Universal dynamic structure factor of a strongly correlated Fermi gas

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Universality of strongly interacting fermions is a topic of great interest in diverse fields. Here we investigate theoretically the universal dynamic density response of resonantly interacting ultracold Fermi atoms in the limit of either high temperature or large frequency. At high temperature, we use quantum virial expansion to derive universal, nonperturbative expansion functions of the dynamic structure factor; at large momentum, we conjecture that the second-order expansion function gives the Wilson coefficient used in the operator product expansion method. The dynamic structure factor is therefore determined by its second-order expansion function with an overall normalization factor given by Tan’s [S. Tan, Ann. Phys. (NY) 323, 2952 (2008); 323, 2971 (2008); 323, 2987 (2008)] contact parameter. We show that the spin-parallel and -antiparallel dynamic structure factors have, respectively, a tail of the form $\sim \omega^{-5/2}$ for $\omega \to \infty$, decaying slower than the total dynamic structure factor found previously ($\sim \omega^{-7/2}$). Our predictions for the dynamic structure factor at high temperature or large frequency are testable using Bragg spectroscopy for ultracold atomic Fermi gases.

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I. INTRODUCTION

Strongly interacting Fermi systems appear in very different areas of physics, including neutron stars, quark-gluon plasmas, high-temperature superconductors, and ultracold atoms. It was recently proposed in the field of ultracold atoms that strongly interacting fermions can have some generic properties that can be classified as universal [1,2]. Universal means that the behavior of the system depends only on scaling factors equal to the average particle separation and/or thermal wavelength, but not on the microscopic details of the interaction potential. To exhibit universal behaviors, the Fermi system must satisfy two requirements: (i) The range of the interaction potential must be much shorter than the mean spacing between particles and (ii) the scattering length that characterizes the strength of the interaction potential must be much larger than the mean spacing between particles.

Recent manipulation of ultracold Fermi gases of $^6$Li and $^{40}$K atoms near a broad collisional (Feshbach) resonance provides an ideal avenue to understand this fermionic universality [3–6]. By taking advantage of precisely controlling the interatomic interactions, the study of universality in ultracold Fermi gases may help in the understanding of other strongly interacting Fermi systems. To date, the universal thermodynamics of strongly interacting fermions has been clearly demonstrated [7] by measuring the static equation of state.

The purpose of this paper is to show that universality appears in dynamical properties as well. We derive exact results for the universal dynamic structure factor (DSF) at high temperatures or at large momentum and frequency. The DSF is the Fourier transform in the separation vector and the time interval between two density operators. It encodes information about the degrees of freedom that can be excited by density fluctuations.

Exact results are valuable for strongly interacting fermions due to their nonperturbative, strongly correlated nature. There is no small interaction parameter to control the accuracy of theories for strongly interacting fermions [8]. In specific cases, $ab\ initio$ calculations possibly use Monte Carlo methods [9–12]. However, in general, this approach suffers from the fermion sign problem [13].

Tan derived a set of exact, universal relations for two-component (spin-1/2) Fermi gases with a large $s$-wave scattering length $a$ [14]. These universal relations all involve a many-body parameter called the contact $I$, which measures the density of pairs within short distances. Tan’s relations can be understood using the short-distance and/or short-time operator product expansion (OPE) method [15–18], which separates in a natural way few-body from many-body physics. At high temperatures, quantum virial expansion provides another rigorous means to bridge few-body and many-body physics [19–24]. It was shown that static thermodynamic properties of a strongly correlated Fermi gas can be expanded nonperturbatively in fugacity using some universal, temperature-independent virial coefficients [6,21,24], which are exactly calculable from few-fermion solutions. Both OPE and virial expansion give useful insight into the challenging many-body problem in the limits of short distance and high temperature, respectively. However, their connection is yet to be understood.

In this paper we investigate theoretically the universal dynamic properties of a strongly correlated Fermi gas in the limit of either high temperature or large momentum or frequency. In the former limit we show that the DSF can be virially expanded in fugacity, using some universal, temperature-implicit virial expansion functions. We derive these universal virial expansion functions for spin-parallel and -antiparallel DSFs of a homogeneous Fermi gas in the resonance (unitarity) limit, where the scattering length diverges ($a \to \pm \infty$). In the latter limit of large momentum or frequency, we show that there is a natural relation between the virial expansion and the OPE. The Wilson coefficient in the OPE is given by the second-order virial expansion function; see, for example, Eq. (13). Therefore, the large-momentum DSF is universally determined by the second-order virial expansion, together with a many-body prefactor: the contact.
Our results can be easily examined using Bragg spectroscopy for ultracold Fermi gases of \(^6\)Li or \(^{40}\)K atoms [25,26].

We note that in the field of ultracold atoms the Wilson coefficients were already used by Braaten et al. as a tool to derive sets of universal relations beyond the original work by Tan [17]. In this respect, our interpretation of Wilson coefficients might be useful for a better understanding of strongly interacting Fermi systems.

The paper is structured as follows. In the following section we recall briefly the virial expansion and the OPE for the DSF. In Sec. III we present the second-order virial expansion function of the DSF of a homogeneous Fermi gas at unitarity. In Sec. IV we discuss the Wilson coefficient of the DSF and build the relation between the Wilson coefficient and the second-order virial expansion function. We then show the universal behavior of the DSF at large momentum and frequency and discuss how to observe it experimentally. Finally, in Sec. V we give a summary and some further remarks.

II. VIRTUAL EXPANSION VS OPERATOR PRODUCTION EXPANSION

The DSF \(S(q,\omega)\) is the Fourier transform of the density-density correlation functions at two different space-time points. For an atomic Fermi gas with an equal number of atoms \(N/2\) in two hyperfine states (referred to as spin-up \(\sigma = \uparrow\) and spin-down \(\sigma = \downarrow\) states), the total DSF is given by \(S(q,\omega)\equiv 2[S_{\uparrow\uparrow}(q,\omega) + S_{\uparrow\downarrow}(q,\omega)]\), where \(S_{\uparrow\uparrow}(q,\omega) = S_{\downarrow\downarrow}(q,\omega)\) and \(S_{\uparrow\downarrow}(q,\omega) = S_{\downarrow\uparrow}(q,\omega)\). According to quantum field theory, the DSF can be conveniently calculated from the dynamic susceptibility \([27]\) \(\chi_{\sigma\sigma}^{\text{DSF}}(\mathbf{r},\mathbf{r}';\tau) = -\langle T_\tau \rho_\sigma(\mathbf{r},\tau)\rho_\sigma(\mathbf{r}',0)\rangle\), where \(\rho_\sigma(\mathbf{r},\tau)\) is the density (fluctuation) operator in spin channel \(\sigma\) and \(\tau\) is an imaginary time in the interval \(0 < \tau \leq \beta = 1/k_BT\). The DSF is then obtained \([27]\) from the Fourier components \(\chi_{\sigma\sigma}^{\text{DSF}}(\mathbf{r},\mathbf{r}';i\omega_n)\) at discrete Matsubara imaginary frequencies \(i\omega_n = i2n\pi k_BT\) \((n = 0,\pm1,\ldots)\) via analytic continuation and the fluctuation-dissipation theorem:

\[
S_{\sigma\sigma}(\mathbf{r},\mathbf{r}';\tau) = -\frac{\operatorname{Im}\chi_{\sigma\sigma}(\mathbf{r},\mathbf{r}';i\omega_n \rightarrow \omega + i0^+)}{\pi(1 - e^{-\beta\omega})}, \quad \tau = \frac{\pm 1}{2}\pi
\]

By performing a Fourier transform with respect to the relative spatial coordinate \(\mathbf{r} - \mathbf{r}'\), one obtains \(S_{\sigma\sigma}(q,\omega)\).

Virial expansion is very useful to determine the high-temperature behavior of the DSF. It expresses the DSF as an expansion in fugacity with some expansion functions since the fugacity \(z \equiv \exp(\mu/k_BT) \ll 1\) is a controllable small parameter at high temperatures. For the interaction DSF \(S^{\text{int}}_{\sigma\sigma}(q,\omega)\equiv S_{\sigma\sigma}(q,\omega) - S_{\sigma\sigma}^{(1)}(q,\omega)\), which is the difference between the interacting DSF \(S_{\sigma\sigma}(q,\omega)\) and a noninteracting DSF \(S_{\sigma\sigma}^{(1)}(q,\omega)\) with the same fugacity, the virial expansion can be written as \([22]\)

\[
S^{\text{int}}_{\sigma\sigma}(q,\omega) = z^2\Delta S_{\sigma\sigma;2} + z^3\Delta S_{\sigma\sigma;3} + \cdots,
\]

where \(\Delta S_{\sigma\sigma;\ell}(q,\omega)\) \((n = 2,3,\ldots)\) is the \(n\)th expansion function involving few-body physics. The expansion is nonperturbative as the few-body problem could be solved exactly, no matter how strong the interaction. Virial expansion is anticipated to work for temperatures down to the superfluid transition, although a nontrivial resummation of expansion terms may be required if \(z \gg 1\). In Ref. [22], the lowest second-order expansion function \(\Delta S^{\text{int}}_{\sigma\sigma;2}\) for a trapped Fermi gas was calculated using two-fermion solutions in traps. However, the universal aspect of expansion functions was not realized. As a result, for different temperatures or momenta the expansion functions had to be recalculated.

The OPE gives another powerful tool to help in the understanding of the strongly correlated many-body system in the short-distance and short-time limit. It is a hypothesis that was independently conjectured by Wilson, Kadanoff, and Polyakov [15]. The OPE expands the product of local operators at different space-time points in local operators with coefficients that are functions of the separation in space and time. For the density correlation, it takes the form

\[
\hat{\rho}_\sigma(r,\tau)\hat{\rho}_\sigma(r',\tau') = \sum C_S^{\text{C}}(r-r' - \tau + \tau')O_C,
\]

where the sum is over infinitely many local operators \(O_C\). The Wilson coefficient of the DSF has been determined by Son and Thompson [18]. Consequently, using the OPE hypothesis (4), the behavior of the DSF in the limits of large momentum and frequency is determined. In Sec. IV we will derive it briefly using Feynman diagrams.

III. UNIVERSAL VIRIAL EXPANSION FUNCTION

Here we consider the expansion functions of a homogeneous Fermi gas in the unitarity limit and emphasize their universal aspect, which is not yet known. As the scattering length diverges, all microscopic scales of the interaction are absent [1]. For this few-body problem, the only energy scale is \(k_BT\) and the length scale is the thermal wavelength \(\lambda \equiv h/(2\pi mk_BT)^{1/2}\). Dimensional analysis leads to

\[
\Delta S_{\sigma\sigma;\ell}(q,\omega) = \frac{V}{k_BT\lambda^3}\Delta S_{\sigma\sigma;\ell}(\tilde{q},\tilde{\omega}),
\]

where \(V\) is the volume, \(\tilde{q} = (h\bar{q}^2/2mk_BT)^{1/2}\), \(\tilde{\omega} = h\omega_0/k_BT\), and \(\Delta S_{\sigma\sigma;\ell}(\cdot)\) is a dimensionless expansion function. The temperature \(T\) is now implicit in the variables \(\tilde{q}\) and \(\tilde{\omega}\). This universal form implies a simple relation between the trapped and homogeneous expansion function. In a shallow harmonic trap \(V_{\text{trap}}(r) = m(a_{\text{h}}^2 x^2 + a_{\text{h}}^2 y^2 + a_{\text{h}}^2 z^2)/2\), where \(\omega_0 \equiv (\alpha_{\text{h}} v_{\text{tr}} a_{\text{h}}^2)^{1/3} \rightarrow 0\), the system may be viewed as a collection of many cells with a local chemical potential \(\mu(r) = \mu - V_{\text{trap}}(r)\) and fugacity \(z(r) = \exp[-V_{\text{trap}}(r)/k_BT]\), so that the trapped DSF is given by \(\Delta S_{\sigma\sigma;\ell}^{\text{trap}}(q,\omega) = \int d^3r [\Delta S_{\sigma\sigma;\ell}(q,\omega,r)/V]\).
Owing to the universal  $\tilde{q}$ and $\tilde{\omega}$ dependences in the expansion functions, the spatial integration can be easily performed, giving rise to
\[
\Delta \tilde{S}_{\sigma\sigma'}(\tilde{q}, \tilde{\omega}) = n^{3/2} \frac{(\hbar\omega_0)^3}{(k_B T)^2} \Delta S_{\sigma\sigma'}^{(\text{traps})}(q, \omega).
\] (6)

The (nonuniversal) correction to the above local-density approximation is of $O((\hbar\omega_0)^2/(k_B T)^2)$. Equation (6) is vitally important because the calculation of expansion functions in harmonic traps is much easier than in free space.

Figure 1 reports the homogeneous expansion function $\Delta \tilde{S}_{2}$ at three different momenta, using $\Delta S_{\sigma\sigma'}^{(\text{traps})}$ in Ref. [22] as the input. One observes a quasistatic peak at $\tilde{\omega} = \tilde{q}^2/2$ or $\omega = h\tilde{q}^2/4m$ as a result of the formation of fermionic pairs. We note that the third expansion function $\Delta \tilde{S}_{\sigma\sigma'}^{(1)}$ or $\Delta S_{\sigma\sigma'}^{(2)}$ can also be calculated straightforwardly using exact three-fermion solutions [28,29].

We may derive sum rules that constrain the universal expansion functions, using the well-known $f$-sum rules satisfied by the DSF: $\int_{-\infty}^{+\infty} \omega S_{\sigma\sigma'}(q, \omega) d\omega = Nq^2/4m$ [30] and $\int_{-\infty}^{+\infty} \omega S_{\sigma\sigma'}(q, \omega) d\omega = 0$ [31]. The latter immediately leads to
\[
\int_{-\infty}^{+\infty} \tilde{\omega} \Delta \tilde{S}_{\sigma\sigma'}(\tilde{q}, \tilde{\omega}) d\tilde{\omega} = 0.
\] (7)

In contrast, virial expansion of the total number of fermions $N$ implies that
\[
\int_{-\infty}^{+\infty} \tilde{\omega} \Delta \tilde{S}_{\sigma\sigma'}(\tilde{q}, \tilde{\omega}) d\tilde{\omega} = n\tilde{q}^2 \Delta b_n,
\] (8)

where $\Delta b_n$ is the $n$th virial coefficient and in the unitarity limit $\Delta b_1 = 1/\sqrt{2}$ and $\Delta b_2 \simeq -0.3551$ [6,21].

At large momentum, the spin-antiparallel static structure factor satisfies the Tan relation [26,32]: $\int S_{\sigma\sigma'}(q, \omega) d\omega \simeq I/8hq$. This indicates a virial expansion of the contact $I = 16\pi^2 V[z^2c_2 + z^3c_3 + \ldots]/\lambda^4$, where the contact coefficients $c_n$ are given by
\[
\Delta \tilde{S}_{1,2}(\tilde{q}) \equiv \int_{-\infty}^{+\infty} \Delta \tilde{S}_{1,2}(\tilde{q}, \tilde{\omega}) d\tilde{\omega} = \frac{\pi^{3/2} c_n}{\tilde{q}}.
\] (9)

The expansion of the contact was alternatively obtained using the adiabatic sweep relation [24]. In the unitarity limit, it was shown that $c_2 = 1/\pi$ and $c_3 \simeq -0.1399$ [24]. In the same limit of large momentum, the spin-parallel static structure factor is nearly unity, so $\int S_{\sigma\sigma'}(q, \omega) d\omega \simeq N/2h$ [22,26]. This leads to
\[
\Delta \tilde{S}_{1,2}(\tilde{q}) \equiv \int_{-\infty}^{+\infty} \Delta \tilde{S}_{1,2}(\tilde{q}, \tilde{\omega}) d\tilde{\omega} = n\Delta b_n.
\] (10)

For the second expansion function $\Delta \tilde{S}_{\sigma\sigma'}^{(2)}$, we have checked numerically that all the above-mentioned sum rules are strictly satisfied.

IV. WILSON COEFFICIENT

We now turn to the large-momentum or -frequency limits, where the OPE (4) is assumed to be applicable. It is clear from the equation that the Wilson coefficient determines entirely the DSF at large $(q, \omega)$ as long as the many-body contact is known.

A. Wilson coefficient from diagrammatic theory

The Wilson coefficient $W_{\sigma\sigma'}$ can be calculated using Feynman diagrams [18] for dynamic susceptibility $\chi_{\sigma\sigma}(r, r) = -(T, \rho_{\sigma\sigma}(r, r) \rho_{\sigma\sigma}(0,0))$ as $S_{\sigma\sigma}(q, \omega) = -\text{Im} \chi_{\sigma\sigma}(q, \omega)/|\pi(1 - e^{-\hbar\omega/k_B T})|$. In the limit of $(q, \omega) \rightarrow \infty$, the diagrams contributing to $\chi_{\sigma\sigma}(q, \omega)$ are sketched in Fig. 2 [18]. Diagrammatically, the contact may be identified as the vertex function at short distance and time [24,33]: $\Gamma = -m^2 \Gamma(0, r) \equiv M_{\sigma\sigma}(q, \omega)$, where $\delta$ and $\omega$ are the shadow of the vertex function $\Gamma(0, r) \equiv M_{\sigma\sigma}(q, \omega)$ represents the contact $I$. These diagrams are well known in the condensed-matter community. In the context of calculating the change in conductivity due to conducting fluctuations, the middle and right-hand diagrams are called the Maki-Thompson [34] and Aslamazov-Larkin contributions [35], respectively, while the left-hand diagram gives the self-energy correction. At zero temperature, we calculate these diagrams in vacuum at $\mu = 0$ and obtain that $W_{\sigma\sigma} = (f_s - f_{\text{AL}})/4\pi^2 \sqrt{m\hbar\omega})^{1/2}$ and

\[
\begin{align*}
&\int_{-\infty}^{+\infty} \tilde{\omega} \Delta \tilde{S}_{\sigma\sigma'}(\tilde{q}, \tilde{\omega}) d\tilde{\omega} = 0, \\
&\int_{-\infty}^{+\infty} \tilde{\omega} \Delta \tilde{S}_{\sigma\sigma'}(\tilde{q}, \tilde{\omega}) d\tilde{\omega} = n\tilde{q}^2 \Delta b_n.
\end{align*}
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\Delta \tilde{S}_{1,2}(\tilde{q}) \equiv \int_{-\infty}^{+\infty} \Delta \tilde{S}_{1,2}(\tilde{q}, \tilde{\omega}) d\tilde{\omega} = n\Delta b_n.
\] (10)

For the second expansion function $\Delta \tilde{S}_{\sigma\sigma'}^{(2)}$, we have checked numerically that all the above-mentioned sum rules are strictly satisfied.
$W_{\uparrow\downarrow} = (f_{\text{MT}} - f_{\text{AL}})/4\pi^2\sqrt{m\hbar\omega^{3/2}}$, where

\[ f_s = \frac{\sqrt{1 - x^2}}{(1 - x)^2}, \quad f_{\text{MT}} = \frac{1}{\sqrt{2x}} \ln \left( 1 + \frac{2x - x^2}{|1 - x|} \right), \]

\[ f_{\text{AL}} = \frac{1}{2x\sqrt{1 - x^2}} \left[ \ln^2 \left( 1 + \frac{2x - x^2}{|1 - x|} \right) - \pi^2\Theta(x - 1) \right], \]

with $x = \hbar^2 q^2/2m\hbar\omega$ and $\Theta$ the step function. These results agree with the previous calculations by Son and Thompson [18], although there the spin-parallel and -antiparallel DSFs were not treated separately. At small $q^2/\omega$, we find that the spin-parallel and -antiparallel DSFs have the tail

\[ W_{\uparrow\downarrow} = -W_{\downarrow\uparrow} = \frac{\hbar^{1/2} q^2}{12\pi^2 m^{3/2} \omega^{3/2}}. \]  

This prediction shows that for $\omega \to \infty$ the spin-dependent DSFs decay an order slower in magnitude than the total DSF. The latter is proportional to $q^4/\omega^{3/2}$, as shown in Refs. [18,36]. The faster decay in the total dynamic structure factor is due to the cancellation of the leading terms in $W_{\uparrow\downarrow}$ and $W_{\downarrow\uparrow}$. We note that the high-frequency $\omega^{-3/2}$ tail appears also in the rf spectral function when the final-state effects are taken into account [17,37].

### B. Wilson coefficient from virial expansion

In the limits of large momentum or frequency and high temperature, where both the OPE and the virial expansion are applicable, we can see an interesting relation between the Wilson coefficient and the virial expansion function from Eq. (4). As $W_{\sigma\sigma}$ involves only the few-body physics and hence does not contain the fugacity $z$, a count of the term $z^n$ on both sides of Eq. (4) leads to

\[ \Delta S_{\sigma\sigma,\omega}(q, \omega) = (c_n/c_2) \Delta S_{\sigma\sigma,\omega}(q, \omega) \]  

and

\[ W_{\sigma\sigma}(q, \omega) = (z^2/\mathcal{I}_2) \Delta S_{\sigma\sigma,\omega}(q, \omega), \]

where $\mathcal{I}_2 = z^2 16\pi^4 V_{c_2}/\lambda^4$ is the contact up to the second-order expansion [24]. Therefore, the Wilson coefficient is given by an expansion of the second virial factor. This result is obtained by applying the OPE and the virial expansion methods. As a result, it should be valid, in principle, at high temperatures above the superfluid transition; however, we may expect that it holds at all temperatures, as both the Wilson coefficient and second expansion function are irrelevant to the many-body pairing in the superfluid phase. The many-body effect enters through the many-body parameter of contact only.

Equation (13) is the main result of this paper. At first glance, it may be a surprising result; however, it could be understood from the proportionality shown in Eq. (12). In contrast, a rigorous proof of Eq. (12) at large $q$ and $\omega$ justifies the hypothesis of the OPE method. We note that for the two-component Fermi gas, our result shows that the Wilson coefficient is determined solely by the two-body physics. This is in agreement with previous work. For example, in Ref. [17] a two-body scattering amplitude was used to calculate the Wilson coefficient for the rf spectroscopy of a strongly interacting Fermi gas. However, in general cases where three- or four-body physics comes into play, we anticipate that the Wilson coefficient should be related to the higher-order virial expansion function. In that case, different universal relations with different many-body contact parameters would appear.

In Fig. 3 we check the validity of Eq. (13) by calculating $(m\hbar)^{3/2} \omega^3 (z^2/\mathcal{I}_2) \Delta S_{\sigma\sigma,\omega}$ at $q = 3, 5, 10$. With increasing momentum and/or frequency, $f_{\sigma\sigma}$ approaches smoothly the result by Son and Thompson [18]: $f_{\uparrow\downarrow} = (f_{\text{MT}} - f_{\text{AL}})/4\pi^2$ and $f_{\downarrow\uparrow} = (f_{\text{MT}} - f_{\text{AL}})/4\pi^2$.

### V. UNIVERSAL DSF AT LARGE $(q, \omega)$

In this limit, using Eqs. (3) and (13), the DSF is approximated by

\[ S_{\sigma\sigma} \simeq S_{\sigma\sigma}^{(1)}(q, \omega) + \frac{T\lambda^4}{16\pi V} \Delta S_{\sigma\sigma,\omega}(q, \omega). \]

This approximate DSF should be quantitatively accurate for sufficiently large momentum and frequency. It holds at all temperatures and the many-body effect is included in the many-body contact. However, the momentum $q$ should be larger than a critical momentum $q_c \gg \text{max}(\lambda^{-1}, k_F)$ in order to validate the use of the OPE (3). Here $k_F$ is the Fermi wave vector. Quantitatively, an estimate of $q_c$ requires the calculation of $\Delta S_{\sigma\sigma,\omega}$ and the examination of Eq. (12). We note that Eq. (14) may not be reliable at small frequency $\omega \sim 0$. As a result, the structure factor sum rules may not be strictly satisfied. We note also that in the limit of high temperatures where the contact $\mathcal{I} \simeq \mathcal{I}_2$, the prefactor of the $\Delta S_{\sigma\sigma,\omega}$ term is
We present in Fig. 4 the temperature dependence of the approximate DSF of a normal, homogeneous unitary Fermi gas at $q = 5k_F$ by assuming that $q_c \sim 5k_F$. The many-body contact and fugacity are calculated by using a strong-coupling pair fluctuation theory [24], which is shown to be accurate for describing the unitary equation of state. Close to the superfluid transition temperature, a quasielastic peak clearly emerges at the recoil energy for pairs $\hbar q^2/4m$, in agreement with the low-temperature experimental observation [25].

Equation (14) indicates that at large $(q, \omega)$, the interaction DSF of a unitary Fermi gas depends on only the reduced moment $\bar{q} = q \lambda/\sqrt{4\pi}$ and reduced frequency $\bar{\omega}$. This universal dependence could be examined using large-momentum Bragg spectroscopy [25,26], with varying momentum and temperature while keeping $\bar{q}$ invariant. One can also extract experimentally the universal second expansion function $\Delta S_{\bar{q}, \bar{\omega}}(\bar{q}, \bar{\omega})$ since the contact $I$ can be determined independently using the $f$-sum rule [26]. These predictions break down below the critical momentum $q_c$. Note that by tuning the transferred momentum in Bragg beams, the value of $q_c \gg \max(\lambda^{-1}, k_F)$ might be determined experimentally.

VI. CONCLUSION

In summary, we have studied the finite-temperature dynamic structure factor of a homogeneous unitary Fermi gas, using the quantum virial expansion and the operator product expansion. We have presented the universal second-order expansion function of the dynamic structure factor of a unitary Fermi gas in homogeneous space. This result provides a nontrivial high-temperature benchmark for future accurate calculations of the homogeneous dynamic structure factor for a unitary Fermi gas.

We have rederived and explained the Wilson coefficient of the dynamic structure factor using Feynman diagrams in a language that is amenable to the condensed-matter community. In addition to confirming the previous result by Son and Thompson for the total dynamic structure factor, we have predicted the Wilson coefficients for the spin-parallel and spin-antiparallel structure factors and predicted a different power-law tail of $\omega^{-5/2}$. This power-law tail is much more amenable to the experimental confirmation than that of the total dynamic structure factor.

We have conjectured that the Wilson coefficient is given by the second-order virial coefficient at large momentum $q$. This conjecture is a direct consequence of the hypothesis on the operator product expansion method.

As a result of the connection between the Wilson coefficient and the virial expansion function, in the large-momentum $q$ limit the thermal wavelength $\lambda$ becomes the only length scale. Therefore, the interaction dynamic structure factor should depend universally on a reduced momentum $q \lambda/\sqrt{4\pi}$. We have proposed that Bragg spectroscopy with a large transferred momentum should be able to confirm this universal dependence. Our results can be extended to other dynamical properties of a strongly correlated Fermi gas such as the rf spectrum and single-particle spectral function.

As a remark for future work, we note that it is an interesting challenge to derive from three- and four-fermion solutions [28,29,38] different universal relations involving a many-body parameter such as Tan’s contact. The determination of Wilson coefficients in that case should be difficult. Our method of calculating the higher-order virial expansion function would provide a natural and convenient way to obtain the Wilson coefficient.

Note added. For identical bosons with a large scattering length in which a three-body Efimov physics occurs, the universal relations and Wilson coefficient have been derived very recently by Braaten, Kang, and Platter [39]. In this case, more complete knowledge of the Wilson coefficient can be obtained from the third virial expansion function of a Bose gas, which is calculable by using the exact three-boson solutions [28].

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