell System Technical Journal*, 61, 661-683.

[29] O'Cinneide, C. and Purdue, P. (1986). The M/M/∞ queue in a random environment. *Journal of Applied Probability*, 23, 175-184.

[30] Pittel, B. (1979). Closed exponential networks of queues with saturation: The Jackson type stationary distribution and its asymptotic analysis. *Mathematics of Operations Research*, 6, 357-378.

[31] Purdue, P.R. (1974). The M/M/1 queue in a random environment. *Operations Research*, 22, 562-569.

[32] Ramanan, K. (2006). Reflected diffusions defined via extended Skorokhod map. *Electronic Journal of Probability*, 11, 934-992.

[33] Skorokhod, A.V. (1961). Stochastic equations for diffusion processes in a bouded region. *Theory of Probability and its Applications*, 6, 264-274.

[34] Tanaka, H. (1979). Stochastic differential equations with reflected boundary conditions in convex regions. *Hiroshima Mathematical Journal*, 9, 163-177.

[35] Whitt, W. (1984). Open and closed models for networks of queues. *AT&T Bell Laboratory Technical Journal*, 63, 1911-1979.

School of Mathematical Sciences, Monash University, Building 28M, Wellington road, Clayton, VIC 3800, Australia

E-mail address: vyacheslav.abramov@sci.monash.edu.au