the algorithm of [4] cannot be computed in place since it requires storing a total of $1.5 \times |\text{det}(N)|$ samples.

The actual value of $C_{U|k|}^{\text{MD}}$ clearly depends on the algorithm used to compute the odd-DFT’s. If a split-radix FFT is used, it is possible to show that the total number of multiplications is

$$C_{U|k|}^{\text{MD}} = \left(\frac{4}{3}M\right)\text{det}(N)\log_2 |\text{det}(N)| - \left(\frac{4}{3}\right)(1 + (2^{3M}/2^{3M} - 1)) |\text{det}(N)|.$$  

If Bruun’s algorithm is used to compute the odd-DFT, the computational weight is still proportional to $|\text{det}(N)|\log_2 |\text{det}(N)|$, but only real arithmetic is required. Finally, if the odd-DFT are computed by means of an algorithm of linear complexity by Duhamel [11], the total multiplicative cost of the proposed MD FFT becomes proportional to $|\text{det}(N)|$.

B. Numerical Comparison

The theoretical results of Section V.A find quantitative confirmation in the results of Table I. This table compares the computational complexity of the algorithm proposed (with odd-DFT’s computed by the 1-D split-radix FFT) with that of the row-column FFT, the vector-radix [6] and, the vector split-radix FFT of [7] in the 2- and 3-D case. In every case, the proposed MD-FFT requires a lower number of additions or multiplications than the other algorithms.

VI. CONCLUSION

This work introduces a new multidimensional FFT algorithm, for signals periodic on generic lattices, based on the idea of weighted-periodic signals which can be interpreted as an ultimate extension of the decimation in frequency scheme.

The algorithm presented is made up of two parts: in the first the input signal is processed with multiplierless butterflies and, in the second, the computation is concluded with 1-D FFT’s. A strong point of this MD FFT is that it inherits the computational characteristics of the 1-D FFT used in the second part: if the 1-D FFT has linear complexity, the proposed method will also have linear complexity, and if the 1-D FFT is efficient with real data, the proposed method will also be efficient with real data, etc.

The computational efficiency of the proposed method is superior to that of any other MD FFT, being paired only by the polynomial transform FFT, which is not, however, as general as the technique proposed.

REFERENCES


Invertibility and Inverses of Linear Time-Varying Digital Filters

Song Wang and Cishen Zhang

Abstract—This brief studies invertibility and inverses of linear time-varying digital filters in the form of ARMA equation with time-varying dynamic order and relative degree. It presents a necessary and sufficient condition for the $\delta$-shift invertibility of linear time-varying filters and a computational procedure for obtaining the $\delta$-shift left and right inverses. A necessary and sufficient condition for stability of inverse filters is given.

Index Terms—ARMA equation, digital filters, inverse, linear time-varying filters, relative degree.

I. INTRODUCTION

Linear time-varying (LTV) digital filters exist abundantly in communications, filtering, decoding, and network synthesis (see for example, [3], [5], [9], [11], and are an active research area [4], [7], [11], [13], [12]). In this area, invertibility of LTV digital filters is a problem of considerable interest and importance, which can find applications in channel equalization, speech scrambling and blind source identification. For example, Fig. 1 depicts the speech scrambling scheme studied in [5]. In this scheme, the scrambling filter is a linear periodic system, i.e., a linear system with periodically time-varying parameters which is a special case of LTV systems. Thus, the scrambling filter and transmission channel in cascade connection can be viewed as an LTV system and the design of the scrambler filter is to find the inverse of this LTV system to recover the input speech signal.

An essential property of LTV filters is that the order and relative degree of the filter can be time-varying (see for example, [7], [11]). The LTV filters with fixed order and relative degree are a special class of general LTV filters.

Finding the inverses of fixed order periodic filters, which are a subset of LTV filters, was studied in [10] for finite impulse response (FIR) filters and in [6] for infinite impulse response (IIR) filters. In [6], a lifting technique is used to transform the periodic filter into an equivalent time invariant model. For general LTV filters, due to the lack of explicit methods in dealing with their time-varying dynamics and structures, there have been no known methods for testing invertibility and computing the inverses. In this brief, we study this problem for LTV filters represented in the form of auto-regressive moving-average (ARMA) equation with time-varying dynamic order and time-varying relative
degree. A necessary and sufficient condition is derived for the $d$-shift invertibility of LTV filters, and a procedure is developed for computation of $d$-shift left and right inverses. We also present a necessary and sufficient condition for stability of inverse LTV filters.

Section II formulates the inversion problem and gives definitions relating to stability, constant relative degree, and $d$-shift invertibility. Section III first discusses the relationship between left and right inverse filters, and then derives conditions for inverting LTV filters with and without constant relative degree and for stability of inverse filters. A procedure for computing the $d$-shift left/right inverse of an LTV filter and an example to illustrate the procedure are given in Section IV.

II. PROBLEM FORMULATION

Consider an LTV IIR digital filter $G : u \mapsto y$ in the form of the following ARMA equation

$$y_k + a_{1,k}y_{k-1} + \ldots + a_{n_k,k}y_{k-n_k} = b_{0,k}u_k + b_{1,k}u_{k-1} + \ldots + b_{m_k,k}u_{k-m_k}$$

(1)

where $u_k, y_k \in \mathbb{R}$ are the input and output of the filter, and $a_{i,k}, b_{j,k} \in \mathbb{R}$ for $1 \leq i \leq n_k, 0 \leq j \leq m_k$ are time-varying coefficients at each time $k$ with $a_{n_k,k}, b_{m_k,k} \neq 0$. Thus, the LTV filter (1) has time-varying order $o_k = \max\{n_k, m_k\}$.

Let $z^{-1}$ be a back-shift operator, such that $z^{-1}u_k = u_{k-1}$ and $z^{-1}y_k = y_{k-1}$. Equation (1) can be alternatively written as

$$A(k, z^{-1})y_k = B(k, z^{-1})u_k$$

(2)

where

$$A(k, z^{-1}) = 1 + a_{1,k}z^{-1} + \ldots + a_{n_k,k}z^{-n_k}$$

$$B(k, z^{-1}) = b_{0,k} + b_{1,k}z^{-1} + \ldots + b_{m_k,k}z^{-m_k}.$$

Let $||s|| = \sqrt{\sum_{k=0}^{\infty} s_k^2}$ be the norm of a discrete-time sequence $s = \{s_0, s_1, \ldots, s_k, \ldots\}$. We define stability of the LTV filter (1) in the norm-bound input-norm-bound-output sense, i.e., as follows.

Definition 1: The LTV filter (1) is stable if there exists a nonnegative constant $\varepsilon$, such that, under the zero initial conditions (i.e., $u_k = 0$ and $y_k = 0$ for all $k < 0$) and for any input $u$ with $||u|| < \varepsilon$, the output $y$ satisfies $||y|| \leq c||u||$.

Definition 2: The LTV filter (1) is inversely stable if there exists a nonnegative constant $\hat{c}$, such that, under the zero initial conditions and for any output $y$ with $||y|| < \hat{c}$, the input $u$ of (1) satisfies $||u|| \leq c||y||$.

The above definition on stability implies that the output of a stable LTV filter converges to zero as time $k$ goes to infinity for any norm bounded input. In particular, the IIR of the filter converges to zero under the impulse input $\delta(k-r)$ at any time $k = r \geq 0$.

A special class of LTV filters with constant relative degree is defined as follows.

Definition 3: The LTV filter (1) has a constant relative degree $\tau$ with $0 \leq \tau \leq m_k$ if its time-varying coefficients satisfy $b_{r,k} \neq 0$ and $b_{r,k} = 0$ for all $k \geq 0$ and $0 \leq \tau \leq r - 1$.

For all $k \geq 0$, let

$$U_k = \{u_0, u_1, \ldots, u_k\}, \quad Y_k = \{y_0, y_1, \ldots, y_k\}$$

be the input and output sequences of the LTV filter (1). We define $d$-shift invertibility of (1) as follows.

Definition 4: The LTV filter (1) is $d$-shift invertible if under the zero initial conditions, any input sequence $u_k$ can be uniquely determined from the measurement of the output sequence $y_k$ at time $k + d$ for all $k \geq 0$ and a smallest possible integer $d \geq 0$.

This definition implies that $d$-shift invertible LTV filter is not $l$-shift invertible for any $l < d$.

If the LTV filter $G$ in the form (1) is $d$-shift invertible, we define $G_{L}^{-1} : y \mapsto u^L$ ($G_{R}^{-1} : y^R \mapsto u$) as the $d$-shift left (right) inverse filter of $G$, such that

$$G_{L}^{-1}y = z^{-d}$$

(3)

In this brief, we aim to find conditions for $d$-shift invertibility of the LTV filter $G$ in the form (1). For a $d$-shift invertible, we use the obtained conditions to design its $d$-shift left (right) inverse filter $G_{L}^{-1}$ ($G_{R}^{-1}$) written as the ARMA equation

$$u_k^L + \hat{a}_{1,k}u_{k-1}^L + \ldots + \hat{a}_{n_k,k}u_{k-n_k}^L = \hat{b}_{0,k}y_k + \hat{b}_{1,k}y_{k-1} + \ldots + \hat{b}_{m_k,k}y_{k-m_k}$$

(4)

$$u_k^R + \hat{a}_{1,k}u_{k-1}^R + \ldots + \hat{a}_{n_k,k}u_{k-n_k}^R = \hat{b}_{0,k}y_k^R + \hat{b}_{1,k}y_{k-1}^R + \ldots + \hat{b}_{m_k,k}y_{k-m_k}$$

(5)

where $y_k, y_k^R, u_k^L, u_k^R \in \mathbb{R}$ are the input and output of $G_{L}^{-1}$ ($G_{R}^{-1}$), $a_{i,k}, b_{j,k}, \hat{a}_{i,k}, \hat{b}_{j,k} \in \mathbb{R}$ for $1 \leq i \leq \hat{n}_k$ ($1 \leq f \leq \hat{n}_k$), $0 \leq j \leq \hat{m}_k$ ($0 \leq g \leq \hat{m}_k$) are time-varying coefficients of $G_{L}^{-1}$ ($G_{R}^{-1}$) at each time $k$ with $\hat{a}_{n_k,k}, \hat{b}_{m_k,k} \neq 0$ and $\hat{a}_{n_k,k}, \hat{b}_{m_k,k} \neq 0$, and the order of $G_{L}^{-1}$ ($G_{R}^{-1}$) is $\hat{o}_k = \max\{\hat{n}_k, \hat{m}_k\}$ ($\hat{o}_k = \max\{\hat{n}_k, \hat{m}_k\}$).

It follows from (3) that the $d$-shift left (right) inverse filter $G_{L}^{-1}$ ($G_{R}^{-1}$) of $G$ satisfy

$$u_k^L = G_{L}^{-1}y_k = z^{-d}u_k^L \quad \text{for all } k \geq d$$

(6)

$$y_k = G_{R}^{-1}y_k^R = z^{-d}y_k^R \quad \text{for all } k \geq d.$$

III. INVERTIBILITY AND INVERSES OF LTV FILTERS

A. Left and Right Inverses of LTV Filters

In this subsection, we derive a relationship between $d$-shift left and right inverses of (1).

Lemma 3.1: (4) is a $d$-shift left inverse of the LTV filter (1) if and only if (5) is a $d$-shift right inverse, with $\hat{a}_{i,k} = \hat{a}_{i,k}, \hat{b}_{j,k} = \hat{b}_{j,k}$ for $1 \leq i \leq \hat{n}_k$ ($1 \leq f \leq \hat{n}_k$), $0 \leq j \leq \hat{m}_k$ ($0 \leq g \leq \hat{m}_k$), and $k \geq 0$, of (1).

Proof: If (4) is a $d$-shift left inverse of (1), we have $u_k^L = u_k^L$ for all $k \geq d$. Rewrite (4) in $z^{-1}$ as

$$A_L(k, z^{-1})u_k^L = B_L(k, z^{-1})y_k$$

(6)

where

$$A_L(k, z^{-1}) = 1 + \hat{a}_{1,k}z^{-1} + \ldots + \hat{a}_{n_k,k}z^{-n_k}$$

$$B_L(k, z^{-1}) = \hat{b}_{0,k} + \hat{b}_{1,k}z^{-1} + \ldots + \hat{b}_{m_k,k}z^{-m_k}.$$

By (2) and (6), we obtain

$$y_k = A^{-1}(k, z^{-1})B_L(k, z^{-1})u_k$$

$$= A^{-1}(k, z^{-1})B(k, z^{-1})u_k$$

$$= A^{-1}(k, z^{-1})B(k, z^{-1})u_k^L$$

$$= A^{-1}(k, z^{-1})B(k, z^{-1})A_L^{-1}(k + d, z^{-1})B_L(k + d, z^{-1})y_{k+d}.$$
With \( \hat{a}_{i,k} = \hat{a}_{i,k+1} \hat{b}_{j,k} \) and \( \hat{b}_{j,k} = \hat{b}_{j,k+1} \) for \( 1 \leq i \leq n_k = \hat{n}_k, 0 \leq j \leq \hat{m}_k = \hat{m}_k \) and all \( k \geq 0 \), (5) can be rewritten in \( z^{-1} \) as
\[
A_k(k + d, z^{-1}) u_k = B_k(k + d, z^{-1}) y_k^R
\]
or equivalently
\[
u_k = A_k^{-1}(k + d, z^{-1}) B_k(k + d, z^{-1}) y_k^R.
\]
Substituting (8) into (2) and using (7), we obtain
\[
y_k = A_1^{-1}(k, z^{-1}) B_1(k, z^{-1}) A_2^{-1}(k, z^{-1}) B_2(k, z^{-1}) y_k^R
= z^{-d} y_k^R.
\]
Thus, (5), with \( \hat{a}_{i,k} = \hat{a}_{i,k+1} \hat{b}_{j,k} = \hat{b}_{j,k+1} \) for \( 1 \leq i \leq n_k = \hat{n}_k, 0 \leq j \leq \hat{m}_k = \hat{m}_k \) and all \( k \geq 0 \), is a \( d \)-shift right inverse of (1).

Following the same line, it can be proven that if (5) is a \( d \)-shift right inverse of (1), with \( \hat{a}_{i,k} = \hat{a}_{i,k+1} \hat{b}_{j,k} = \hat{b}_{j,k+1} \) for \( 1 \leq i \leq n_k = \hat{n}_k, 0 \leq j \leq \hat{m}_k = \hat{m}_k \) and all \( k \geq 0 \), then (4) is a \( d \)-shift left inverse of (1). \( \square \)

B. Inverses of LTV Filters With Constant Relative Degree

We consider the invertibility of LTV filters in the form (1) with constant relative degree.

**Theorem 3.1:** If the LTV filter (1) has a constant relative degree \( r, 0 \leq r \leq m_k \), then it is \( r \)-shift invertible.

**Proof:** The filter satisfies \( b_{r,k} \neq 0 \) for all \( k \geq 0 \) and \( 0 \leq i \leq r - 1 \) since it has a constant relative degree \( r \). For any \( 0 \leq i \leq r - 1 \), the output sequence \( Y_{k+i} = \{y_0,y_1,\ldots,y_{y+1}\} \) is only dependent on the past input sequence \( U_{k+r+1} = \{u_0,u_1,\ldots,u_{y+1}\} \). The input sequence \( U_{k+r} = \{u_0,u_1,\ldots,u_{y}\} \) cannot be determined from the output sequence \( Y_{k+i} = \{y_0,y_1,\ldots,y_{y+1}\} \) for any \( 0 \leq i \leq r - 1 \).

Since \( b_{r,k} \neq 0 \) for all \( k \geq 0 \), we construct an LTI IIR filter with input \( y_k \) and output \( \hat{u}_k \), written as
\[
\hat{u}_k + \frac{b_{r,k+1}}{b_{r,k}} \hat{u}_{k-1} + \ldots + \frac{b_{m,k}}{b_{r,k}} \hat{u}_{k-r+m} = \frac{1}{b_{r,k}} y_k + \frac{a_{1,k}}{b_{r,k}} y_{k-1} + \ldots + \frac{a_{n,k}}{b_{r,k}} y_{k-n}.
\]
Let the initial condition of (9) be \( \hat{u}_0 = 0 \) for all \( i \leq r \). Under the initial condition \( u_0 = 0 \) for all \( i \) for the LTI filter, we have \( y_i = 0 \) for \( 0 \leq i \leq r - 1 \) and \( y_r = b_{r,k} u_0 \). Applying this to the input of (9) at \( k = r \) yields
\[
\hat{u}_r = \frac{1}{b_{r,k}} y_r = \frac{1}{b_{r,k}} b_{r,k} u_0 = u_0.
\]
We now proceed the proof by induction. Assume that \( \hat{u}_{i+r} = u_i \) is satisfied for all \( 1 \leq i \leq k - 1 \). Using this assumption and (1) and (9), it is straightforward to obtain
\[
\hat{u}_{k+r} = \frac{b_{r,k+1} y_{k+r}}{b_{r,k+1}} \hat{u}_{k+r-1} + \ldots + \frac{b_{m,k}}{b_{r,k}} \hat{u}_{k+2-r+m} + \frac{1}{b_{r,k}} y_{k+r} + \frac{a_{1,k}}{b_{r,k}} y_{k+r-1} + \ldots + \frac{a_{n,k}}{b_{r,k}} y_{k+r-n}.
\]

Thus, the input sequence \( U_k \) is uniquely determined from the output sequence \( Y_{k+r} \), and this is true for all \( k \geq 0 \) by induction. It follows that (1) with constant relative degree \( r \) is \( r \)-shift invertible. \( \square \)

The proof of Theorem 3.1 is constructive in that it explicitly gives an \( r \)-shift left inverse filter of (1) in the form of ARMA equation (9). That is, replacing \( \hat{u}_k \) with \( \hat{u}_k^L \) in (9), we obtain the following \( r \)-shift left inverse of (1) for all \( k \geq d \):
\[
\hat{u}_k^L + \frac{b_{r,k+1}}{b_{r,k}} \hat{u}_{k-1} + \ldots + \frac{b_{m,k}}{b_{r,k}} \hat{u}_{k-r+m} = \frac{1}{b_{r,k}} y_k + \frac{a_{1,k}}{b_{r,k}} y_{k-1} + \ldots + \frac{a_{n,k}}{b_{r,k}} y_{k-n}.
\]

Applying Lemma 3.1, the \( r \)-shift right inverse filter of (1) for all \( k \geq 0 \) is immediately obtained as
\[
u_k = \frac{1}{b_{r,k}} y_k + \frac{a_{1,k}}{b_{r,k}} y_{k-1} + \ldots + \frac{a_{n,k}}{b_{r,k}} y_{k-n}.
\]
\(R^{(d_k+1)\times(d_k+1)}\) is linearly independent of the other column vectors to its right, then for each \(k \geq 0\), there exists a nonzero vector \(\beta_{3,k}^{\#} \in R^{1\times(d_k+1)}\), such that

\[
\beta_{3,k}^{\#} \beta_{1,k} d_k = [1, 0, \ldots, 0] \in R^{1\times(d_k+1)}.
\]

(16)

It is observed that one solution of \(\beta_{3,k}^{\#}\) can be the first row vector from the top of the pseudo inverse \([2]\) of the matrix obtained by deleting all the linearly dependent column vectors of \(\beta_{1,k} d_k\).

Let \(d = \max_{k \geq 0} \{d_k\}\). Multiplying (15) with \(d\) from the left by \([\beta_{3,k}^{\#}, 0, \ldots, 0] \in R^{1\times(d_k+1)}\) and using (16), we obtain

\[
u_k + \tilde{a}_{1,k} d_k y_{k-1} + \ldots + \tilde{a}_{m_k,k} d_k y_{k-m_k} = \tilde{b}_0 d_k y_{k+d} + \tilde{b}_1 d_k y_{k+d-1} + \ldots + \tilde{b}_d d_k y_{k+d-n_k}.
\]

(17)

Hence, under the zero initial conditions, any input sequence \(U_k = \{u_0, u_1, \ldots, u_k\}\) can be recursively computed from the measurement of the output sequence \(Y_{k+d} = \{y_0, y_1, \ldots, y_{k+d}\}\) for all \(k \geq 0\).

We now prove that any nonnegative integer \(l < d\) is unable to have any input sequence \(U_k\) uniquely determined from the output sequence \(Y_{k+d}\) for all \(k \geq 0\). By the definition of \(d\), there exists some \(k \geq 0\), such that the first column vector of \(\beta_{1,k,d-1}\) is linearly dependent on the other column vectors to its right. Thus for this \(k\), there exists a nonzero vector \(u = [\tilde{u}_k, \tilde{u}_{k+1}, \ldots, \tilde{u}_{k+d-1}]^T \in R^d\) with \(\tilde{u}_k \neq 0\), such that \(\beta_{1,k,d-1} u = 0\). By (15) and the initial condition \(y_i = 0\) for all \(i < 0\), the output of (1) is \(y_k = 0\) for all \(0 \leq i \leq k + d - 1\) under the nonzero input sequence \([0, \ldots, 0, \tilde{u}_k, \tilde{u}_{k+1}, \ldots, \tilde{u}_{k+d-1}]\). Since the nonzero value \(\tilde{u}_k\) cannot be distinguished from zero using the output sequence \(Y_{k+d-1} = \{y_0, y_1, \ldots, y_{k+d-1}\}\), up to time \(k + d - 1\), the LTV filter (1) is not \((d - 1)\)-shift invertible. By the same argument, it is shown that (1) is not \(l\)-shift invertible for any \(0 \leq l \leq d - 2\).

Applying \(u_k^L = u_{k-d}\) in (17), we obtain the \(d\)-shift left inverse filter of (1) in the form (4) for all \(k \geq d\).

Based on Lemma 3.1 and using (4), the \(d\)-shift right inverse filter of (1) is written as (5), where \(\tilde{a}_{i,k} = \tilde{a}_{i,k+d}\) and \(\tilde{b}_{i,k} = \tilde{b}_{i,k+d}\) for \(1 \leq i \leq \tilde{n}_k = \tilde{n}_k, 0 \leq j \leq \tilde{m}_k = \tilde{m}_k\) and all \(k \geq 0\).

Remarks:

1) If (1) represents a \(d\)-shift invertible periodically time-varying filter with period \(P\), such that \(a_{i,k} = a_{i,k+P}\) and \(b_{i,k} = b_{i,k+P}\), for \(1 \leq i \leq n_k, 0 \leq j \leq m_k\), are satisfied for any integer \(h > 0\), we only need to compute \(k = 0, 1, \ldots, P - 1\) to obtain the \(d\)-shift left/right inverse filter (4)/(5) whose coefficients are periodically time-varying with period \(P\).

2) It is noted from the proof of Theorem 3.2 that there may exist more than one solution to the row vector \(\beta_{3,k}^{\#}\), such that (16) is satisfied. Thus the solution for the \(d\)-shift left/right inverse filter of (1) may be nonunique.

D. Stability of Inverse Filters

We now give a necessary and sufficient condition for stability of inverse LTV filters.

Theorem 3.3: Suppose that the LTV filter (1) is \(d\)-shift invertible. Then its \(d\)-shift left and right inverses are stable if and only if (1) is inversely stable.

Proof:

Sufficiency: If (1) is inversely stable, then there exists a nonnegative constant \(\tilde{c}\), such that \(\|u\| \leq \tilde{c}\|y\|\) for any \(y\) with bounded norm. For any \(d\)-shift right inverse filter \(G_d^{-1} : y^R \mapsto u\) of (1) satisfying \(y^R_{k+d} = y_k\) and \(\|y^R\| = \|y\|\), we have \(\|u\| \leq \tilde{c}\|y^R\|\) for any norm bounded \(y^R\). Hence, the right inverse of (1) is stable.

Necessity: If \(G_d^{-1} : y^R \mapsto u\) is a stable \(d\)-shift right inverse of (1), such that \(\|u\| \leq \tilde{c}\|y^R\|\) for a nonnegative constant \(\tilde{c}\) and any input \(y^R\) with bounded norm, then the input \(u_k\) and output \(y_k = y^R_k\) of
(1) satisfy \(|\|v\| \leq c|\|y\|\) for any norm bounded \(y\). Thus, (1) is inversely stable.

Hence, the \(d\)-shift right inverse of (1) is stable if and only if (1) is inversely stable. It follows from Lemma 3.1 that the \(d\)-shift left inverse of (1) is stable if and only if (1) is inversely stable.

It is noted that there has been no known finite step procedure for testing stability and inverse stability of general LTV filters in system stability theory. In a special case that (1) and its inverses are periodic, their stability can be easily checked using the well known lifting technique [8], [11].

IV. A PROCEDURE AND AN EXAMPLE

For an LTV filter in the form (1) with constant relative degree \(r\), it is straightforward to obtain its \(r\)-shift left/right inverse filter of the form (10)/(11). As to LTV filters with nonconstant relative degree, based on Theorem 3.2 we present the following procedure for computing the \(d\)-shift left/right inverse filter in the form (4)/(5).

Step 1) For each \(k \geq 0\), form matrix \(\beta_k\) in (13) and find the least \(d_k \leq L\), such that the first column vector of \(\beta_{1,k,d_k}\) is linearly independent of the other column vectors of \(\beta_{1,k,d_k}\).

If there exists no \(d_k \leq L\) for some \(k \geq 0\), then the LTV filter (1) is not \(d\)-shift invertible for any \(d \leq L\). Otherwise, set \(d = \max_{k \geq 0}\{d_k\}\) and go to Step 2.

Step 2) Obtain the row vector \(\beta_k^r\) for each \(k \geq 0\), such that (16) is satisfied.

Step 3) For each \(k \geq 0\), form matrices \(\tilde{\beta}_{1,k,d}\) and \(A_{k,d}\) in the form (12) and (14), respectively. For each \(k \geq 0\), multiply (15), with \(f = d\), from the left by \(\tilde{\beta}_k^r 0, \ldots, 0 \in \mathbb{R}^{1 \times (d+1)}\) to obtain the coefficients in (17).

Step 4) For all \(k \geq d\), coefficients \(a_{i,k}\) for \(1 \leq i \leq \hat{n}_k\) and \(\hat{b}_{j,k}\) for \(0 \leq j \leq \hat{m}_k\) determine the \(d\)-shift left inverse (4). For all \(k \geq 0\), let \(\hat{a}_{i,k} = \hat{a}_{i,k} + d\) for \(1 \leq i \leq \hat{n}_k\) and \(\hat{b}_{j,k} = \hat{b}_{j,k} + d\) for \(0 \leq j \leq \hat{m}_k\), which determine the \(d\)-shift right inverse (5).

Consider an example of an LTV filter in the form (1) with time-varying coefficients \(a_{i,k}, b_{i,k}\) for \(1 \leq i \leq n_k\), \(0 \leq j \leq m_k\), given in the equation at the bottom of the page. By Step 1, \(d_0 = 0, d_1 = 3, d_2 = 0, d_3 = 0\), and \(d_4 = 1\) for \(k = 4, 5, \ldots\), are, respectively, the least integer, such that the first column vector of \(\beta_{1,k,d_k}\) is linearly independent of the other column vectors to its right. Thus, \(d = \max_{k \geq 0}\{d_k\} = 3\).

By Step 2, we have

\[
\begin{align*}
\beta_k^r &= 1 \quad \text{for } k = 0 \\
\beta_k^r &= \begin{bmatrix} 0 & -0.2 & -0.2 & 3 & 4/3 \end{bmatrix} \quad \text{for } k = 1 \\
\beta_k^r &= -1 \quad \text{for } k = 2 \\
\beta_k^r &= 1/2 \quad \text{for } k = 3 \\
\beta_k^r &= [0 \ 1] \quad \text{for } k = 4, 5, \ldots
\end{align*}
\]

By Step 3, for each \(k \geq 0\), form matrices \(\tilde{\beta}_{1,k,d}\) and \(A_{k,d}\) as follows:

\[
\begin{align*}
\tilde{\beta}_{1,k,d} &= \begin{bmatrix} 2 & -1 \\
0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \end{bmatrix} \quad \text{for } k = 0 \\
A_{k,d} &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{for } k = 1 \\
\tilde{\beta}_{1,k,d} &= \begin{bmatrix} -2 & -3 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
3 & 2 & 1 & 0 & 0 \end{bmatrix} \quad \text{for } k = 2 \\
A_{k,d} &= \begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -3 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for } k = 3 \\
\tilde{\beta}_{1,k,d} &= \begin{bmatrix} 1 & 1 \\
0 & 0.1 & -0.1 \\
0 & 0.1 & 0 \\
0 & 0 & 0 \end{bmatrix} \quad \text{for } k = 4 \\
A_{k,d} &= \begin{bmatrix} 0.5 & -1 & 1 & 0 & 0 \\
0 & 0.5 & -1 & 1 & 0 \\
0 & 0 & 0 & 0.5 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for } k = 5, \ldots
\end{align*}
\]

By Step 4, the \(3\)-shift left inverse filter in the form (4) of the given LTV filter is determined by the following coefficients \(\hat{a}_{i,k}, \hat{b}_{i,k}\) for \(1 \leq i \leq \hat{n}_k\), \(0 \leq j \leq \hat{m}_k\), and \(k \geq 3\):

\[
\begin{align*}
\hat{a}_{1,k} &= -1, \quad \hat{a}_{2,k} = 2, \quad \hat{b}_{0,k} = 0, \quad \hat{b}_{1,k} = 0 \\
\hat{b}_{2,k} &= 0, \quad \hat{b}_{3,k} = 1, \quad \hat{b}_{4,k} = 1, \quad \text{for } k = 3 \\
\hat{a}_{1,k} &= -0.8/3, \quad \hat{b}_{0,k} = 4/3, \quad \hat{b}_{1,k} = -1.4
\end{align*}
\]
\[ \dot{b}_{2,k} = 0.4, \quad \dot{b}_{3,k} = 0.4, \quad \dot{b}_{4,k} = -0.2, \quad \text{for } k = 4 \]
\[ \dot{a}_{1,k} = -2, \quad \dot{a}_{2,k} = -1, \quad \dot{b}_{0,k} = 0, \quad \dot{b}_{1,k} = 0 \]
\[ \dot{b}_{2,k} = 0, \quad \dot{b}_{3,k} = -1, \quad \dot{b}_{4,k} = 1, \quad \dot{b}_{5,k} = -1 \]
\[ \dot{b}_{6,k} = -2, \quad \text{for } k = 5 \]
\[ \dot{a}_{1,k} = 1/2, \quad \dot{a}_{2,k} = -1/2, \quad \dot{a}_{3,k} = 1/2, \quad \dot{b}_{0,k} = 0 \]
\[ \dot{b}_{1,k} = 0, \quad \dot{b}_{2,k} = 0, \quad \dot{b}_{3,k} = 1/2, \quad \dot{b}_{4,k} = 1/2, \quad \dot{b}_{5,k} = -3/2, \quad \text{for } k = 6 \]
\[ \dot{a}_{1,k} = -0.1, \quad \dot{a}_{2,k} = 0.1, \quad \dot{b}_{0,k} = 0, \quad \dot{b}_{1,k} = 0 \]
\[ \dot{b}_{2,k} = 1, \quad \dot{b}_{3,k} = -1, \quad \dot{b}_{4,k} = 0.5, \quad \text{for } k = 7,8, \ldots \]

Likewise, for \( k \geq 0 \), the 3-shift right inverse filter of the form (5) is determined by \( \dot{a}_{i,k} = \dot{a}_{i,k+j} \) for \( 1 \leq i \leq \dot{n}_{k} \) and \( \dot{b}_{i,k} = \dot{b}_{i,k+j} \) for \( 0 \leq j \leq \dot{m}_{k} \).

Since the LTV filter and its inverses become time invariant after a certain time, it can be easily verified that both of them are stable filters.

V. CONCLUSION

In this brief, a necessary and sufficient condition for the \( d \)-shift invertibility of LTV filters with time-varying dynamic order and relative degree has been developed. A procedure resulting from this condition is presented for computing the \( d \)-shift left and right inverses of a given LTV filter. It is shown that the order of the inverses is also time-varying and that there may exist nonunique solutions for the inverses. Moreover, stability of the given filter and its inverses is discussed and a necessary and sufficient condition for stability of inverse filters is given.

The general results of this paper are applicable to periodic filters with time-varying order and relative degree, which is an extension of the existing results on inverting fixed order periodic filters. Our computation of inverses is straightforward using directly the filter parameters, thus avoiding transformations between time-varying and time invariant models as used in [6].

REFERENCES


A DCT-Based D-FANN for Nonlinear Adaptive Time Series Prediction

Torbjørn Eltoft and Rui J. P. deFigueiredo

Abstract—A nonlinear adaptive time-series predictor has been developed using a new type of artificial neural network called dynamical-functional artificial neural network (D-FANN) for its underlying model structure. D-FANNs are two-layer neural systems in which the synaptic weights of the first layer are “functions” rather than numbers, and where the action of a synapse on a signal passing through it takes place in the form of a scalar product in \( L^2 \) between the functional weight and the signal. The second layer of these networks is a combiner, which optimally linearly combines the weighted outputs of the zero-memory nonlinear elements comprising the neurons. In this brief, we introduce a neural network which we call a DCT-based D-FANN. This is a D-FANN where the functional weights of the first layer is a filter bank built up of discrete cosine transform basis functions. We show that this system can successfully be used to model and predict an important class of highly dynamic and nonstationary signals, namely speech signals.

Index Terms—Adaptive signal processing, discrete cosine transform, neural networks, nonlinear adaptive prediction, time series prediction.

I. INTRODUCTION

In recent years, several structures have been developed for identification of nonlinear systems and modeling and prediction of nonlinear time series. Among these, the conventional [1] and Generalized Fock Space\(^1\) (GFS) [2]–[4] models of the Volterra series, the multi-layer perceptron (MP) [5], and the radial basis functions (RBF) networks [6] are the most common. Previous work in applying neural networks for time series prediction also include [7]–[10].

According to [11], the formulation of the general category of functional artificial neural networks (FANN) can be derived from the best approximation of the input-output map of a generic nonlinear system in a GFS [3], subject to input-output training data constraints. The use of a GFS framework for modeling artificial neural networks was introduced in [12], and further developments and applications are described in some of the references in [11]. D-FANNs [11] are neural networks in which the synaptic weights are “functions,” and where the effect of a synapse is to calculate the integral of the point-wise product of the functional weight and the incoming signal. The identification of the functional weights of the first layer can be accomplished by simply modeling the weights as impulse responses \( h_i(t), i = 1, 2, \ldots, L \) of a set of \( L \) linear time-invariant differential dynamical systems (hence, the name dynamical-functional artificial neural networks).

In this brief, we present a D-FANN model where the linear filter bank consists of the orthogonal basis set of an \( L \)-point DCT, and we test the model by applying it in single-step prediction of a speech signal.

Manuscript received March 1999; revised June 2000. This work was supported by the Research Council of Norway, by NIH under Grant AG05142, by the National Science Foundation under Grant CCR97-04262, and by funding from Neural Computing Systems, Irvine, CA. This paper was recommended by Associate Editor J. Bioucas-Dias.

T. Eltoft is with the Department of Physics, Faculty of Science, University of Tromsø, N-9037 Tromsø, Norway (e-mail: torbjorn.eltoft@phys.uio.no).

R. J. P. deFigueiredo is with the Department of Electrical and Computer Engineering and the Department of Mathematics, University of California, Irvine, CA 92697-2625 USA (e-mail: rui@uci.edu).

Publisher Item Identifier S 1057-7130(00)09338-1.

A GFS is defined as a Hilbert space of arbitrarily weighted functional power series in one or more scalar variables.