Thermal elastic–plastic stress analysis of an anisotropic structure

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Abstract: In this paper, a polynomial stress function is utilized to satisfy both the governing differential equation for an anisotropic plane stress problem and the corresponding boundary conditions for plastic deformation. A theoretical solution for the thermal elastic–plastic problem of composite structure is obtained by means of the Tsai–Hill strength theory of anisotropic material. The composite structure is composed of a steel fibre-reinforced aluminium metal–matrix with a linear hardening material property. On the other hand, an elastic–plastic finite element analysis for the same problem is also carried out by using ABAQUS. The theoretical solution is in good agreement with the results from the finite element analysis. Finally, some examples are given and the corresponding results are discussed.

Keywords: composite structure, elastic–plastic thermal stress, residual stress

NOTATION

- \( a_{ij} \) components of composite compliance matrix
- \( f \) potential function
- \( K \) hardening constant
- \( T_0 \) maximum value of temperature change
- \( \Delta T \) temperature change (absolute temperature minus reference temperature)
- \( u \) and \( v \) \( x \) direction and \( y \) direction displacements in the \( xy \) plane of structure
- \( V_f \) volume percentages of the fibres
- \( X, Y \) and \( Z \) yield strengths in the three main directions (1, 2 and 3) of the fibre-reinforced material
- \( \alpha_1 \) and \( \alpha_2 \) thermal expansion coefficients of the material’s main directions (1–2)
- \( \varepsilon_{ij} \) strains in the \( xy \) plane of structure
- \( \varepsilon^e_{ij} \) elastic strains in the \( xy \) plane of structure
- \( \varepsilon^p_{ij} \) plastic strains in the \( xy \) plane of structure
- \( \varepsilon_p \) equivalent plastic strain
- \( \theta \) angle between the fibre reinforced direction and geometry axis \( x \)
- \( \sigma_{ij} \) stresses in the \( xy \) plane of structure
- \( (\sigma_x)_y \) residual stress component in the \( x \) axis
- \( \sigma_0 \) yield stress in the first principal direction of the reinforced material

1 INTRODUCTION

Because thermoplastic composites possess many unique characteristics, such as they may be re-melted, re-processed and reformed, they are easily repaired and remelted for repairing the local cracks and delamination. Experimental investigations on the forming of thermoplastic composites can be found in references [1] to [6]. Residual stresses in thermoplastic composites are particularly interesting because they may lead to structural premature failure. Prediction and measurement of residual stresses are, therefore, important in relation to production, design and application of thermoplastic composite structure, so many researches have been done in references [7] to [10].

Sayman and Bektas [11] carried out an elastic–plastic stress analysis on different angle-ply thermoplastic laminated plates with a simply supported boundary, which were subjected to average temperature change through the thickness of the structure. They found that the magnitudes of residual stress increase gradually depending on the temperature increment. Sayman [12]
presented an elastic–plastic thermal stress analysis on steel fibre-reinforced aluminium metal–matrix composite beams. In their paper, temperature is considered as a linear variation, which is from zero at the upper surface of the beam to \( T_0 \) at the lower surface. The distribution of residual stresses and deformations in the beam is obtained. For composite structures under general thermal loading, relative temperature change can be positive or negative. It is therefore important to research the thermal elastic–plastic problem of the composite beam subjected to thermal loading, in which temperature changes from a negative value to a positive one along the height of the beam.

The paper presents a polynomial stress function to solve the governing differential equation with the corresponding boundary conditions for the anisotropic plane stress case. A theoretical solution for the thermal elastic–plastic problem of composite structure is obtained by means of the Tsai–Hill strength theory of anisotropic material. The composite structure is composed of a steel fibre-reinforced aluminium metal–matrix with a linear hardening material property and different stacking angles. Some practical examples for steel fibre-reinforced aluminium metal–matrix structures that have a linear strain-hardening property are given. It is found that thermal residual stresses in a thermoplastic composite are not only dependent on the thermal loading forms but also on the volume percentage of the fibres. Finally, other valuable results are presented.

In order to further validate the method and computing process of the theoretical solution, an elastic–plastic finite element solution for the same problem is also carried out by using the finite element software ABAQUS. Comparing the results from the theoretical and finite element analyses, it is found that the two kinds of results are very close.

2 ELASTIC SOLUTION

A steel fibre-reinforced aluminium metal–matrix beam with a linear hardening material property is shown in Fig. 1. The two ends of the beam are considered as a fixed boundary condition. Temperature variation in the beam is linear and from \(- T_0 \) to \( T_0 \) along the height of the beam. The equilibrium differential equation for the composite beam is given by [13]

\[
a_{22} \frac{\partial^4 \phi}{\partial x^4} - 2a_{26} \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + (2a_{12} + a_{66}) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} - 2a_{16} \frac{\partial^2 \phi}{\partial x \partial y} + a_{11} \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{1}
\]

where \( \phi \) represents a stress function. The relation between stress and strain is written as

\[
\begin{align*}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} &= \begin{bmatrix}
a_{11} & a_{12} & a_{16} \\
a_{12} & a_{22} & a_{26} \\
a_{16} & a_{26} & a_{66}
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} \\
&+ \begin{bmatrix}
x_x \\
x_y \\
x_{xy}
\end{bmatrix} \Delta T
\end{align*}
\tag{2}
\]

where

\[
\begin{align*}
a_{11} &= s_{11}m^4 + (2s_{12} + s_{66})m^2n^2 + s_{22}n^4 \\
a_{12} &= s_{12}(m^4 + n^4) + (s_{11} + s_{22} - s_{66})m^2n^2 \\
a_{22} &= s_{11}n^4 + (2s_{12} + s_{66})m^2n^2 + s_{22}n^4 \\
a_{16} &= (2s_{11} - 2s_{12} - s_{66})mn^3 - (2s_{22} - 2s_{12} - s_{66})n^3m \\
a_{26} &= (2s_{11} - 2s_{12} - s_{66})n^3m - (2s_{22} - 2s_{12} - s_{66})mn^3 \\
a_{66} &= 2(s_{11} + 2s_{22} - 4s_{12} - s_{66})m^2n^2 + s_{66}(m^4 + n^4)
\end{align*}
\]

\[
s_{11} = 1/E_1, \quad s_{22} = 1/E_2, \quad s_{12} = -\mu_{12}/E_1, \quad s_{66} = 1/G_{12}, \quad m = \cos \theta, \quad n = \sin \theta \tag{3}
\]

and

\[
\begin{align*}
x_x &= x_1 \cos \theta + x_2 \sin \theta \\
x_y &= x_1 \sin \theta + x_2 \cos \theta \\
x_{xy} &= 2(x_1 - x_2) \sin \theta \cos \theta
\end{align*}
\tag{4}
\]

The boundary conditions and end conditions for this beam (as shown in Fig. 1) can be expressed as

\[
\begin{align*}
\sigma_x &= 0, \quad \tau_{xy} = 0, \quad \text{at} \quad y = \pm c \tag{5a} \\
\varepsilon_c &= 0, \quad \text{at} \quad x = 0, y = 0 \tag{5b}
\end{align*}
\]

The stress function satisfying both the differential equation (1) and boundary conditions (5a) can be written as

\[
\phi = \frac{s}{6} y^3 + \frac{e}{2} y^2 \tag{6}
\]
The relationships between the strain and displacement

For small deformations are

\[ \varepsilon_x = \frac{\partial u}{\partial x} = 0 \]  
(16a)

\[ \varepsilon_y = \frac{\partial v}{\partial y} = 0 \]  
(16b)

\[ \gamma_{xy} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \]  
(16c)

Because the structure of the composite beam and loading form are independent of the coordinate in the \( x \) direction, equation (16a) is written as

\[ \varepsilon_x = \frac{\partial u}{\partial x} = 0 \]  
(17)

Integrating equation (17) yields

\[ u = C_1(y) \]  
(18)

Substituting equations (15) into equation (16b) yields

\[ \varepsilon_y = \frac{\partial v}{\partial y} = sa_{12}y + x_sT_0 \frac{y}{c} \]  
(19)

Integrating equation (19) gives

\[ v = \frac{sy^2}{2} + \frac{y^2}{2c}x_sT_0 + C_2(x) \]  
(20)

Differentials of equations (18) and (20) yield

\[ \frac{\partial u}{\partial y} = \frac{dC_1(y)}{dy}, \quad \frac{\partial v}{\partial x} = \frac{dC_2(x)}{dx} \]  
(21)

Substituting equations (21) and (15) into equation (16c) gives

\[ \frac{dC_2(x)}{dx} + \frac{dC_1(y)}{dy} - sa_{16}y - x_sT_0 \frac{y}{c} = 0 \]  
(22)

Letting

\[ \frac{dC_2(x)}{dx} = K_1, \quad \frac{dC_1(y)}{dy} - sa_{16}y - x_sT_0 \frac{y}{c} = G_1 \]  
(23)

gives equation (22) as

\[ K_1 + G_1 = 0 \]  
(24)

Integrating equations (23) gives

\[ C_2(x) = K_1x + C_3 \]

\[ C_1(y) = \frac{s}{2}a_{16}y^2 + x_sT_0 \frac{y^2}{2c} - K_1y + C_4 \]  
(25)

Substituting equations (25) into equations (18) and (20)
For the composite beam with fixed ends and only under thermal load, in the plastic region of the beam, the equations of equilibrium are written as
\[
\begin{align*}
\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \quad (32a) \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0 \quad (32b)
\end{align*}
\]
For the composite beam with fixed ends and only under thermal load \(\Delta T(y)\), \(\tau_{xy}\) and \(\sigma_y\) can be neglected in comparison with \(\sigma_x\), and so equation (32b) is satisfied automatically. From equation (32a), \(\sigma_x\) is expressed as a function of \(y\).

3 ELASTIC–PLASTIC SOLUTIONS

Because the yield strengths of the thermoplastic composite beams under tension and compression loads are identical, the Tsai–Hill strength theory of composite structure is used as a yield criterion in this solution. \(X\), \(Y\), \(Z\) represent, respectively, the yield strengths in the three different main directions of the fibre-reinforced material; \(Y\) is assumed to be equal to \(Z\). \(S\) is the shear yield strength in the 1–2 plane. The yield condition according to this criterion can be written as
\[
\frac{\sigma_x^2}{X^2} + \frac{\sigma_y^2}{Y^2} - \sigma_1 \sigma_2 + \frac{\tau_{xy}^2}{S^2} = 1
\]
Equation (30) is rewritten as
\[
\sigma_{eq} = X = \frac{\sigma_x^2}{X^2} - \frac{\sigma_y^2}{Y^2} + \frac{\tau_{xy}^2}{S^2}
\]
In the plastic region of the beam, the equations of equilibrium are written as
\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0
\end{align*}
\]
For a linear strain-hardened material, the yield stress is given by the Ludwik equation
\[
X = \sigma_0 + K \varepsilon_p
\]
The stress components in the material’s principal directions are
\[
\sigma_1 = \sigma_x \cos^2 \theta, \quad \sigma_2 = \sigma_x \sin^2 \theta, \quad \tau_{12} = -\sigma_x \sin \theta \cos \theta
\]
Substituting equations (34) into equation (31), the yield condition of a composite beam with the stacking angle \(\theta\) is given by
\[
X_1 = \frac{X}{N}
\]
where
\[
N = \sqrt{\cos^4 \theta - \sin^2 \theta \cos^2 \theta + \frac{X^2 \sin^4 \theta}{Y^2} + \frac{X^2 \sin^2 \theta \cos^2 \theta}{S^2}}
\]
By using the potential function \(f = \sigma - \sigma(\varepsilon_p)\), the strain increments in the material’s principal directions are represented as
\[
\begin{align*}
\{ d \varepsilon^p_1 \} &= \{ d \varepsilon^p_2 \} = \frac{\partial f}{\partial \sigma_1} d \lambda \\
\{ d \varepsilon^p_2 \} &= \frac{\partial f}{\partial \sigma_2} d \lambda \\
\{ d \varepsilon^p_{12} \} &= \frac{\partial f}{\partial \tau_{12}} d \lambda
\end{align*}
\]
The total strain increments in the material’s principal directions are written as
\[
\begin{align*}
d \varepsilon_1 &= d \varepsilon_1^p + d \varepsilon_1^e \\
&= s_{11} \sigma_1 + s_{12} \sigma_2 + \frac{2 \sigma_1 - \sigma_2}{2\sigma} \varepsilon_p d \lambda + x_1 d (\Delta T) \\
d \varepsilon_2 &= d \varepsilon_2^p + d \varepsilon_2^e \\
&= s_{12} \sigma_1 + s_{22} \sigma_2 + \frac{-\sigma_1 + 2 \sigma_2 X^2}{2\sigma} \varepsilon_p d \lambda + x_2 d (\Delta T) \\
d \varepsilon_{12} &= d \varepsilon_{12}^p + d \varepsilon_{12}^e \\
&= \frac{S_{66}}{2} \tau_{12} + \frac{2 \tau_{12} X^2}{2\sigma} \tau_{12} d \lambda
\end{align*}
\]
where \( d\lambda = d\epsilon_p \). For the stacking angle \( \theta \), the stress component \( \sigma_x \) is written as

\[
\sigma_x = \frac{X}{N} = \frac{\sigma}{N}
\]

(39)

Substituting \( \sigma \) in equation (39) and equations (34) into equations (38) gives

\[
d\epsilon_1 = d\epsilon^1_x + d\epsilon^1_p
\]
\[
= s_{11} \ d\sigma_1 + s_{12} \ d\sigma_2 + \frac{2 \cos^2 \theta - \sin^2 \theta}{2N} \ d\lambda
\]
\[
+ \sigma_1 \ d(\Delta T)
\]
\[
de\epsilon_2 = d\epsilon^2_x + d\epsilon^2_p
\]
\[
= s_{12} \ d\sigma_1 + s_{22} \ d\sigma_2
\]
\[
+ \frac{-\cos^2 \theta + 2 \sin^2 \theta \ X^2/Y^2}{2N} \ d\lambda + \sigma_2 \ d(\Delta T)
\]
\[
de\epsilon_12 = d\epsilon^1_{12} + d\epsilon^2_{12}
\]
\[
= \frac{S_{66}}{2} \ d\tau_{12} + \frac{-2 \cos \theta \sin \theta \ X^2/S^2}{2N} \ d\lambda
\]

(40)

Substituting \( d\lambda = d\epsilon_p \) into equations (40) and integrating gives

\[
e_1 = e^1_x + e^1_p
\]
\[
= s_{11} \ e_x + s_{12} \ e_2 + \frac{2 \cos^2 \theta - \sin^2 \theta}{2N} \ e_p + \sigma_1 (\Delta T) + C_5
\]
\[
e_2 = e^2_x + e^2_p
\]
\[
= s_{12} \ e_x + s_{22} \ e_2 + \frac{-\cos^2 \theta + 2 \sin^2 \theta \ X^2/Y^2}{2N} \ e_p
\]
\[
+ \sigma_2 (\Delta T) + C_6
\]
\[
e_{12} = e^1_{12} + e^2_{12}
\]
\[
= \frac{S_{66}}{2} \ e_{\tau_{12}} + \frac{-2 \cos \theta \sin \theta \ X^2/S^2}{2N} \ e_p + C_7
\]

(41)

In the above formulae, let \( e_p = 0 \) and the strain components at the interface between the elastic region and plastic region be equal. The integration constants \( C_5, C_6 \) and \( C_7 \) in equations (41) are given by

\[
C_5 = X_1[(a_{11} - s_{11}) \cos \theta + (a_{12} - s_{12}) \sin \theta + a_{16} \sin \theta \cos \theta]
\]
\[
C_6 = X_1[(a_{11} - s_{12}) \sin \theta + (a_{12} - s_{12}) \cos \theta - a_{16} \sin \theta \cos \theta]
\]
\[
C_7 = X_1[(a_{12} - a_{11}) \sin \theta \cos \theta + \frac{a_{16}}{2} \cos \theta + \frac{s_{66}}{2} \sin \theta \cos \theta]
\]

(42)

The corresponding strain components are

\[
e_1 = s_{11} \sigma_1 + s_{12} \sigma_2 + \frac{2 \cos^2 \theta - \sin^2 \theta}{2N} \ e_p + \sigma_1 (\Delta T)
\]
\[
+ X_1[(a_{11} - s_{11}) \cos \theta + (a_{12} - s_{12}) \sin \theta]
\]
\[
+ a_{16} \sin \theta \cos \theta]
\]
\[
e_2 = s_{12} \sigma_1 + s_{22} \sigma_2 + \frac{-\cos^2 \theta + 2 \sin^2 \theta \ X^2/Y^2}{2N} \ e_p + \sigma_2 (\Delta T)
\]
\[
+ X_1[(a_{11} - s_{22}) \sin \theta + (a_{12} - s_{12}) \cos \theta]
\]
\[
- a_{16} \sin \theta \cos \theta]
\]
\[
e_{12} = \frac{S_{66}}{2} \ e_{\tau_{12}} + \frac{-2 \cos \theta \sin \theta \ X^2/S^2}{2N} \ e_p
\]
\[
+ X_1[(a_{12} - a_{11}) \sin \theta \cos \theta + \frac{a_{16}}{2} \cos \theta
\]
\[
+ \frac{S_{66}}{2} \sin \theta \cos \theta]
\]

(43)

Utilizing the strain transformation formula

\[
\begin{bmatrix}
  e_x \\
  e_y \\
  e_{xy}
\end{bmatrix}
= \begin{bmatrix}
  \cos^2 \theta & \sin^2 \theta & -2 \sin \theta \cos \theta \\
  \sin^2 \theta & \cos^2 \theta & 2 \sin \theta \cos \theta \\
  \sin \theta \cos \theta & -\sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
  e_1 \\
  e_2 \\
  e_{12}
\end{bmatrix}
\]

(44)

the strain components in the system of geometry axis (xy) are

\[
e_x = a_{11} \sigma_x + B_1 \epsilon_p + \sigma_x (\Delta T)
\]
\[
e_y = a_{12} \sigma_x + B_2 \epsilon_p + \sigma_y (\Delta T)
\]
\[
e_{xy} = \frac{a_{16}}{2} \sigma_x + B_3 \epsilon_p + \frac{a_{xy}}{2} (\Delta T)
\]

(45)

where

\[
B_1 = \frac{1}{2N} \left[ 2 \cos^4 \theta - 2 \cos^2 \theta \sin^2 \theta + 2 \sin^4 \theta \left( \frac{X^2}{Y^2} \right) \right]
\]
\[
+ 4 \sin^2 \theta \cos^2 \left( \frac{X^2}{Y^2} \right)
\]
\[
B_2 = \frac{1}{2N} \left[ 2 \cos^2 \theta \sin^2 \theta - \sin^4 \theta - \cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta \left( \frac{X^2}{Y^2} \right) - 4 \sin^2 \theta \cos^2 \theta \left( \frac{X^2}{Y^2} \right) \right]
\]
\[
B_3 = \frac{1}{2N} \left[ 3 \sin^3 \theta \cos \theta - \sin^3 \theta \cos \theta - 2 \sin^3 \theta \cos \theta \left( \frac{X^2}{Y^2} \right) \right]
\]
\[
+ (-2 \cos^3 \theta \sin \theta + 2 \sin^3 \theta \cos \theta) \left( \frac{X^2}{Y^2} \right) \right]
\]

(46)

Plastic deformation occurs first at the upper and lower surfaces and then expands from the lower and upper surfaces to \( h \) and \(-h\) respectively. At the interface between the elastic region and the plastic region, the
axial stress is
\[ \sigma_x = \pm sh = X_1 \] (47)

The interface position between the elastic region and the plastic region, \( h \), is given by
\[ h = \frac{a_{11}c}{\alpha_s T_0} X_1 \] (48)

The thermal load \( T_0 \) is expressed as
\[ T_0 = \frac{a_{11}c}{\alpha_s h} X_1 \] (49)

Because the two ends of the composite beam are fixed, the thermal strain component \( \varepsilon_x \) in the beam should equal zero, so the first equation (45) gives
\[ \varepsilon_x = -\frac{\sigma_0 + K e_p}{N} a_{11} + B_1 e_p + \alpha_x T_0 \frac{y}{c} = 0 \] (50)

Equation (50) is simplified to
\[ \varepsilon_p = a + by \] (51)

where
\[ a = \frac{a_{11} \sigma_0}{-a_{11}K + B_1 N} \]
\[ b = -\frac{\alpha_x T_0}{c(-a_{11}K + B_1 N)} \] (52)

The stress component \( \sigma_x \) is expressed as
\[ \sigma_x = -\frac{\sigma_0 + K e_p}{N} = -\frac{\sigma_0 + K(a + by)}{N} \] (53)

Utilizing the geometric relation between displacement components and strain components
\[ \varepsilon_x = \frac{\partial u}{\partial x} = 0 \]
\[ \varepsilon_y = \frac{\partial v}{\partial y} = a_{12} \sigma_x + B_2 e_p + \alpha_x \Delta T \] (54)

gives
\[ u = C_8(y) \]
\[ v = -\frac{a_{12} \sigma_0}{N} \frac{y}{h^2} + \left( -\frac{a_{12} K}{N} + B_2 \right) \left( ay + \frac{by^2}{2} \right) + \frac{\alpha_x T_0 y^2}{2c} + C_9(x) \] (55)

Substituting \( u \) and \( v \) in equations (55) into the expression of shear strain \( \varepsilon_{xy} \) in equations (45) gives
\[ \frac{a_{16} \sigma_x}{2} + B_3 e_p + \frac{\alpha_{xy} T_0}{2} = \frac{1}{2} \left[ \frac{d C_8(y)}{dy} + \frac{d C_9(x)}{dx} \right] \] (56)

Integrating equation (56) over \( x \) and \( y \) leads to the expressions of \( C_8(y) \) and \( C_9(x) \). Substituting the expressions of \( C_8(y) \) and \( C_9(x) \) into equations (55) yields
\[ u = -\frac{a_{16} \sigma_0}{N} y + \left( \frac{a_{16} K}{N} + 2B_3 \right) \left( ay + \frac{by^2}{2} \right) + \alpha_x T_0 \frac{y^2}{2c} + C_{10} \]
\[ v = -\frac{a_{12} \sigma_0}{N} y + \left( -\frac{a_{12} K}{N} + B_2 \right) \left( ay + \frac{by^2}{2} \right) + \frac{\alpha_x T_0 y^2}{2c} + C_{11} \] (57)

Utilizing the continuity condition at the interface between the elastic region and the plastic region
\[ u_e(x, h) = u_p(x, h), \quad v_e(x, h) = v_p(x, h) \] (58)

the displacement components \( u \) and \( v \) in the elastic–plastic solution are represented as
\[ u = -\frac{a_{16} \sigma_0}{N} y + \left( -\frac{a_{16} K}{N} + 2B_3 \right) \left( ay + \frac{by^2}{2} \right) + \alpha_x T_0 \frac{y^2}{2c} + C_{10} \]
\[ v = -\frac{a_{12} \sigma_0}{N} y + \left( -\frac{a_{12} K}{N} + B_2 \right) \left( ay + \frac{by^2}{2} \right) + \frac{\alpha_x T_0 y^2}{2c} + C_{11} \] (59)

To find out the residual stresses, it is necessary to superimpose the elastic–plastic stresses in equation (53) into the elastic stresses solution in equation (14), subjected to the same external forces. The resultants of the stresses at the fixed ends for the elastic region and the plastic region are, respectively, \( F_1 \) and \( F_2 \) or \(-F_1 \) and \(-F_2 \). The resultant force and bending moment about the middle layer of composite beam are \( F = F_1 + F_2 + (-F_1) + (-F_2) = 0 \) and a bending moment \( M = M_1 + M_2 \), as shown in Fig. 2. The bending moment is represented as
\[ M = M_1 + M_2 = 2 \left[ \frac{X_1 h^2}{3} + \frac{\sigma_0 + Ka}{2N} (c^3 - h^3) \right] + \frac{Kbt}{3N} (c^3 - h^3) \] (60)

The elastic stress component \( \sigma_x \) in the composite beam subjected to moment \( M \) is given by
\[ \sigma_x = -\frac{M y}{I} \] (61)
where \( I = \frac{2tc^3}{3} \) is the inertia moment of the beam and \( t \) is the thickness of the beam.

4 EXAMPLES AND DISCUSSION

As an example, a steel fibre-reinforced aluminum metal–matrix composite is considered. Two sets of mechanical properties corresponding to the volume percentage of the fibres are shown in Table 1.

The geometric parameters of the composite beam are, respectively, taken as \( L = 100 \text{ mm} \), \( t = 6 \text{ mm} \), \( 2c = 12 \text{ mm} \). Fibre-reinforced orientation angles are taken as 0°, 30°, 45°, 60° and 90° respectively. In this section, in order to further validate the present theoretical method, an elastic–plastic finite element solution for the same example is also obtained by using the finite element analysis software ABAQUS. The corresponding finite element mesh is shown in Fig. 3. The geometric dimensions and material properties are the same as those in the analytical solution. The finite element mesh consists of three-dimensional solid laminar elements of eight nodes. For accuracy, 7200 elements are used in the calculating model.

The theoretical solution and the finite element solution of composite beams with two ends fixed and subjected to thermal load \( \Delta T \) are shown, respectively, in Table 2. \( T_0 \) represents the value of the maximum temperature change that causes initial plastic yielding of the beam. From Table 2, it can be seen that the two results obtained by using the theoretical method and

<table>
<thead>
<tr>
<th>Volume percentage of the fibres ( V_f(%) )</th>
<th>( E_1 ) (GPa)</th>
<th>( E_2 ) (GPa)</th>
<th>( G_{12} ) (GPa)</th>
<th>( \mu_{12} )</th>
<th>( X ) (MPa)</th>
<th>( Y ) (MPa)</th>
<th>( S ) (MPa)</th>
<th>( K ) (MPa)</th>
<th>( x_1 \left(10^{-6}^\circ C\right) )</th>
<th>( x_2 \left(10^{-6}^\circ C\right) )</th>
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<td>74</td>
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<td>230</td>
<td>24.0</td>
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<td>32.8</td>
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<td>24.0</td>
<td>48.9</td>
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<td>20.2</td>
<td>23</td>
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</tbody>
</table>

Fig. 3 The elastic–plastic finite element model of the composite beam with two fixed ends, subjected to thermal load \( V_f = 20 \) per cent
Table 3 Elastic, plastic and residual stresses at the lower surface of the beam with \( V_i = 20 \) per cent

<table>
<thead>
<tr>
<th>( \theta ) (deg)</th>
<th>( T_0 (^\circ \text{C}) )</th>
<th>( h ) (mm)</th>
<th>( (\sigma_z)_p )</th>
<th>( (\sigma_y)_p )</th>
<th>( (\sigma_z)_a )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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<td>229.3134</td>
<td>264.9768</td>
<td>35.6633</td>
</tr>
<tr>
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</table>

Table 4 Displacements and equivalent plastic strains in the beam with \( V_i = 20 \) per cent

<table>
<thead>
<tr>
<th>( \theta ) (deg)</th>
<th>( h ) (mm)</th>
<th>( r_p )</th>
<th>( u ) (mm)</th>
<th>( v ) (mm)</th>
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<td>(-0.0014)</td>
<td>0</td>
<td>0.0078</td>
</tr>
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<td>3</td>
<td>(-0.0027)</td>
<td>0</td>
<td>0.0058</td>
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<tr>
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<td>2</td>
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<tr>
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<tr>
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<td>5</td>
<td>(-1.7434 \times 10^{-5})</td>
<td>(-8.1549 \times 10^{-5})</td>
<td>0.0018</td>
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<tr>
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</table>
that because the two ends of the composite beam are fixed, the transversal displacement $v$ is larger than the axial displacement $u$. For the $0^\circ$ stacking orientation angle and the elastic–plastic interface $h = 2\,mm$, the transversal displacement $v$ at the surface of the beam reaches the maximum of 0.0371 mm. The corresponding equivalent plastic strain at the lower surface of the beam equals $-0.0055$.

Elastic, plastic and residual stresses in the $0^\circ$ stacking composite beams with two different values of volume percentage of the fibres are shown in Table 5. This shows that elastic, plastic and residual stresses in the composite beams are related to the volume percentage of the fibre. For a given value of the temperature change $T_0$ in the composite beams, as the volume percentage of the fibres increases, the residual stresses in the composite beams decrease gradually.

<table>
<thead>
<tr>
<th>$V_f$ (%)</th>
<th>$T_0$ (°C)</th>
<th>$h$ (mm)</th>
<th>$(\sigma_x)_p$ (MPa)</th>
<th>$(\sigma_x)_e$ (MPa)</th>
<th>$(\sigma_x)_r$ (MPa)</th>
</tr>
</thead>
<tbody>
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</table>

The distribution of the residual stress component $(\sigma_x)_r$ in the composite beam with the $0^\circ$ stacking orientation angle is shown in Fig. 4. From this figure, the value of the residual stress component is maximum at the upper and lower surfaces of the beams, with the elastic–plastic interface $h = 5, 4$ and $3\,mm$. However, as the position of the interface reduces to $h = 2\,mm$, the maximum of the residual stress component $(\sigma_x)_r$ appears at this interface of the elastic–plastic regions.

The distribution of the residual stress components $(\sigma_x)_r$ in the composite beam with $30^\circ, 45^\circ, 60^\circ$ and $90^\circ$ stacking orientation angles are shown, respectively, in Figs 5 to 8. From these figures, the maximum value of the residual stress component $(\sigma_x)_r$ appears, respectively, at the upper and lower surfaces or at the interface of the elastic–plastic regions of the beams, with the elastic–plastic interface $h = 5, 4$ and $3\,mm$. However, for

![Fig. 4](image-url)  
**Fig. 4** Distributions of the residual stress component $(\sigma_x)_r$ along the height of the composite beam with a $0^\circ$ stacking orientation angle. $V_f = 20$ per cent

![Fig. 5](image-url)  
**Fig. 5** Distributions of the residual stress component $(\sigma_x)_r$ along the height of the composite beam with a $30^\circ$ stacking orientation angle. $V_f = 20$ per cent
the beam with the elastic–plastic interface $h = 2 \text{ mm}$, the maximum value of the residual stress component $(\sigma_x)_r$ always appears at the interface of the elastic–plastic regions. As the stacking orientation angle $\theta$ of the composite beam increases, the value of the residual stress component $(\sigma_x)_r$ in the beam decreases gradually; it is minimum with an orientation angle of $90^\circ$. Moreover, as the plastic region extends further, the value of the residual stress component $(\sigma_x)_r$ becomes large.

5 CONCLUSIONS

The following conclusions can be drawn from the present elastic–plastic solutions of a composite beam subjected to antisymmetric thermal loads:

1. The value of the residual stress component $(\sigma_x)_r$ reaches a maximum either at the upper and lower surfaces of composite beam or at the interface of the elastic–plastic regions. As the plastic region increases further, the maximum value occurs at the interface of the elastic–plastic regions. The larger the stacking angle of the composite beam, the lower is the temperature that causes the beam to produce plastic yielding. The residual stress component $(\sigma_x)_r$ reaches a maximum with the stacking angle $0^\circ$. The larger the stacking angle, the lower is the residual stress component $(\sigma_x)_r$.

2. The transversal displacement $v$ of the beam is always larger than the axial displacement $u$. The maximum value of the transversal displacement $v$ occurs in a beam with the stacking angle $0^\circ$ and the interface of the elastic–plastic region $h = 2 \text{ mm}$, and so does the maximum equivalent plastic strain.
3. Residual stresses in thermoplastic composites are related to the volume percentage of fibres. From Table 1, besides the volume percentage of fibres, the mechanical properties and yield strengths of composite structures also depend on the properties of the matrix and the reinforced fibre. In the example, the temperature $T_0$ and the residual stresses in the composite beams are directly dependent on the effective mechanical properties and yield strengths of the composite beams, such as $E_1, E_2, G_{12}, v_{12}, X, Y, S, K$ shown in Table 1. A difference exists between the values of the effective material’s properties of composite structures predicted from the volume percentage of fibres and those measured from experiments. A deeper discussion about the volume effect of fibres should be interesting, but is difficult using the macrothermal elastic–plastic method presented in the paper. Therefore, a microthermal elastic–plastic method should be applied in order to show how the temperature $T_0$ and the residual stresses in the composite beams are affected by the volume percentage of fibres.

ACKNOWLEDGEMENTS

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REFERENCES


