The Further Study of Sliding Mode Observers for Nonlinear Systems

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Abstract

In this report, our new research results on the error sign propagation, the finite error convergence and the learning mechanism of the sliding mode observer systems, developed by Slotine et al. in 1980s, for a class of high-order nonlinear systems are presented. It will be shown that, as the switching component of the output tracking error, between the output estimate and the measurable system output, drives the output tracking error dynamics (leading subsystem) to converge to zero in a finite time, this switching component is also used as the driving force of the other subsystems (followers) in the observer error dynamics, to force them to learn the dynamic behaviours of the leading subsystem, for achieving the finite error convergence. Our new findings are that (i) on the sliding surfaces, the signs of the error derivatives of the followers are the same as the one of the leading subsystem; (ii) The error derivatives of all followers can be treated as the modulations of the error derivatives of leading subsystem by the bounded positive functions. Based on these observations and considerations, the error sign propagation rule on the sliding surfaces is formulated, the supervised learning mechanism embedded in the sliding mode observer systems is explored in detail, and the finite error convergence of the sliding mode observer systems is also proved.

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1. Introduction

Since the pioneer works of the sliding mode observers were developed by Slotine and Walcott et al. in the middle of 1980s [1] and [2] based on the variable structure system theory [3], sliding mode observer technique has been widely investigated by researchers in control engineering and applied to the state estimations, parameter monitoring, fault detections and reconstruction of input fault signals for many industrial systems [4]-[14]. Because of the strong robustness property against system uncertainties, sliding mode observers are particularly suitable for the state estimation and fault detections of industrial systems where the systems have complex nonlinearities and large uncertain dynamics [3]-[8].

It has been noted that, in most recent industrial applications, the sliding mode observer methodology, proposed by Slotine et al. in 1987, has been extensively adopted to develop the on-line state estimation software with the excellent performance against uncertainties and disturbances [9]-[12]. However, many simulation results have shown that, for most second-order systems, if the observer gains are designed properly, the estimated state variables, by using sliding mode observer methodology, exhibit the finite error convergence property. It looks that the concept about the asymptotic convergence of closed-loop sliding mode observer error dynamics on the “sliding patches” presented in [1] may need to be further studied.

Also, it has long been observed by engineers and researchers that, in a sliding mode observer system for a class of nth-order nonlinear system in companion form as seen in [1], the finite error convergence and a strong robustness against the interaction disturbance of the first subsystem in the error dynamics can be achieved, by properly choosing the observer gain and using the measurable position data. However, why do all of other subsystems in the sliding mode observer use the sign function $\text{sgn}(e_1)$ as the switching input component? Is this because of the reason that the high-order data of the observed system are difficult to be measured?

Motivated by the above issues, we recently constructed a second-order sliding mode observer-like system to learn a given second order system, where each subsystem is controlled by a weighted sign function of the corresponding output error. Initially, we assume that all of state variables are measurable. After all of errors reach and are attained on the corresponding sliding mode surfaces, it is observed that, in order to keep the error dynamics...
on the sliding mode surfaces, all of signs of errors are fund to become the same. In other words, on the sliding mode surfaces, the sliding mode observer-like system is equivalent to the sliding mode observer system, and thus only the measurable position error is required for estimating all of system states in practice.

The above finding, in fact, has established a theoretical foundation for sliding mode observer systems and also, from the theoretical viewpoint, explained why all of subsystems in a sliding mode observer use the the sign function $\text{sgn}(e_1)$ as the switching input component. Our new finding about the signs of errors is generalized as the sign propagation rule on the sliding mode surfaces, as formulated in following section of this report.

In addition, when studying the structure of a sliding mode observer for a class of $n$th-order systems, we have observed that a sliding mode observer system is actually a supervised learning system. It is noted that the convergence rate and robustness property of the first subsystem of the error dynamics can be well specified by properly choosing the observer gain. After that, each $i$th subsystem, for $i = 2, \ldots, n$, is supervised to learn the error dynamics of the first subsystem in the way that (i) the same switching component or sign function $\text{sgn}(e_1)$ is used as the $i$th input that is weighted by a corresponding observer gain $K_i$; (ii) each observer gain $K_i$ for $i = 2, \ldots, n$, is chosen in the sense that the input switching component is capable of eliminating the effects of the “disturbance term” from the interaction with another subsystem and ensuring that the error changing rate of each subsystem for $i = 2, \ldots, n$, has the same sign with the one of the first subsystem.

Furthermore, considering the fact as stated in point (ii) in the above, we can obtain the error changing rate of each subsystem for $i = 2, \ldots, n$, through the modulation of the error changing rate of the first subsystem by a positive bounded function. It is because of such a finding that the finite error convergence of the error dynamics can be easily proved.

The report is structured as follows: In Section 2, the sign propagation rule on the sliding mode surfaces for a class of sliding mode observer type systems is discussed. In Section 3, the sliding mode observer-like system is constructed and, the equivalence of the sliding mode observer-like system and the sliding mode observer (Slotine 1987) on the sliding surfaces are discussed by using the sign propagation rule, In Section 4, the learning nature of the sliding mode observer system is studied from the view point of the supervised learning mechanism and the finite convergence of the sliding mode observer error dynamics is proved. In Section 5, the conclusions are given.
2. Preliminary

Consider the following second-order system [1]:

\[
\begin{align*}
\dot{x}_1 &= x_2 - K_1 \text{sgn}(x_1) \\
\dot{x}_2 &= -K_2 \text{sgn}(x_2)
\end{align*}
\]  

(2.1)  

(2.2)

where the system states \(x_1\) and \(x_2\) are assumed to be measurable, \(K_1\) and \(K_2\) are the positive constants, \(\text{sgn}(\cdot)\) is the sign function of the form:

\[
\text{sgn}(x) = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 
\end{cases}
\]  

(2.3)

Defining a Lyapunov function \(V_1 = \frac{1}{2} x_1^2\) for the subsystem (2.1) and differentiating it with respect to time \(t\) leads

\[
\dot{V}_1 = x_1 \dot{x}_1 = x_1 (x_2 - K_1 \text{sgn}(x_1))
\]

\[
= x_1 x_2 - K_1 |x_1| \leq |x_1||x_2| - K_1 |x_1|
\]  

(2.4)

If \(K_1\) is chosen such that

\[
K_1 \geq K_{10} + |x_2| \quad \text{with} \quad K_{10} > 0
\]  

(2.5)

(2.4) becomes

\[
\dot{V}_1 \leq K_{10} |x_1| < 0 \quad \text{for} \quad |x_1| \neq 0
\]  

(2.6)

(2.6) means that \(x_1\) converges to zero in a finite time, say \(t_1\).

Similarly, for the second subsystem in (2.2), defining a Lyapunov function \(V_2 = \frac{1}{2} x_2^2\) and differentiating it, we have

\[
\dot{V}_2 = x_2 \dot{x}_2 \leq -K_2 |x_2| < 0
\]  

(2.7)

(2.7) means that \(x_2\) also converge to zero in a finite time.

**Remark 2.1:** After the dynamics of the subsystem (2.1) reaches and then is kept on the sliding surface \(x_1 = 0, \dot{x}_1 = 0\) (for \(t \geq t_1\)) implies

\[
x_2 = K_1 \text{sgn}(x_1)
\]  

(2.8)

and

\[
\text{sgn}(x_2) = \text{sgn}(x_1)
\]  

(2.9)

Now the problems are that

- Can (2.9) be held, in practice, for the second subsystem (2.2) under the condition (2.5)?
Can we use $K_1 sgn(x_1)$ to replace $K_1 sgn(x_2)$ in the subsystem (2.2) to derive the solution of the subsystem (2.2) for $t \geq t_1$?

**Remark 2.2:** From (2.2) and (2.7) we can see that both the convergence and the values of the state variable $x_2$ are nothing related to $x_1$, because the second subsystem in (2.2) is driven by the switching component $-K_2 sgn(x_2)$. In addition, after $x_1$ is maintained on the sliding surface $x_1 = 0$, any sign change of $x_2$ will correspondingly result in the sign change of $x_1$ in order to ensure that the dynamics of the first subsystem in (2.1) to be attained on the sliding mode surface $x_1 = 0$. For instance, if the sign of $x_2$ is positive, the sign of $x_1$ must be the positive, to ensure $\dot{V}_1 < 0$. Alternatively, if the sign of $x_2$ is negative, the sign of $x_1$ must be the negative, to ensure that $\dot{V}_1 < 0$. Thus, the sign relationship in (2.9) is held under the condition (2.5).

**Remark 2.3:** It will be seen from the subsequent discussions that (2.9) will play an important role for designing and analysing sliding mode observer systems. We call (2.9) as the *sign propagation rule*.

**Remark 2.4:** From Remarks 2.2 and 2.3, we can conclude that, under the condition (2.5), the sign function $sgn(x_2)$ in the subsystem (2.2) can be replaced by $sgn(x_1)$, which gives

$$\dot{x}_2 = -K_2 sgn(x_1) \quad (2.10)$$

that is, the subsystem (2.2) is equivalent to (2.10).

**Remark 2.5:** In [1], the following second-order system is considered:

$$\dot{x}_1 = x_2 - K_1 sgn(x_1) \quad (2.11)$$
$$\dot{x}_2 = -K_2 sgn(x_1) \quad (2.12)$$

Obviously, if $K_1$ is chosen to satisfy (2.5) and $K_2$ is a positive constant, the system in (2.11) and (2.12) are equivalent to the one in (2.1) and (2.2), and both $x_1$ and $x_2$ in (2.11) and (2.12) converge to zero in finite time.

The above discussions are summarized in the following theorem:

**Theorem 2.1:** Consider the system in (2.1) and (2.2). If $K_1$ is chosen to satisfy (2.5) and $K_2$ is a positive constant, the system in (2.1) and (2.2) is equivalent to the system in (2.11) and (2.12), and also, both the state variables $x_1$ and $x_2$ of the system in (2.1) and (2.2) and the state variables $x_1$ and $x_2$ of the equivalent system in (2.11) and (2.12) converge to zero in finite time.
3. Theoretical framework of sliding mode observer systems

Consider the following second-order system:

\[ \dot{x}_1 = x_2 \]  
\[ \dot{x}_2 = f(x_1, x_2, u) \]

where the system state variables \( x_1 \) and \( x_2 \) are assumed to be measurable at this moment, \( u \) is the scalar input, \( f(x_1, x_2, u) \) is the linear or nonlinear uncertain function of \( x_1, x_2 \) and \( u \). Let us first consider the following sliding mode observer-like system:

\[ \dot{\hat{x}}_1 = \hat{x}_2 - K_1 \text{sgn}(e_1) \]  
\[ \dot{\hat{x}}_2 = \hat{f}(\hat{x}_1, \hat{x}_2, u) - K_2 \text{sgn}(e_2) \]

where \( e_1 = \hat{x}_1 - x_1, e_2 = \hat{x}_2 - x_2, \hat{f}(\hat{x}_1, \hat{x}_2, u) \) is the estimate of \( f(x_1, x_2, u) \), \( K_1 \) and \( K_2 \) are chosen to satisfy the following inequalities:

\[ K_1 \geq K_{10} + |x_2| \]  
\[ K_2 \geq K_{20} + |\hat{f}(\hat{x}_1, \hat{x}_2, u) - f(x_1, x_2, u)| \]

with the positive constants \( K_{10} \) and \( K_{20} \).

The error dynamics can then be expressed as:

\[ \dot{e}_1 = \dot{e}_2 - K_1 \text{sgn}(e_1) \]  
\[ \dot{e}_2 = \hat{f}(\hat{x}_1, \hat{x}_2, u) - f(x_1, x_2, u) - K_2 \text{sgn}(e_2) \]

Defining a Lyapunov function \( V_1 = \frac{1}{2} e_1^2 \) for the subsystem (3.7), we then have

\[ \dot{V}_1 = e_1 \dot{e}_1 = e_1 (e_2 - K_1 \text{sgn}(e_1)) \]

\[ = e_1 e_2 - K_1 |e_1| \leq |e_1||e_2| - K_1 |e_1| \]

\[ \leq K_{10} |e_1| < 0 \quad \text{for} \quad |e_1| \neq 0 \]

(3.9) indicates that \( e_1 \) converges to zero in finite time, say \( t_{e1} \). Then, on the sliding surface \( e_1 = 0 \) (for \( t > t_{e1} \)), \( \dot{e}_1 = 0 \) leads the equivalent dynamics of the subsystem (3.7) has the following properties:

\[ \text{P.1} \quad e_2 = K_1 \text{sgn}(e_1) \]

\[ \text{P.2} \quad \text{sgn}(e_2) = \text{sgn}(e_1) \]
Therefore, after $t > t_{e1}$, the sliding mode observer-like system in (3.3) and (3.4) can be equivalent to

$$\dot{x}_1 = \dot{x}_2 - K_1 \text{sgn}(e_1) \quad (3.12)$$
$$\dot{x}_2 = \dot{f}(\dot{x}_1, \dot{x}_2, u) - K_2 \text{sgn}(e_1) \quad (3.13)$$

(3.12) and (3.13) are actually the famous sliding mode observer proposed by Slotine in [1]. Also, the error dynamics in (3.7) and (3.8) can be expressed as

$$\dot{e}_1 = e_2 - K_1 \text{sgn}(e_1) \quad (3.14)$$
$$\dot{e}_2 = (\dot{f}(\dot{x}_1, \dot{x}_2, u) - f(x_1, x_2, u)) - K_2 \text{sgn}(e_1) \quad (3.15)$$

**Remark 3.1:** The discussions from (3.3) to (3.15) have revealed the following characteristics of the sliding mode observer-like system in (3.3) and (3.4):

(i) After the dynamics of the subsystem (3.7) reaches and then is attained on the sliding mode surface $e_1 = 0$ for $t \geq t_{e1}$, the sign of the error $e_1$ is forced to be the same as the one of the error $e_2$. Such an “error sign propagation” on the sliding mode surface $e_1 = 0$ shows how the closed-loop error dynamics of the first subsystem in (3.7) is coordinated with the one of the second subsystem in (3.8). In fact, before the error $e_1$ reaches the sliding surface $e_1 = 0$, if the initial values $\dot{x}_1(t_0)$ and $x_1(t_0)$ have the same sign, (3.11) holds.

(ii) Because of the sign propagation property on the sliding mode surface $e_1 = 0$, we can always use $\text{sgn}(e_1)$ to replace $\text{sgn}(e_2)$ in the sliding mode observer-like system in (3.3) and (3.4). Such a merit makes the sliding mode observer-like system in (3.3) and (3.4) to play the role of the observer without using the measurements of the error $e_2$. From this viewpoint, we can confirm that the sliding mode observer-like system in (3.3) and (3.4) are equivalent to the sliding mode observer in (3.14) and (3.15) on the sliding surface $e_1 = 0$.

Generally, the above discussions on the sign propagation property can be extended to the high-order system. Consider the following nonlinear system in companion form:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
\[ \dot{x}_n = f(x_1, x_2, \ldots, x_n, u) \]  
(3.16)

where \( x_1, x_2, \ldots, x_n \) are system state variables, \( u \) is the scalar input, \( f(x_1, x_2, \ldots, x_n, u) \) is the uncertain nonlinear function of both the system state variables and input.

The sliding mode observer-like system can be constructed as:

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 - K_1 \text{sgn}(e_1) \\
\dot{x}_2 &= \dot{x}_3 - K_2 \text{sgn}(e_2) \\
&\vdots \\
\dot{x}_{n-1} &= \dot{x}_n - K_{n-1} \text{sgn}(e_{n-1}) \\
\dot{x}_n &= (\hat{f} - f) - K_n \text{sgn}(e_n)
\end{align*}
\]  
(3.17)

where \( e_i = \hat{x}_i - x_i \) for \( i = 1, 2, \ldots, n, \hat{f} \) is the estimate of \( f \), and the positive constants \( K_1, K_2, \ldots, K_{n-1} \) and \( K_n \) are chosen such that

\[
\begin{align*}
K_1 &\geq K_{10} + |x_2| \\
K_2 &\geq K_{20} + |x_3| \\
&\vdots \\
K_{n-1} &\geq K_{n-1,0} + |x_n| \\
K_n &\geq K_{n0} + |\hat{f} - f|
\end{align*}
\]  
(3.18)

with the positive constants \( K_{10}, K_{20}, \ldots, K_{n0} \).

The error dynamics can then be obtained as:

\[
\begin{align*}
\dot{e}_1 &= e_2 - K_1 \text{sgn}(e_1) \\
\dot{e}_2 &= e_3 - K_2 \text{sgn}(e_2) \\
&\vdots \\
\dot{e}_i &= e_{i+1} - K_i \text{sgn}(e_i) \\
&\vdots \\
\dot{e}_{n-1} &= e_n - K_{n-1} \text{sgn}(e_{n-1}) \\
\dot{e}_n &= (\hat{f} - f) - K_n \text{sgn}(e_n)
\end{align*}
\]  
(3.22)

For the first subsystem of the error dynamics in (3.22), defining the Lyapunov function \( V_1 = e_1^2/2 \) and differentiating it with respect to time leads
\[
\dot{V}_1 = e_1 \dot{e}_1 = e_1 (e_2 - K_1 sgn(e_1)) \\
= e_1 e_2 - K_1 |e_1| \leq |e_1||e_2| - K_1 |e_1| \\
\leq K_{10} |e_1| < 0 \quad for \ |e_1| \neq 0 \quad (3.27)
\]

(3.27) shows that \( e_1 \) converges to zero in a finite time, say \( t_{e_1} \), and on the sliding surface \( e_1 = 0 \) (for \( t > t_{e_1} \)), \( \dot{e}_1 = e_2 - K_1 sgn(e_1) = 0 \). Thus,
\[
\dot{e}_2 = K_1 sgn(e_1) \quad (3.28)
\]
and
\[
sgn(e_2) = sgn(e_1) \quad (3.29)
\]
For the second subsystem of the error dynamics in (3.23), defining the Lyapunov function \( V_2 = e_2^2/2 \) and differentiating it with respect to time leads
\[
\dot{V}_2 = e_2 \dot{e}_2 = e_2 (e_3 - K_2 sgn(e_2)) \\
= e_2 e_3 - K_2 |e_2| \leq |e_2||e_3| - K_2 |e_2| \\
\leq K_{20} |e_2| < 0 \quad for \ |e_2| \neq 0 \quad (3.30)
\]
Then \( e_2 \) converges to zero in a finite time, say \( t_{e_2} \), and on the sliding surface \( e_2 = 0 \) (for \( t > t_{e_2} \)), \( \dot{e}_2 = e_3 - K_2 sgn(e_2) = e_3 - K_1 sgn(e_1) = 0 \) with the properties that
\[
e_3 = K_1 sgn(e_1) \quad (3.31)
\]
and
\[
sgn(e_3) = sgn(e_1) \quad (3.32)
\]
Continuously, for the \( i \)th subsystem of the error dynamics in (3.24), for \( i \leq n - 1 \), defining the Lyapunov function \( V_i = e_i^2/2 \) and differentiating it with respect to time leads
\[
\dot{V}_i = e_i \dot{e}_i = e_i (e_{i+1} - K_i sgn(e_i)) \\
= e_i e_{i+1} - K_i |e_i| \leq |e_i||e_{i+1}| - K_i |e_i| \\
\leq K_{i0} |e_i| < 0 \quad for \ |e_i| \neq 0 \quad (3.33)
\]
Then \( e_i \) converges to zero in a finite time, say \( t_{e_i} \), and on the sliding surface \( e_i = 0 \) (for \( t > t_{e_i} \)), \( \dot{e}_i = e_{i+1} - K_i sgn(e_i) = e_{i+1} - K_i sgn(e_1) = 0 \) with the properties that
\[
e_{i+1} = K_i sgn(e_i) \quad (3.34)
\]
and
\[
sgn(e_{i+1}) = sgn(e_i) = sgn(e_1) \quad for \ i \leq n - 1 \quad (3.35)
\]
Finally, for the $n$th subsystem of the error dynamics in (3.26), defining the Lyapunov function $V_n = e_n^2/2$ and differentiating it with respect to time leads

$$\dot{V}_n = e_n \dot{e}_n = e_n \left( (\dot{f} - f) - K_n sgn(e_n) \right)$$

$$= e_n (\dot{f} - f) - K_n |e_n| \leq |e_n||\dot{f} - f| - K_n |e_n|$$

$$\leq K_{n0} |e_n| < 0 \quad \text{for } |e_n| \neq 0 \quad (3.36)$$

Thus, $e_n$ converges to zero in a finite time.

**Remark 3.2:** It is seen from (3.27) to (3.36) that, with the errors $e_i$ (for $1 \leq i \leq n$) converging to zero in finite time, on the sliding surfaces $e_i = 0$ (for $1 \leq i \leq n$), all of sign functions $sgn(e_i)$ (for $2 \leq i \leq n$) are the same as $sgn(e_1)$. Therefore, on the sliding mode surfaces $e_i = 0$ (for $1 \leq i \leq n$), the sliding mode observer-like system in (3.17) is equivalent to the following sliding mode observer system:

$$\dot{x}_1 = \dot{x}_2 - K_1 sgn(e_1)$$

$$\dot{x}_2 = \dot{x}_3 - K_2 sgn(e_1)$$

$$\vdots$$

$$\dot{x}_{n-1} = \dot{x}_n - K_{n-1} sgn(e_1)$$

$$\dot{x}_n = (\dot{f} - f) - K_n sgn(e_1) \quad (3.37)$$

The error dynamics can then be obtained as:

$$\dot{e}_1 = e_2 - K_1 sgn(e_1)$$

$$\dot{e}_2 = e_3 - K_2 sgn(e_1)$$

$$\vdots$$

$$\dot{e}_i = e_{i+1} - K_i sgn(e_1)$$

$$\vdots$$

$$\dot{e}_{n-1} = e_n - K_{n-1} sgn(e_1)$$

$$\dot{e}_n = (\dot{f} - f) - K_n sgn(e_1) \quad (3.38)$$

**Remark 3.3:** It is seen, from the $i$th subsystem of the error dynamics in (3.24), that the high-gain, $K_i$, plays an important role of eliminating the effect of the “disturbance” $e_{i+1}$, ensuring the finite convergence of $e_i$ as well as controlling the finite error convergence times $t_{ei}$ (for $1 \leq i \leq n$).
Remark 3.4: Another important characteristic of the sliding mode observer systems is that, because of the “error sign propagation” on the sliding mode surfaces, only the measurements of $x_1$ and $\hat{x}_2$ are required. This merit, together with the finite error convergence property, sliding mode observer systems have been playing the role of the state estimations, parameter monitoring, fault detections and reconstruction of input fault signals, where the systems have nonlinearities and uncertainties.

In the following, we will explore the sliding mode observer system in (3.37) from the respective of learning mechanism.

4. Sliding mode observer from the viewpoint of learning mechanism

Consider the following third-order system:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= f(x_1, x_2, x_3, u)
\end{align*}
\]

where $x_1, x_2,$ and $x_3$ are state variables, $u$ is the control input, $f(x_1, x_2, x_3, u)$ is the linear or nonlinear uncertain function of $x_1, x_2, x_3$ and $u$. It is assumed that only the system states $x_1$ and the input $u$ are measurable. The corresponding sliding mode observer system is of the form:

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 - K_1 \text{sgn}(e_1) \\
\dot{\hat{x}}_2 &= \hat{x}_3 - K_2 \text{sgn}(e_1) \\
\dot{\hat{x}}_3 &= \hat{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3, u) - K_3 \text{sgn}(e_1)
\end{align*}
\]

where $e_1 = \hat{x}_1 - x_1$, $e_2 = \hat{x}_2 - x_2$, $e_3 = \hat{x}_3 - x_3$, $\hat{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3, u)$ is the estimate of $f(x_1, x_2, x_3, u)$, $K_1, K_2$ and $K_3$ are chosen to satisfy the following inequalities:

\[
\begin{align*}
K_1 &\geq K_{10} + |e_2| \\
K_2 &\geq K_{20} + |e_3| \\
K_3 &\geq K_{30} + |\hat{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3, u) - f(x_1, x_2, x_3, u)|
\end{align*}
\]

with the positive constants $K_{10}$, $K_{20}$ and $K_{30}$.

The corresponding error dynamics can be expressed as:
\[
\dot{e}_1 = e_2 - K_1 sgn(e_1) \tag{4.10}
\]
\[
\dot{e}_2 = e_3 - K_2 sgn(e_1) \tag{4.11}
\]
\[
\dot{e}_3 = \left( \hat{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3, u) - f(x_1, x_2, x_3, u) \right) - K_3 sgn(e_1) \tag{4.12}
\]

The supervised learning mechanism of the sliding mode observer in (4.4)-(4.6) can be formulated as follows:

(i) With the gain \( K_1 \) chosen as in (4.7), the closed-loop error dynamics of the first subsystem in (4.10) converges to zero in a finite time as discussed in (3.7). In the learning process, the first subsystem in (4.10) is the sample or a teacher for the second and the third subsystems to learn.

(ii) During the learning process, the second and the third subsystems in (4.11) and (4.12) are required to learn the finite error convergence property of the closed-loop error dynamics of the first subsystem in (4.10) in the sense of

- Using the same switching signal \( sgn(e_1) \) as the input component of both the second and the third subsystems in (4.11) and (4.12), respectively.
- The observer gains \( K_2 \) and \( K_3 \) chosen, as seen in (4.7) and (4.8), to first eliminate the effects of \( e_3 \) and \( (\hat{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3, u) - f(x_1, x_2, x_3, u)) \), respectively, and then continuously forcing the error changing rates \( \dot{e}_2 \) and \( \dot{e}_3 \) in (4.11) and (4.12) to have the same sign with \( \dot{e}_1 \).

(iii) Through learning, the error dynamics of the second and the third subsystems in (4.11) and (4.12) will converge to zero in finite time.

Considering the fact that the error changing rates \( \dot{e}_2 \) and \( \dot{e}_3 \) in (4.11) and (4.12) are forced to be equal to the sign of \( \dot{e}_1 \), the error changing rates \( \dot{e}_2 \) and \( \dot{e}_3 \) can be obtained by the modulated \( \dot{e}_1 \) as follows:

\[
\dot{e}_2 = g_{1m}(e_1, e_2, e_3)\dot{e}_1 \tag{4.13}
\]
\[
\dot{e}_3 = g_{2m}(e_1, e_2, e_3)\dot{e}_1 \tag{4.14}
\]

where \( g_1(e_1, e_2, e_3) \) and \( g_2(e_1, e_2, e_3) \) are the modulation functions with the following properties:
(i) \[ g_{im}(e_1, e_2, e_3) = \begin{cases} g_i(e_1, e_2, e_3) > 0 & t > t_0 \\ 0 & t \leq t_0 \end{cases} \quad \text{for } i = 1, 2 \quad (4.15) \]

(ii) \[ 0 < g_i(e_1, e_2, e_3) \leq g_{i0} \quad (4.16) \]

Then, \[
|e_2(t)| = \left| \int_{-\infty}^{t} g_{1,m}(e_1(\tau), e_2(\tau), e_3(\tau)) \dot{e}_1(\tau) d\tau \right|
\]
\[
= \left| \int_{-\infty}^{t_0} g_{1,m}(e_1(\tau), e_2(\tau), e_3(\tau)) \dot{e}_1(\tau) d\tau + \int_{t_0}^{t} g_{1,m}(e_1(\tau), e_2(\tau), e_3(\tau)) \dot{e}_1(\tau) d\tau \right|
\]
\[
= \left| \int_{t_0}^{t} g_1(e_1(\tau), e_2(\tau), e_3(\tau)) \dot{e}_1(\tau) d\tau \right|
\]
\[
\leq g_{10} \left| \int_{t_0}^{t} \dot{e}_1(\tau) d\tau \right| = g_{10} |e_1(t)| \quad (4.17)
\]

Similarly, we can prove that
\[
|e_3(t)| \leq g_{20} \left| \int_{t_0}^{t} \dot{e}_1(\tau) d\tau \right| = g_{20} |e_1(t)| \quad (4.18)
\]

Therefore, both \(e_2(t)\) and \(e_3(t)\) converge to zero in finite time.

**Remark 4.1:** It has been seen from the above discussions that, the sliding mode observer system has a typical supervised learning characteristic. This point can be seen from the following facts: (i) The desired closed-loop error dynamics of the first subsystem in (4.10) has been specified in terms of the selection of the observer gain \(K_1\) in (4.7) and the sufficient condition for the closed-loop error dynamics to converge to zero in a finite time in (3.27); (ii) The second and the third subsystems in (4.11) and (4.12) use the same switching function \(sgn(e_1)\) as the input signal, weighted with different observer gains \(K_2\) and \(K_3\), respectively, to overcome the effects of the disturbances \(e_3\) and \(\hat{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3, u) - f(x_1, x_2, x_3, u)\), respectively, in the sense that the error rates \(\dot{e}_2\) and \(\dot{e}_3\) in (4.11) and (4.12) have the same sign with \(\dot{e}_1\). Through such a supervised learning, the finite convergence of both \(e_2(t)\) and \(e_3(t)\) can be achieved.

**Remark 4.2:** In practice, the initial values \(e_i(t_0)\) of the errors \(e_i(t_0)\), for \(i = 1, 2\), may not be zero or may not have the same sign. The property (i) of the modulation functions \(g_{im}(e_1, e_2, e_3)\), for \(i = 1, 2\), as seen in (4.15), has ensured that, after the integration in (4.17), the effects of the initial values of the errors on the error convergence are eliminated.
Remark 4.3: The learning mechanism embedded in the sliding mode observer system implements a parallel learning process. This point can be seen that three subsystems in (4.10)-(4.12) are coordinated with the same switching function $sgn(e_1)$ as the subsystems’ input. Also, the observer gains $K_2$ and $K_3$ are chosen to make the error rates $\dot{e}_2$ and $\dot{e}_3$ in (4.11) and (4.12) have the same sign with $\dot{e}_1$.

The above discussions can easily be extended to a class of high-order nonlinear systems in (3.16) with the sliding mode observer system in (3.37). Especially, the finite convergence of the observer error dynamics in (3.38) is described in the following theorem:

Theorem 4.1: Consider a class of high-order systems in companion form described by (3.16) with the sliding mode observer system in (3.37). If the observer gains $K_i$, for $i = 1, 2, \ldots, n$, are designed such that

\begin{equation}
K_i \geq K_{i0} + |e_{i+1}| \quad \text{for } i = 1, 2, \ldots, n-1 \quad (4.19)
\end{equation}

and

\begin{equation}
K_n \geq K_{n0} + |\hat{f}(\bar{x}_1, \ldots, \bar{x}_n, u) - f(x_1, \ldots, x_n, u)| \quad (4.20)
\end{equation}

then, the error dynamics in (3.38) will converge to zero in finite time.

5. Conclusion

In this work, some new research results on the sliding mode observer systems have been presented. It has been shown that the design of sliding mode observer is formulated from the viewpoint of learning, and through the selection of observer gains, the signs of the error derivatives of followers are the same as the one of the leading subsystem in the sliding mode observer error dynamics. Based on this work, many further researches are under the authors’ investigation.

References


