

USING A GRAPHICAL CALCULATOR FOR THE VCAB COMMON ASSESSMENT TASK IN SPACE AND NUMBER

Stephen R. Clarke & Kaye B. Clarke
Mathematics Department
Swinburne Institute of Technology

Graphical calculators are gaining a foothold in Australian classrooms. The March issue of Vinculum contained an insert advertising the TI-81 Graphics calculator purchased in bulk by the Mathematics Association of Victoria. This should make them more accessible to schools. At Swinburne, these calculators were introduced as a compulsory purchase for all students in the first year of the Mathematics/Computer Science degree. By drawing graphs of cartesian, polar and parametric equations, the calculators allow alternative methods of investigating problems to those of traditional calculators. In this paper we show how a graphics calculator such as the TI-81, could be used to investigate the properties of a spirograph¹.

The 1991 Victorian Curriculum and Assessment Board (VCAB) Space and Number Common Assessment Task 1: Investigative Project was based on the theme Circles in Design. One section of this suggested investigating the patterns drawn by a spirograph and finding the equations of the curves produced. This suggestion raised the possibility of using the TI-81 to simulate the operations of a spirograph, and so produce an electronic spirograph.

The Cycloid

When a circle of radius a rolls on a straight line, a point on the circumference traces out a cycloid. The derivation of the equations is easily found in most undergraduate mathematics texts (for example, Hunt, 1981, p. 773). In fact this is usually the standard example used for the introduction to parametric equations. Whilst we cannot find an equation explicitly connecting x and y , we can find both x and y in terms of a third variable or parameter, in this case the angle t radians through which the circle has turned. The equations for the point originally on top of the circle are derived using the formula for arc length and some geometry:

$$x = at + a \sin t$$

$$y = a + a \cos t$$

Note that we cannot eliminate t to obtain y in terms of x . Instead, we have to choose values of t , and then evaluate the equations to find the values of x and y for one point on the curve. However in addition to graphing cartesian equations where we have y as a function of x , the TI-81 also handles parametric equations.

Graphing Parametric Equations on the TI-81

Suppose we wish to graph these equations on the TI-81 for $a = 1$. We need to go through the following steps:

* Use **MODE** button to ensure the calculator is in **Parametric mode**, **Connected mode** and **Radian mode**.

* Use **Y=** button to enter the above equations into the x_{IT} and y_{IT} variables. i.e. enter

$$x_{IT} = t + \sin t$$

$$y_{IT} = 1 + \cos t$$

* Use **RANGE** button to enter required ranges. In this case $T_{min} = 0$, $T_{max} = 13$, $T_{step} = .1$, $X_{min} = 0$, $X_{max} = 13$.

* Use **ZOOM 5** to 'Square' up the graph paper, so the same scales will be used on the x and y axis.

* Use **GRAPH** to draw the graph.

This takes a relatively inexperienced user about 2 minutes to produce Figure 1².

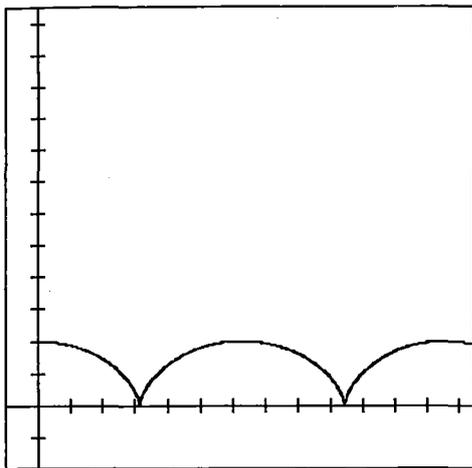


FIGURE 1

At this point we see a big difference in the way the TI-81 presents the graph from how we usually see them in texts. Here we see the graph as it is being drawn - the graph has movement. In fact the speed of movement can be varied by altering the value of Tstep. Thus for example we see the point moving vertically downwards just prior to touching the axis, and momentarily stop on the axis. This has a greater impact than merely seeing the completed graph.

An observant student (with a little prompting if necessary), would see that each coordinate is made up of 2 terms i.e. $(x,y) = (at,a) + (a.\sin t,a.\cos t)$. The first term gives the coordinates of the centre of the circle, while the second is the equation of a circle; i.e the path of the point on the circumference relative to the centre. In fact students could use this argument to guess the form of the equations without going through the formal derivation. Could we demonstrate this with the calculator? The calculator actually allows simultaneous graphing of up to 3 graphs. Thus returning to Y= window, enter the equations of the centre $x_{2T}=t, y_{2T}=1$. Use **MODE** button to enter **Simultaneous** mode. Pressing **GRAPH** we now see the path of the centre traced out, with the cycloid moving around it.

The Extended Cycloid

The cycloid is the path traced out by a point on the circumference of a rolling circle. In a spirograph, holes inside the rolling circle allow points inside the circle to be selected. If the hole is at radius h from the centre, a similar derivation to the above or some guess-work can be used to obtain

$$\begin{aligned} x &= a.t + h \sin t \\ y &= a + h \cos t \quad h < a \end{aligned}$$

The TI-81 allows editing of equations, so we could

enter the Y = window and insert a value for a and h . This could now be used to reproduce the various patterns of the spirograph and investigate the effect on the graph of altering h . However we can now use the calculator to 'extend' the spirograph. Because of the construction of the spirograph, the pen holes are inside the circle. While it is not physically possible to have the holes outside the circle, mathematically it is easy to let h take values greater than a in the above equations. In this case the graph produced has loops as shown in Figure 2. This was produced using $a=2, h=3$.

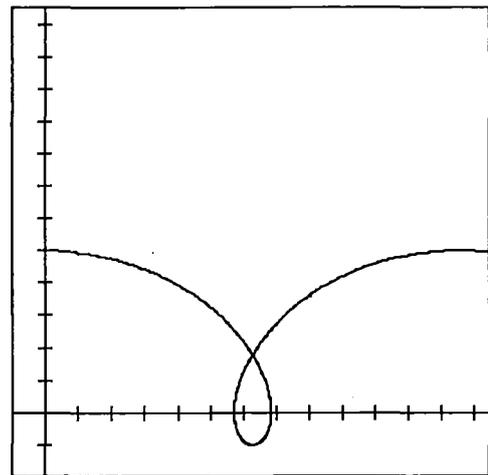


FIGURE 2

In fact this situation arises in rail travel, where because of the flange used to keep the wheel on the rails, the rim of the wheel extends past the rail on which the wheel rolls. This means that parts of the wheel actually move backwards. This can be verified by differentiating the x equation with respect to t . However with the TI-81 this is not necessary. As the curve is drawn out students actually see the point going backwards. The TRACE button can be used to gain control over this movement. The cursor keys can now be used to step the cursor through the graph. In this mode, the current values of t, x and y are shown on the bottom of the screen.

Circles Rolling on Circles

The spirograph also allows circles to roll on circles, rather than straight lines. Bright students could calculate the relevant equations from a suitable diagram in much the same way as the equations for the cycloid, although it is more complicated. However, the TI-81 allows a 'trial and error' approach.

We saw above that we could obtain the equation by adding the motion relative to the centre to the motion of the centre. Now if a circle of radius a rolls on a

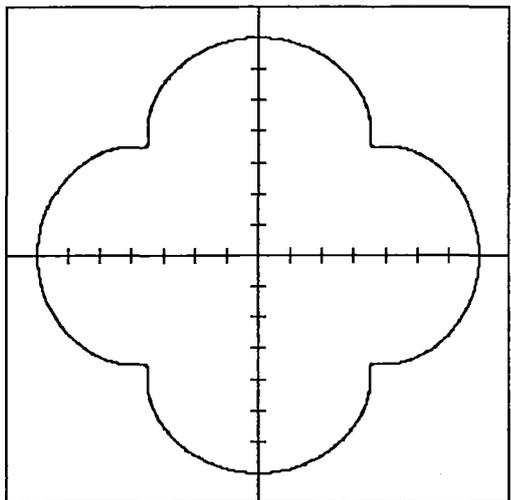


FIGURE 3

circle of radius b , the centre of the first circle moves in a circle of radius $(b+a)$. Thus by analogy with the cycloid, we might try for the equations

$$\begin{aligned} x &= (b+a) \sin \theta + a \sin \phi \\ y &= (b+a) \cos \theta + a \cos \phi \end{aligned}$$

where θ and ϕ are the angles the two circles turn through. As before, since there is no slipping the arc lengths are the same and $a\phi = b\theta$, or $\phi = (b/a)\theta$. Using the calculator students could then try and produce the curves obtained with a spirograph. An example is shown in Figure 3 for $a = 1$, $b = 5$. Unfortunately we are not quite there. Using spirograph wheels with radii in the ratio of 5:1 produces a figure similar to Figure 3, but with 5 sections rather than 4. By comparison with the spirograph, students would notice that they always get fewer 'spikes' using the above equation. (In fact, one for each revolution of the stationary circle). Where do the extra spikes come from?

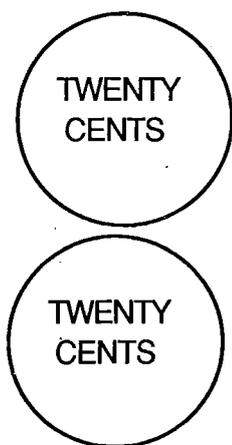


FIGURE 4

Try placing two twenty cent pieces as in Figure 4. If the top coin is rolled around the second, what will it

look like when it gets to the bottom? Many people say upside down. Try it. Now try sliding it around, where the bottom of the top coin remains in contact. The first case produces an extra 180° of rotation. This is similar to what we did before. To get the total rotation of the moving circle (360°), we add the rotation relative to its centre (180°) to the rotation of its centre (180°). It is clear the angle θ has to be added to ϕ to produce the total rotation of the rotating circle.

Thus we might try

$$\begin{aligned} x &= (b+a) \sin \theta + a \sin (\phi + \theta) \\ y &= (b+a) \cos \theta + a \cos (\phi + \theta) \end{aligned}$$

Substituting for ϕ , and allowing the hole to be distance h from the centre as before, and using t for θ we obtain

$$\begin{aligned} x &= (b+a) \sin t + h \sin (b/a + 1)t \\ y &= (b+a) \cos t + h \cos (b/a + 1)t \end{aligned}$$

Note that some students may arrive here more easily just by adding the 1 to ϕ for the spike they were missing.

Examples can now be tried and verified as similar to those produced by the spirograph. In trying these, students will often find they obtain only part of the picture. With a real spirograph, one just continues drawing until the curve joins up. With the TI-81 one needs to enter the range for t . This could be 0 to 6.3 (approximately 2π) if one revolution is required, or 0 to 63 if 10 revolutions are required. Using the calculator forces students to consider the range explicitly, and to consider the relationship between the radii of the circles, the number of points on the graph, and the number of turns necessary.

Circles within Circles

We leave it to those with TI-81's to experiment with this case. For those who wish to try we suggest the equations below.

$$\begin{aligned} x &= (b-a) \sin t + h \sin (b/a - 1)t \\ y &= (b-a) \cos t - h \cos (b/a - 1)t \end{aligned}$$

We can again use the facility of drawing 2 plots simultaneously to advantage. By drawing at the same time the graphs produced by the circle rolling on the outside and the inside, students see the similarities between the two curves. For example the diagram in Figure 5 is produced with $b=5$, $a=4$, $h=3$. This is actually not possible on the real spirograph, because of the thickness of the rings used; the internal radius of

the stationary ring is different from the outside radius. This always masks the pattern that is obvious if the same diameters are used.

We might think a has to be less than b (Can a large circle roll inside a small one?). Of course for the TI-81, there is no need to restrict ourselves to the cases where $a < b$. We might ask students to investigate if $a > b$ corresponds to any physical case. Again by experimenting, they may find they obtain with the calculator similar shapes to certain cases with the spirograph.

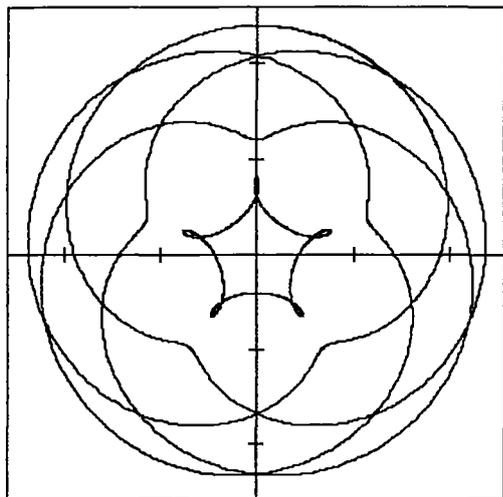


FIGURE 5

Circles rolling on Circles rolling on Circles and beyond.

While this would be difficult with a spirograph, it is very easy to extend the above equations in all sorts of

weird and wonderful ways; by adding extra terms, by changing signs, by changing operations, by altering parameters. The calculator itself becomes an interesting device on which to experiment; it is more flexible than the spirograph. In fact, we have produced some interesting curves by making typing errors.

Conclusion

It is a mistake to imagine that the uses of graphical calculators are confined to traditional areas of the syllabus where static graphs are required. The dynamic nature of the TI-81 produces insights not apparent from the completed graph. In addition, the availability of such a tool allows the use of alternative methods of looking at problems. Although most students would not have had available such a tool to assist in the recent Space & Number Common Assessment Task, in a few years such aids will be common. Teachers need to begin thinking of the uses to which such tools can be put.

Notes: 1. Spirographs are made by Toltoys Ltd.
2. Because of the difficulties in getting hard copy suitable for reproduction from a TI-81, all the graphs shown here have actually been produced using Master Grapher (1989) on a Macintosh.

References

- Hunt, R. A. (1988) *Calculus with Analytic Geometry*, Harper and Row.
- Texas Instruments. (1990). *TI-81 Guidebook*.
- Waits, B., Demana, F., & New, D. (1989) *Master Grapher V1.0*. Addison Wesley.

DATES TO NOTE

★ NATIONAL MATHEMATICS WEEK ★

August 5th - 9th

Theme: Mathematics in the Making

★ MATHS TALENT QUEST ★

Display: June 24th - 28th at

Plaza Conference Room, Melbourne University