A Differential Vector Space Approach to Nonlinear System Regulation

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Abstract—Nonlinear system regulation is an active research area and is important for disturbance rejection and reference signal tracking. Existing results using differential geometry or Lyapunov theory are restricted to bounded exogenous signals. This paper uses a differential vector space approach to develop solutions for nonlinear system regulation with both bounded and unbounded exogenous signals. The solutions are given in terms of observable dual state spaces in the differential vector space of the system and are well suited to symbolic and numerical computation.

Index Terms—Differential vector space, disturbance rejection, nonlinear systems, observability, state feedback, tracking systems.

I. INTRODUCTION

REGULATION is a problem of fundamental importance in control system design. Basically, it consists of finding feedback control laws that regulate the closed-loop system response about a specified value. There are essentially two aspects to the problem: rejecting the effect of exogenous disturbances and controlling the system to track a specified reference signal.

Most of the research on control system regulation has dealt with linear systems, for example, [7], [17], and [4], and only recently have results for nonlinear systems begun to appear, for example, [12], [16], and [2]. The work in [12] presents a necessary and sufficient condition for nonlinear system regulation where the exosystem satisfies a Poisson stability condition. The conditions in [16] for output regulation require the exogenous signal and its derivative to belong to a compact set. In [2], a number of necessary conditions for nonlinear system asymptotic tracking are presented. These results are based on differential geometry or Lyapunov theory. All existing results on nonlinear system regulation require the exogenous signal, being tracked or rejected, to be bounded and, in some cases, constant. There are currently no results on nonlinear system regulation for unbounded exogenous signals.

This paper presents solutions to the problem of state feedback nonlinear system regulation subject to possibly unbounded exogenous signals. While the the system state can be unbounded

when tracking an unbounded exogenous signal, the solution presented here is local in the sense that the system error output is regulated in a neighborhood of the equilibrium point. We give a sufficient condition for the solution of such a regulation problem and further show that this sufficient condition is also necessary under an additional condition on system observability. Finally, we refine the necessary and sufficient conditions when the exogenous signal is bounded.

The approach to nonlinear system regulation developed here is based on the theory of differential linear algebra. Early work on the application of differential linear algebra to control systems is presented in [1] and [8], where it is shown that a dynamic system defines a differential field and the system properties can be represented by a differential vector space over this field. Motivated by linear system regulation in the dual space as defined in [17], this paper defines dual spaces for the system state, input, and output in the differential vector space of a nonlinear system. Moreover, the results in [20] and [21] on nonlinear system observability in differential vector space are further developed here to facilitate the solution to the nonlinear system regulation problem.

Our conditions for nonlinear system regulation are given in terms of the relationship between the system observable dual state space and the dual exogenous signal space. Such a dual state space representation addresses the problem of regulation via system observability properties and provides a clear physical interpretation of the conditions and design procedures. The dual state space approach to system observability allows the system state to be unbounded, thus resolving the difficulty experienced by some other approaches like center manifold theory, which requires the system state to be bounded. A further advantage of the dual state space approach is that integrability of the dual spaces leads directly to constructive conditions and computational procedures for nonlinear system regulation.

The results in this paper cover regular properties of nonlinear systems but do not apply on some singular (irregular) points. By regular properties, we mean analytical properties of the system over an open dense set of the system state, input, and output spaces. System properties on singular points deserve further study but are beyond the scope of this paper.

The organization of this paper is as follows. Section II formulates the nonlinear system regulation problem. Section III reviews linear system regulation in the dual space and presents fundamental results from the theory of differential vector spaces of nonlinear systems. Section IV develops results on observability of nonlinear systems in the differential vector space. The main results of this paper, in Section V, include nonlinear system regulation results for both stable and unstable systems with both
bounded and unbounded exogenous signals. Section VI illustrates the procedure for nonlinear state feedback design through two examples. Section VII concludes the paper.

II. PROBLEM FORMULATION

Throughout this paper, \( \mathbb{R}^n \) denotes an \( n \)-dimensional Euclidean space with the usual norm. \( \mathcal{D} \) denotes the differential operator. \( \partial \) denotes the derivative and \( \partial^k \) denotes the \( k \)-th order derivative with respect to time \( t \) of a function \( g \). \( (\partial/\partial x_i) \cdots (\partial/\partial x_m) \) denotes the differential derivative with respect to an \( n \)-dimensional vector \( x \). \( L_x \) denotes the Lie derivative along the vector field \( v \). \( \dim \mathcal{L} \) denotes the dimension of a vector space \( \mathcal{L} \). \( \mathcal{L}^\perp \) denotes the orthogonal (annihilator) of a linear vector space \( \mathcal{L} \). \( \mathcal{L}^* \) denotes the dual space of a vector space \( \mathcal{L} \). \( v^* \in \mathcal{L}^* \) and \( w \in \mathcal{L} \) denotes the closed integer set \([1 \cdots n]\).

The nonlinear system under consideration is

\[
\begin{align*}
x' & = f(x) + g(x)u + p(x)w \\
w & = s(u) \\
y & = h(x) \\
e & = h(x) + q(w)
\end{align*}
\]  

(1)-(4)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), and \( y \in \mathbb{R}^c \) with \( c \leq n \) are the system state, control input, and output, respectively, and where \( w \in \mathbb{R}^r \) is the exogenous signal of the system; \( e \in \mathbb{R}^c \) is the error output to be regulated; \( f(\cdot), g(\cdot), p(\cdot) \) are analytic vector fields in an open set of \( \mathbb{R}^n \); \( s(\cdot) \) is an analytic vector field in an open set of \( \mathbb{R}^r \); \( h(\cdot) \) and \( q(\cdot) \) are analytic mapping in an open set of \( \mathbb{R}^n \) and \( \mathbb{R}^c \), respectively.

The state, input, output, and exogenous signals of the system are written as \( x = (x_1, x_2, \ldots, x_n)^T \), \( u = (u_1, u_2, \ldots, u_m)^T \), \( y = (y_1, y_2, \ldots, y_c)^T \), \( w = (w_1, w_2, \ldots, w_r)^T \), and \( e = (e_1, e_2, \ldots, e_c)^T \) is the error output to be regulated. In the nonlinear system (1)-(4), (1) represents the controlled nonlinear plant affine in control input \( u \) and exogenous signal \( w \) and (2) is an exosystem generating the exogenous signal \( w \), which can represent a disturbance to the system or a specified reference signal to be tracked. The error output \( e \) is the difference between the actual plant output \( h(x) \) and a function \( q(w) \) of the exogenous signal.

In this paper, the continuous feedback control input of the system uses the full system state \( (x(t), w(t)) \), which is continuous for all \( t \geq 0 \). Under this condition, we describe stability and stabilizability of the system as follows.

A nonlinear system \( \dot{x} = f(x) \) is globally asymptotically stable if for any initial condition \( x(0) \in \mathbb{R}^n \) the state \( x \) of the system satisfies \( \lim_{t \to \infty} x(t) = 0 \).

A nonlinear system \( \dot{x} = f(x) + g(x)u \) is globally asymptotically stabilizable if there exists a state feedback control \( u = \alpha(x) \) such that the closed-loop system \( \dot{x} = f(x) + g(x)\alpha(x) \) is globally asymptotically stable.

A nonlinear system \( \dot{x} = f(x) \) is locally asymptotically stable if there exists a neighborhood of the origin \( X \subset \mathbb{R}^n \) such that for any initial condition \( x(0) \in X \) the state \( x \) of the system satisfies \( \lim_{t \to \infty} x(t) = 0 \).

A nonlinear system \( \dot{x} = f(x) + g(x)u \) is locally asymptotically stabilizable if there exists a state feedback control \( u = \alpha(x) \) such that the closed-loop system \( \dot{x} = f(x) + g(x)\alpha(x) \) is locally asymptotically stable.

For the nonlinear system (1), a Taylor series expansion applied at the origin yields

\[
\dot{x} = A_x x + A_u w + B_g u + \psi(x, w) + \phi(x) u
\]

(5)

where \( A_x = (\partial f(0)/\partial x) \), \( A_u = (\partial f(0)/\partial u) \). \( B_g = (\partial f(0)/\partial g) \), the term \( \psi(x, w) \) represents the nonlinear higher order terms in \( x \) and \( w \) of \( f(x) + g(x)u \), and the term \( \phi(x) \) represents the linear and nonlinear terms in \( x \) of \( g(x) \). It follows that a linear approximation of (1) is

\[
\dot{x} = A_x x + A_u w + B_g u.
\]

(6)

A nonlinear system \( \dot{x} = f(x) \) is locally asymptotically stable if the pair \( (A_x, B_g) \) is stabilizable.

Given a full state feedback control \( u = \alpha(x, w) \), we obtain the closed-loop system

\[
\dot{x} = f(x) + g(x)\alpha(x, w) + p(x)w \\
w = s(w) \\
e = h(x) + q(w).
\]

III. DIFFERENTIAL VECTOR SPACE OF NONLINEAR SYSTEMS

A. Linear System Regulation in the Dual Vector Space

This section provides a brief review of a well-known result on linear system regulation [7] and relates it to the notion of dual vector space presented by Wonham [17]. It will turn out that the differential vector space of a nonlinear system is a natural extension of the dual vector space of a linear system.

Consider the linear system

\[
\begin{align*}
\dot{x} & = Ax + Bu \\
y & =Cx
\end{align*}
\]

(7)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), and \( y \in \mathbb{R}^c \) are the system state, input, and output, respectively, and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \) and \( C \in \mathbb{R}^{c \times n} \) are the system state, input, and output matrices,
respectively. The dual state space of the system (7) as defined in [17] is
\[ \lambda^* = \text{span}_R \{ z \} = \text{span}_R \{ x_1, x_2, \ldots, x_n \}. \]

Thus, \( \lambda^* \) includes all linear functionals of the state variables \( x_1, x_2, \ldots, x_n \). The dual input space and dual output space of the system are defined, respectively, as
\[ \mathcal{U}^* = \text{span}_R \{ u_j, \dot{u}_j, \ldots, u_j^{(k)}, \ldots, j \in \mathcal{U}, k \geq 0 \} \]
\[ \mathcal{V}^* = \text{span}_R \{ v_j, \dot{v}_j, \ldots, v_j^{(k)}, \ldots, j \in \mathcal{V}, k \geq 0 \}. \]

While \( \lambda^* \) is a finite-dimensional vector space, \( \mathcal{U}^* \) and \( \mathcal{V}^* \) may be infinite dimensional spaces.

The dual output space can be computed as follows:
\[ y = Cx \]
\[ \dot{y} = CAx + CBu \]
\[ \ldots \]
\[ y^{(k)} = CA^kx + CA^{k-1}Bu + CA^{k-2}B\dot{u} + \cdots + CBu^{(k-1)} \]
\[ \ldots. \]
(8)

Clearly
\[ \mathcal{V}^* \subset \lambda^* + \mathcal{U}^* \]
and it can be shown that the system (7) is observable if and only if
\[ \lambda^* \subseteq \mathcal{V}^* + \mathcal{U}^*. \]

For linear system regulation, we consider the following state equations:
\[ \dot{x} = Ax + Bu + Pw \]
\[ \dot{w} = Sw \]
\[ e = Dx + Dww \]
(9) (10) (11)

where \( w \in \mathcal{R}^r \) is the exogenous signal and \( e \in \mathcal{R}^c \) is the error output of the system.

In [7], it is assumed that the exogenous signal \( w \) is generated by an unstable exosystem (10) where all the eigenvalues of the state matrix \( A \) are assumed to be in the closed right half of the complex plane. For the purposes of this section, we only consider the problem of stable linear system regulation, i.e., for \( w = 0 \), the system (9) is stable and the input matrix \( B = 0 \).

Under these conditions, the result in [7] can be stated as follows.

**Proposition 3.1:** The error output of the linear system (7) is regulated if and only if
\[ E^* \cap \mathcal{W}^* = 0. \]
(12)

Proposition 3.2 is a dual space equivalent of Proposition 3.1. It follows from (8) that condition (12) can be readily checked by taking a finite number of derivative operations on the error output \( e \). Thus, the dual vector space is a useful notion for linear system regulation. In this paper, we further extend these ideas to describe the differential vector space of nonlinear systems and apply it to nonlinear system regulation.

### B. Differential Vector Space of Nonlinear Systems

We now present basic definitions and results from the theory of differential fields and differential vector spaces [1], [8].

A differential field \( K \) is a field together with a derivative operation \( \frac{d}{dx} \) on \( K \), which satisfies the following rules:
\[ \frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}, \]
\[ \frac{d}{dx}(cf) = c \frac{df}{dx}, \]
\[ \frac{d}{dx}(fg) = \frac{df}{dx}g + f \frac{dg}{dx}, \]
for \( f, g \in K \).

A vector space \( \mathcal{A} \) over a differential field \( K \) is a vector space together with a derivative operation \( \frac{d}{dx} \) on \( \mathcal{A} \), which satisfies the following rules:
\[ (a + b) = \dot{a} + \dot{b} \]
and
\[ (ka) = k\dot{a} \]
for \( a, b \in \mathcal{A} \) and \( k \in K \).

Let \( K \) be the field of meromorphic functions of \( x \), \( u^{(k)} \), and \( u_i \) for \( i \in \mathcal{I}, j \in \mathcal{J}, k \geq 0 \), and \( l \in \mathcal{L} \), and let \( \mathcal{K}(x) \in K \) be the set of meromorphic functions of \( x \), \( i \in \mathcal{I} \). For a function \( \phi(x, u_i^{(k)}), \ldots, u^{(k)}, w) \in K \), we have \( \dot{\phi} \in K \) since
\[ \dot{\phi} = \frac{\partial \phi}{\partial x} \dot{x} + \frac{\partial \phi}{\partial u_i} \dot{u}_i + \cdots + \frac{\partial \phi}{\partial u^{(k)}} \dot{u}^{(k)} + \frac{\partial \phi}{\partial w} \dot{w} \]
\[ = \frac{\partial \phi}{\partial x} (f(x) + x \dot{x} + \cdots + u^{(k)} \dot{u}^{(k)} + \cdots + u_i \dot{u}_i + \cdots + u \dot{u} + \cdots + w \dot{w})(w). \]
(13)

Further, it can be shown using (13) that if \( \phi_\alpha \in K \) and \( \phi_\beta \in K \),
then \( (\phi_\alpha + \phi_\beta) \in K \) and \( (\phi_\alpha \phi_\beta) \in K \). Thus, \( K \) is a differential field defined by the dynamical system (1)–(4).
Over the differential field $\mathcal{K}$, the system (1)–(4) also defines a linear vector space $\mathcal{D}$ written as

$$\mathcal{D} = \text{span}_\mathcal{K} \left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial u_l} ; i \in \mathbb{N}, j \in \mathbb{M}, k \geq 0, l \in \mathbb{I} \right\}.$$  

The dual space of $\mathcal{D}$ over $\mathcal{K}$ is defined as

$$\mathcal{D}^* = \text{span}_\mathcal{K} \left\{ dx_i, du_j, dy; i \in \mathbb{N}, j \in \mathbb{M}, k \geq 0, l \in \mathbb{I} \right\}.$$  

The dual space $\mathcal{D}^*$ defined by the nonlinear system (1)–(4) is a differential linear vector space over $\mathcal{K}$.

**Remark 3.1:** The definition of the dual space over the differential field $\mathcal{K}$ of meromorphic functions follows from standard results [18], [1], [8]. The linear vector space and its dual space are induced by the inner product of vectors in these two spaces, and the inner product can be described in terms of the dual bases of the two spaces. For example, $\langle dx_i, (\partial/\partial x_j) \rangle = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta function. For a subspace $\mathcal{L}$ of the vector space $\mathcal{D}$, any vector in its orthogonal space $\omega \in \mathcal{L}^\perp$ satisfies $\langle \omega, v \rangle = 0$ for any vector $v \in \mathcal{L}$. Moreover, the orthogonal space $\mathcal{L}^\perp$ is a subspace of the differential vector space $\mathcal{D}^*$, i.e., $\mathcal{L}^\perp \subset \mathcal{D}^*$.

Over the differential field $\mathcal{K}$, the state space, input space, output space, exogenous signal space, and error output space of the nonlinear system (1)–(4), denoted by $\mathcal{X}, \mathcal{U}, \mathcal{Y}, \mathcal{W}, \mathcal{E}$, respectively, are defined as follows:

$$\mathcal{X} = \text{span}_\mathcal{K} \left\{ \frac{\partial}{\partial x_i} ; i \in \mathbb{N} \right\}$$  

$$\mathcal{U} = \text{span}_\mathcal{K} \left\{ \frac{\partial}{\partial u_l} ; i \in \mathbb{M}, k \geq 0 \right\}$$  

$$\mathcal{Y} = \text{span}_\mathcal{K} \left\{ \frac{\partial}{\partial y_j} ; i \in \mathbb{N}, k \geq 0 \right\}$$  

$$\mathcal{W} = \text{span}_\mathcal{K} \left\{ \frac{\partial}{\partial u_l} ; i \in \mathbb{I} \right\}$$  

$$\mathcal{E} = \text{span}_\mathcal{K} \left\{ \frac{\partial}{\partial x_i} ; i \in \mathbb{I}, k \geq 0 \right\}.$$  

The dual space, dual input space, dual output space, dual exogenous signal space, and dual error output space of the system (1)–(4) over $\mathcal{K}$, denoted by $\mathcal{X}^*, \mathcal{U}^*, \mathcal{Y}^*, \mathcal{W}^*, \mathcal{E}^*$, respectively, are defined as

$$\mathcal{X}^* = \text{span}_\mathcal{K}\{ dx_i ; i \in \mathbb{N} \}$$  

$$\mathcal{U}^* = \text{span}_\mathcal{K}\{ du_j, dy; i \in \mathbb{M}, k \geq 0 \}$$  

$$\mathcal{Y}^* = \text{span}_\mathcal{K}\{ dy_i; i \in \mathbb{N}, k \geq 0 \}$$  

$$\mathcal{W}^* = \text{span}_\mathcal{K}\{ dv_i ; i \in \mathbb{I} \}$$  

$$\mathcal{E}^* = \text{span}_\mathcal{K}\{ dx_i, dy; i \in \mathbb{I}, k \geq 0 \}.$$  

**Remark 3.2:**

a) When the system is linear, the dual spaces defined above are equivalent to the corresponding dual spaces of the linear system over the real field defined in Section III-A. Thus the dual spaces defined for nonlinear systems over $\mathcal{K}$ are consistent with those for linear systems and are extensions of the linear system results to nonlinear systems.

b) The state space $\mathcal{X}$ as defined in (14) can be described using the ideas of differential geometry. Over an open set $\mathcal{X} \subset \mathbb{R}^n$, the state variables $x_1, x_2, \ldots, x_n$ establish a coordinate system over $\mathcal{X}$ and form a differential manifold. Thus the space $\mathcal{X}$ represents the tangent bundle over the differential manifold. Consequently, a subspace of $\mathcal{X}$ forms a distribution over $\mathcal{X}$, and a vector in $\mathcal{X}$ is a vector field over $\mathcal{X}$.

c) For a differential manifold in the local coordinate system over $\mathcal{X}$, the dual state space $\mathcal{X}^*$ presents the cotangent bundle over the manifold since the dual bases of $\mathcal{X}^*$ and $\mathcal{X}$, as defined in (16) and (14), respectively, are orthonormal. Further, a subspace of $\mathcal{X}^*$ forms a codistribution, and a vector in $\mathcal{X}^*$ is a differential one-form in the dual state space $\mathcal{X}^*$. The integrability of $\mathcal{X}^*$ follows from the well-known Frobenius theorem, which states that a nonsingular $\eta$-dimensional subspace $\mathcal{X}^*$ of $\mathcal{X}^*$ is integrable if there exist $\eta$ analytic functions $\phi_i(x), k \in \mathbb{N}_\eta$, such that $\mathcal{X}^* = \text{span}_{\mathcal{K}(\mathcal{X})}\{ d\phi_1, d\phi_2, \ldots, d\phi_\eta \}$. Specifically, a vector $\psi$ in $\mathcal{X}^*$ is called an exact one-form if there exists a function $\phi(x)$ of $x$ such that $\psi = d\phi(x)$ and a vector $\eta$ of exact one-forms is integrable.

It follows from the definitions of dual spaces that the dual state space, dual input space, and dual exogenous signal space of the system (1)–(4) are subspaces of the differential vector space $\mathcal{D}^*$, and these spaces satisfy

$$\mathcal{X}^* \oplus \mathcal{U}^* \oplus \mathcal{W}^* = \mathcal{D}^*.$$  

The dual output space and dual error output space of the system (1)–(4) are the vector spaces generated by the dual state space, dual input space, and dual exogenous signal space. Thus the dual output space and dual error output space are subspaces of the differential vector space $\mathcal{D}^*$, namely

$$\mathcal{Y}^* \subset \mathcal{X}^* \oplus \mathcal{U}^* \oplus \mathcal{W}^* = \mathcal{D}^*$$  

$$\mathcal{E}^* \subset \mathcal{X}^* \oplus \mathcal{U}^* \oplus \mathcal{W}^* = \mathcal{D}^*.$$  

We now define local regularity of a subspace of the differential vector space.

A finite-dimensional subspace $\mathcal{L}^*$ of the differential vector space $\mathcal{D}^*$ is a locally regular subspace if $\dim \mathcal{L}^* = \text{constant}$ in a neighborhood of the system equilibrium point in $\mathbb{R}^{n+(n \times \infty)+\mathbb{I}}$.

Throughout this paper and unless otherwise stated, we assume the following: each finite-dimensional subspace of the differential vector space of the nonlinear system under consideration satisfies the local regularity condition.

**Remark 3.3:**

a) The local regularity condition is a standard condition for nonlinear system analysis using the differential vector...
space approach [1], [8]. Since the differential vector space is defined over the meromorphic function field \( K \), this condition is necessary to deal with difficulties caused by singular points of the system. Under the local regularity condition, the results in this paper are generic for nonlinear system regulation but may not cover certain special cases that occur on singular (irregular) points of the nonlinear system.

b) The local regularity condition does not imply that there exist no globally regular subspaces in the differential vector space. In other words, the existence of globally regular subspaces in the differential vector space for the nonlinear system does not contradict the local regularity condition.

c) The system equilibrium point referred to in the definition of the local regularity is not restricted to the origin of the system.

IV. OBSERVABILITY IN THE DIFFERENTIAL VECTOR SPACE

For linear systems, observability is represented by a factor state space, which is defined by the unobservable subspace of the system [17]. Such a situation also exists for nonlinear systems, where the unobservable states are described by a non-singular involutive (or integrable) distribution [11]. We follow the results in [20] to show that in differential vector space, observability can be represented by an “observable” subspace in \( A^* \).

A. Observable Dual State Spaces

For the following nonlinear system without an exogenous signal:

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]  

(19)

we define observability as follows.

The system (19) is observable if for any two initial states \( x_0 \neq x_1 \) there exists a control \( u(t) \), \( t > 0 \) such that the system output satisfies

\[
g(t, x_0, u(t)) \neq g(t, x_1, u(t)), \quad \text{for } t \geq 0.
\]

The system (19) is strongly observable if for any state feedback control \( \alpha(x) \), the following closed-loop system is observable:

\[
\begin{align*}
\dot{x} &= f(x) + g(x)\alpha(x) + g(x)u_s \\
y &= h(x)
\end{align*}
\]  

(20)

where \( u_s \) is a new control input to the system.

The system (19) is absolutely observable if for any state feedback control \( \alpha(x) \) the following autonomous closed-loop system is observable:

\[
\begin{align*}
\dot{x} &= f(x) + g(x)\alpha(x) \\
y &= h(x).
\end{align*}
\]  

(21)

Remark 4.1: This definition of observability accounts for the dependence of system observability on the input matrix \( g(x) \) and the choice of control input \( u \). The definitions of strong observability and absolute observability account for the dependence of system observability on the state feedback control [20], [14].

We also need the following observability definitions for functions of the system state.

A function of the system state \( \varphi(x) \in K(x) \) is observable for system (19) if under the condition \( \varphi(x_0(0)) \neq \varphi(x_1(0)) \), for two initial states \( x_0(0) \) and \( x_1(0) \), there exists a control \( u(t) \) such that the system output satisfies

\[
g(t, x_0(0), u(t)) \neq g(t, x_1(0), u(t)).
\]

A function of the system state \( \varphi_a(x) \in K(x) \) is strongly observable for system (19) if for any state feedback control \( \alpha(x) \) it is observable for the closed-loop system (20).

A function of the system state \( \varphi_a(x) \in K(x) \) is absolutely observable for system (19) if for any state feedback control \( \alpha(x) \) it is observable for the autonomous closed loop system (21).

Remark 4.2: An observable function \( \varphi(x) \) can be regarded as a new coordinate system over \( R^n \). Thus \( \varphi(x) \) is equivalent to a state variable being observed from the output \( y \), and observability of \( \varphi(x) \) implies that any two states with different location in \( \varphi(x) \) can be distinguished from measurement of the output subject to a suitably chosen control input \( u \).

For system (19), let \( O \) be the set of all observable functions, \( O_s \) be the set of all strongly observable functions, and \( O_a \) be the set of all absolutely observable functions in \( K(x) \), namely

\[
\begin{align*}
O &= \{ \varphi(x) \in K(x); \varphi(x) \text{ is observable} \} \\
O_s &= \{ \varphi_a(x) \in K(x); \varphi_a(x) \text{ is strongly observable} \} \\
O_a &= \{ \varphi_a(x) \in K(x); \varphi_a(x) \text{ is absolutely observable} \}
\end{align*}
\]

The observable dual state space, strongly observable dual state space, and absolutely observable dual state space of the system (19), denoted by \( O^* \), \( O_s^* \), and \( O_a^* \), respectively, are defined as follows:

\[
\begin{align*}
O^* &= \text{span}_K \{ \delta \varphi(x); \varphi(x) \in O \} \\
O_s^* &= \text{span}_K \{ \delta \varphi_a(x); \varphi_a(x) \in O_s \} \\
O_a^* &= \text{span}_K \{ \delta \varphi_a(x); \varphi_a(x) \in O_a \}
\end{align*}
\]

We now state some fundamental results on observability of nonlinear systems in the dual space.

Proposition 4.1 [21]:

a) The observable dual state space of the system (19) is a subspace of the dual state space \( A^* \), i.e., \( O^* \subseteq A^* \), and is described by

\[
O^* = A^* \cap (J^* + u^*).
\]

b) The system (19) is observable if and only if

\[
O^* = A^* \subseteq J^* + u^*.
\]

Proposition 4.2 [20]: The observable dual state space, strongly observable dual space, and absolutely observable dual state space of the system (19) are subspaces of the system observable dual state space, and these spaces satisfy

\[
O_s^* \subseteq O_a^* \subseteq O^* \subseteq A^*.
\]
Proposition 4.3 [20]: A state feedback control $\alpha(x)$ with $\alpha(0) = 0$ and a nonsingular input transformation $u_s = \beta(x)u$ can be found such that the strongly observable dual state space of the system (19) is identical to the observable dual state space of the following closed-loop system

$$\dot{x} = f(x) + g(x)\alpha_0(x) + g(x)u_s$$
$$y = h(x),$$

(22)

Proposition 4.4 [20]: A state feedback control $\alpha(x)$ with $\alpha(0) = 0$ can be found such that the absolutely observable dual state space of the system (19) is identical to the observable dual state space of the following closed-loop system:

$$\dot{x} = f(x) + g(x)\alpha_0(x)$$
$$y = h(x).$$

(23)

Following from Propositions 4.3 and 4.4, the observable dual state space of system (22) is $\mathcal{O}_s^k$, and the observable dual state space of system (23) is $\mathcal{O}_a^k$, where $\mathcal{O}_s^k$ and $\mathcal{O}_a^k$ are, respectively, the strongly observable dual state space and absolutely observable dual state space of system (19).

Remark 4.3: It is shown in [20] that the strongly observable dual state space $\mathcal{O}_s^k$ can be related to the dual space of the maximal controlled invariant distribution in kernel $(dh)$ as defined in [11]. In differential geometry, such an invariant distribution can be interpreted as the dynamics associated with the maximal loss of observability [13]. Also, the notion of absolute observability can be associated with the notion of transmission zeros. In [20], it is shown that the absolutely observable dual state space can be related to the zero output constrained dynamics, as defined in [13].

For (22) in Proposition 4.3, the observable dual state space and strongly observable dual state space are identical. Based on this, a strongly observable canonical form for system (22) can be constructed using the dual spaces $\mathcal{X}^*$ and $\mathcal{O}_s^k$, as given in the following proposition.

Proposition 4.5 [20]: New state coordinates $\{\xi_1, \ldots, \xi_{n_0}, \xi_{n_0+1}, \ldots, \xi_n\}$, can be found for (22) such that the dual state space and the strongly observable dual state space of the system can be written as

$$\mathcal{X}^* = \text{span}_k\{d\xi_o, d\xi_u\}$$
$$\mathcal{O}_s^k = \text{span}_k\{d\xi_0\}$$

and under the new state coordinates, the system can be written in a strongly observable canonical form as

$$\begin{bmatrix}
\xi_o \\
\xi_u
\end{bmatrix} = \begin{bmatrix}
\psi_o(\xi_o) \\
\psi_u(\xi_o, \xi_u)
\end{bmatrix} + \begin{bmatrix}
0 \\
\phi_u(\xi_o, \xi_u)
\end{bmatrix} \begin{bmatrix}
u_0 \\
u_u
\end{bmatrix}$$

(24)

$$y = h(\xi_o)$$

(25)

where $(\xi_o, \xi_u)^T = (\xi_1, \ldots, \xi_{n_0}, \xi_{n_0+1}, \ldots, \xi_n)^T$, and $(\nu_o, \nu_u)^T$ are, respectively, the new system state and new control input for the closed-loop system.

B. Computation of Observable Dual State Spaces

The observable dual spaces of the system (19) and the state feedback controls $\alpha_0(x)$ and $\alpha_o(x)$ for (22) and (23) can be readily computed [20]. To compute the observable dual state space $\mathcal{O}^*$, write the vector field $g(x)$ and the output mapping $h(x)$ as $g(x) = [g_1(x), \ldots, g_m(x)]$ and $h(x) = [h_1(x), \ldots, h_n(x)]^T$. Let $\eta_k$, $k \geq 1$ be the vector fields belonging to the set \{f, g_j; j \in \mathbb{N}\}. Using these, a set of exact one-forms can be constructed as

$$\{dL_{\eta_1}L_{\eta_2} \ldots L_{\eta_k}h(x); k \geq 1, i \in \mathbb{N}\}.$$

Then, the observable dual state space can be obtained as

$$\mathcal{O}^* = \mathcal{X}^* \cap (\mathcal{U}^* + \mathcal{U}^k)$$
$$= \text{span}_k\{dL_{\eta_1}L_{\eta_2} \ldots L_{\eta_k}h_i; k \geq 1, i \in \mathbb{N}\}.$$

Since there are only a finite number of independent vectors in the set $\{dL_{\eta_1}L_{\eta_2} \ldots L_{\eta_k}h_i; k \geq 0, i \in \mathbb{N}\}$, the observable dual state space $\mathcal{O}^*$ can be computed in a finite number of steps.

Several approaches are available for computing the strongly observable dual state space. One approach is based on the duality between the strongly observable dual state space and the maximal controlled invariant distribution in kernel $(dh)$ [20], [11]. To compute the absolutely observable dual state space, duality between the absolutely observable dual state space and the zero output constrained dynamics [20], [13] can be used, and computation follows the procedure in [13]. In [20], two algorithms, called $SO$ and $AO$, are presented for computing the state feedback controls $\alpha_0(x)$ and $\alpha_o(x)$ in Propositions 4.3 and 4.4 above.

V. NONLINEAR SYSTEM REGULATION

This section presents our main results on nonlinear system regulation. We first present necessary and sufficient conditions for stable nonlinear system regulation subject to both bounded and unbounded exogenous signals. Then follows a sufficient condition for possibly unstable nonlinear system regulation subject to unbounded exogenous signals. This sufficient condition is also shown to be necessary under an additional condition on system observability. Finally, the conditions for regulation subject to possibly unbounded exogenous signals are refined to obtain a necessary and sufficient condition for nonlinear system regulation subject to bounded exogenous signals. All of these results are stated in terms of the dual spaces of the nonlinear system and make extensive use of the differential vector space theory in Sections III and IV.

A. Preliminaries

Without loss of generality, assume that the error output vector $e$ in (4) of the nonlinear system has $c$ independent variables written as $e = (e_1, e_2, \ldots, e_c)^T$. We order the differential vectors of the error output as follows:

$$de_1, de_2, \ldots, de_c, de_1, \ldots, de_c, \ldots, de_1^{(k)}, \ldots, de_1^{(k)}, \ldots$$

(26)

It can be shown that if a vector $de_i^{(j)}$ is dependent on the vectors to its left, then each vector $de_i^{(j+k)}$ with $k \geq 1$ is also depen-
dent on the vectors to its left. Thus we can choose a set of \( n_o \) independent vectors denoted by \( dz_i \) for \( i \in \mathbb{Z}_o \) from the left of (26) to form the dual error output space

\[
\mathcal{E}^* = \text{span}_K \{dz_1, dz_2, \ldots, dz_{n_o}\}.
\]  

(27)

This yields the augmented error output

\[
z_0 = (z_1, z_2, \ldots, z_{n_o})^T
\]

(28)

We assume, without loss of generality, that \( n_0 \leq n \). If \( n_0 < n \), we further choose a set of \( n_\mu = n - n_0 \) vectors denoted by \( dz_{n_\mu+1}, \ldots, dz_n \) from the set \( \{dx_1, \ldots, dx_n\} \), which are independent of \( \mathcal{E}^* \). Further assume, without loss of generality, that these independent vectors are the last \( n_\mu \) vectors in \( \{dx_1, \ldots, dx_n\} \). Thus, we have

\[
\dim \text{span}_K \{dz_1, \ldots, dz_{n_\mu}, dx_{n_\mu+1}, \ldots, dx_n\} = \dim \text{span}_K \{dz_1, \ldots, dz_{n_\mu}, dx_{n_\mu+1}, \ldots, dx_n\} = n.
\]

(29)

We use this to define the augmented error state of the nonlinear system as

\[
z = (z_0^T, z_0^T)^T
\]

(30)

where \( z_0 \) is called the internal state of the system and written as

\[
z_0 = (x_{n_\mu+1}, x_{n_\mu+2}, \ldots, x_n)^T = (z_{n_\mu+1}, z_{n_\mu+2}, \ldots, z_n)^T.
\]

(31)

Suppose \( Z_o \subset \mathbb{R}^{n_\mu} \) and \( Z_\mu \subset \mathbb{R}^{n_\mu} \) are neighborhoods of the origin, in the sense of the induced topology, in \( \mathbb{R}^n \). We now state the problem of nonlinear system regulation for unbounded exogenous signals as follows.

The nonlinear system (1)–(4) is locally regulated if a full state feedback control \( u = \alpha(x, w) \) can be found such that for any initial conditions of the augmented error output \( z_0(0) \in Z_0 \) and the internal state \( z_\mu(0) \in \mathbb{R}^{n_\mu} \):

1) The closed-loop system \( \dot{z} = f(x) + g(x)\alpha(x, 0) \) with \( w = 0 \) is asymptotically stable;

2) The error output \( e \) satisfies \( \lim_{t \to \infty} e(t) = 0 \), \( j \geq 0 \).

For the problem of nonlinear system regulation for bounded exogenous signals, the above statement can be relaxed as follows.

The nonlinear system (1)–(4) is locally regulated if a full state feedback control \( u = \alpha(x, w) \) can be found such that a) and b) of the above statement hold locally for any initial conditions of the augmented error output \( z(0) \in Z_0 \) and the internal state \( z_\mu(0) \in Z_\mu \).

Remark 5.1:

a) The problem statement for unbounded exogenous signals is in terms of the augmented error output \( z_0 \) and the internal state \( z_\mu \). Local regulation is with respect to the augmented error output \( z_0 \) and requires the initial condition \( z_0(0) \) to be small, which implies that \( e(0) = h(x(0)) - q(w(0)) \) is small. Under this local condition, the corresponding continuous exogenous signal \( w \) and state \( x \) are constrained by the error output equation \( e = h(x) - q(w) \). Thus the ranges of \( w \) and \( x \) are constrained to lie in the region where the error output is small. In this sense, \( w \) and \( x \) are not global and do not range over the whole space \( \mathbb{R}^{n \times n} \), although they are not necessarily bounded in a neighborhood of the origin.

b) For unbounded exogenous signals, the meaning of asymptotic stability in the problem statement is twofold. With respect to \( z_\mu \), which is a function of the continuous exogenous signal \( w \) and state \( x \), the stability is local and requires \( z_\mu(0) \in Z_\mu \). With respect to \( z_0 \), which is independent of the augmented error output \( z_\mu \), the stability is global for \( z_0 \in \mathbb{R}^{n_\mu} \). This global condition accounts for the unbounded exogenous signal case and is an extension of the standard local stability condition [12] for nonlinear system regulation in the bounded exogenous signal case. Moreover, this global condition is an additional condition to the local regularity condition. It implies that \( \text{span}_K \{dz_{n_\mu+1}, \ldots, dz_n\} \) is a globally regular subspace in \( \mathbb{R}^n \), and it does not contradict the local regularity condition.

c) For bounded exogenous signals, the stability with respect to \( z_\mu \) is local that requires \( z_\mu(0) \) to be bounded in a neighborhood of the origin. This local stability problem statement follows from that in [12].

Since the solution for regulation is an asymptotic result on the error output, a trivial case is the exogenous signal asymptotically converging to zero. For linear system regulation, this situation is excluded without loss of generality by assuming that the exosystem is unstable with all its poles in the closed right-half part of the complex plane [7]. For nonlinear system regulation, we now define areas of attraction and normal functions and use these to present assumptions that are equivalent to the unstable exosystem assumption used in the linear case.

A regular submanifold of the exosystem \( \mathcal{L} = \mathcal{L}(w) \), denoted by \( \mathcal{M} \), through the origin of the state space is defined as an area of attraction if for any initial state \( z(0) \notin \mathcal{M} \) the state trajectory \( \psi(t) \) of the system always converges to the submanifold, i.e.,

\[
\lim_{t \to \infty} \|w(t) - \mathcal{M}\| = 0.
\]

A function \( \eta(x, w) \) is a normal function of the state \( (z_\mu, x) \) if \( \eta(x, w) = 0 \) defines an \( n + r - 1 \)-dimensional regular submanifold of \( \mathbb{R}^{n+r} \).

The state of the exosystem is bounded if \( w = 0 \) is a stable equilibrium, in the sense of Lyapunov stability, of the vector field \( s(\mathcal{L}) \).

Remark 5.2: Not every analytic function of \( x \) and \( w \) defines a regular submanifold of \( \mathbb{R}^{n+r} \). For example, functions satisfying \( \eta(0, w) \neq 0 \) for bounded \( w \) and \( \eta(0, w) = 0 \) when \( w \) is infinitely large, such as \( \eta(x, w) = (\sum_{i=1}^r w_i)/(1 + \sum_{i=1}^r w_i^2) \), are not normal functions.

We now use the preceding definitions to present assumptions on the exosystem and error output. These will be used for the regulation results in Theorems 5.1–5.4.

Assumption 5.1 (Regulation Under Unbounded Exogenous Signals):

a) The dual error output space in (27) is a locally regular subspace of the differential vector space, i.e., \( \dim \mathcal{E}^* = n_o \) for all possible \( x \in \mathbb{R}^n \) and \( w \in \mathbb{R}^r \) such that \( z_o \in Z_o \).
b) Each \( z_i(x, w) \), for \( i \in \mathbb{N}_0 \), of the augmented error output \( z_0 \) is a normal function of the system state \( x \) and the exogenous signal \( w \).

c) The nonlinear exosystem (2) has no areas of attraction.

Assumption 5.2 (Regulation Under Bounded Exogenous Signals): The nonlinear exosystem (2) has bounded state \( w \) and has no areas of attraction.

Remark 5.3:

a) Assumption 5.1a) concerns local regularity of the dual error output space \( \mathcal{E}^* \) and takes into account the local condition for the augmented error output \( z_0 \in \mathbb{Z}_0 \) and the unboundedness of the exogenous signal \( w \) and system state \( x \).

b) The condition on the area of attraction in Assumption 5.1c) implies that each variable \( w_i \), for \( i \in \mathbb{N}_0 \) of the exogenous signal \( w \) remains away from zero asymptotically and may become unbounded. Assumption 5.1b) and c) together imply that each function \( z_i(0, w) \) with \( i \in \mathbb{N}_0 \) of \( z_0 \) remains away from zero asymptotically for any nonzero bounded or unbounded \( w \). Thus Assumption 5.1b) and c) remove the trivial possibility of regulation where the contribution of a nonzero exogenous signal to the error output asymptotically becomes zero. Assumption 5.1b) and c) are only used for establishing necessary conditions for nonlinear system regulation. They are not used for establishing sufficient conditions. When the system is linear, these assumptions are precisely equivalent to assuming that all the poles of the exosystem are in the closed right-half part of the complex plane [7].

c) Assumption 5.2 implies that all eigenvalues of the matrix \( \frac{\partial \bar{t}(0)}{\partial w} \) are on the imaginary axis.

To facilitate derivation of the main results, we now present two technical lemmas.

Lemma 5.1 [15]: If \( f(t) \) is a uniformly continuous function, such that \( \lim_{t \to \infty} \int_0^t f(s) \, ds \) exists and is finite, then \( f(t) \to 0 \) as \( t \to \infty \).

For the zero input \( u = 0 \), write the system (1)–(4) as

\[
\begin{align*}
\dot{z} &= f(x) + p(x)w \\
\dot{w} &= s(w) \\
\dot{e} &= h(x) + q(w).
\end{align*}
\]

With respect to the error output \( e \), we suppose that the observable dual state space of the system (32) is \( \mathcal{O}^* \). We have the following lemma.

Lemma 5.2: The observable dual state space of the autonomous system (32) is identical to the dual error output of the system, i.e., \( \mathcal{O}^* = \mathcal{E}^* \).

Proof: The dual state space of the overall system is \( \mathcal{E}^* \oplus \mathcal{W}^* \). Since \( u = 0 \), the observable dual state space is \( \mathcal{O}^* = \mathcal{E}^* \cap (\mathcal{X}^* \oplus \mathcal{W}^*) \). Then the result follows from \( \mathcal{E}^* \subseteq \mathcal{X}^* \oplus \mathcal{W}^* \).

B. Nonlinear System Regulation for Stable Systems

This subsection presents results on stable nonlinear system regulation, respectively, for bounded and unbounded exogenous signals.

Theorem 5.1 (Stable Systems and Unbounded Exogenous Signals): Suppose that the nonlinear system (32) satisfies Assumptions 5.1 and is asymptotically stable for \( w = 0 \) and any initial conditions \( z_0(0) \in \mathbb{Z}_0 \) and \( z_0(0) \in \mathbb{R}^{n_u} \). Then the system is locally regulated if and only if

\[
\mathcal{O}^* \cap \mathcal{W}^* = \emptyset.
\]

Proof:

a) Sufficiency: The condition \( \mathcal{O}^* \cap \mathcal{W}^* = \emptyset \) and the result of Lemma 5.2 lead to \( \mathcal{E}^* \cap \mathcal{W}^* = \emptyset \). Since the order of the whole system is \( n + r \), we can use (33), (29), and Assumption 5.1a) to form a set of new state coordinates \( \{z_1, \ldots, z_{n_0}, z_{n_0+1}, \ldots, z_n, w_1, \ldots, w_r\} \), which satisfies

\[
\text{rank } \frac{\partial (z_1(x, w), \ldots, z_n(x, w))}{\partial x} = n_*.
\]

b) Necessity: If \( \mathcal{O}^* \cap \mathcal{W}^* = \emptyset \) with \( \mathcal{E}^* \) in the form (27), the system (32) can be transformed into the following realization:

\[
\begin{align*}
\dot{z}_0 &= \Psi_0(z_0) \\
\dot{z}_1 &= \Psi_1(z_0, z_1, w) \\
w &= s(w) \\
e &= (z_1, \ldots, z_{n_0})^T.
\end{align*}
\]

This system is asymptotically stable for \( w = 0 \) and any initial conditions \( z_0(0) \in \mathbb{Z}_0 \) and \( z_0(0) \in \mathbb{R}^{n_u} \). Hence the assumption that the system is regulated is not true.

Remark 5.4:

a) For the autonomous nonlinear system (32) with the full state \( (x, w) \) and error output \( e \), the necessary and sufficient condition (33) is an intersection of the observable dual
state space and the dual exogenous signal space. When the exogenous signal \( w \) is viewed as a substate of the system (32) and following from the results on the observable dual state space in Proposition 4.1, condition (33) implies that the substate \( w \) is not observable and is decoupled from the error output \( e \). Thus the error output converges to zero while the exogenous signal is nonzero and possibly unbounded. Hence, condition (33) relates the problem of regulation to system observability in the differential vector space and provides a clear interpretation of the regulation result.

b) Theorem 5.1 shows that the necessary and sufficient condition for regulation is equivalent to examining the dependence of the dual error output space on the dual exogenous signal space. While local regularity of the dual error output space requires that the augmented error state is in a neighborhood of the origin, the intersection of the dual error output space and the dual exogenous signal space is independent of the boundedness of the system state \( x \) and the exogenous signal \( w \). Thus conditions for nonlinear system regulation with unbounded exogenous signals can be established using the differential vector space approach. In contrast, the well-known center manifold approach requires the system state \( x \) and the exogenous signal \( w \) to be bounded.

c) Theorem 5.1 expressed in terms of the intersection between the dual error output space and the dual exogenous signal space is a constructive result in the sense that it provides a computational procedure for testing if regulation is possible. Since the observable dual state space is integrable, the dependence of the observable dual state space on the dual exogenous signal space can be checked by a finite number of derivative operations on the error output \( e \).

Theorem 5.2 (Stable Systems and Bounded Exogenous Signals): Suppose that the nonlinear system (32) satisfies Assumption 5.2 and is locally asymptotically stable for any \( z_0(0) \in Z_0 \) and \( z_0(0) \in Z_u \). Then the system is locally regulated if and only if

\[
\{F^* \cap \mathcal{W}\} \big|_{z=0} = \{E^* \cap \mathcal{W}\} \big|_{z=0} = 0 \tag{37}
\]

where \( \mathcal{O}^* \cap \mathcal{W}\) is the transversal section of the cotangent bundle \( \mathcal{O} \cap \mathcal{W}\) at \( z = 0 \).

Proof:

a) Sufficiency: Under the new state coordinates \( \{z_1, \ldots, z_{n_e}, z_{n_e+1}, \ldots, z_n, w_1, \ldots, w_r\} \), we express system (32) using a Taylor series expansion in terms of \( z = (z_0^T, z_w^T)^T \) about \( z = 0 \) as

\[
\begin{align*}
z & = Az + \psi(z, w) \\
\dot{w} & = s(w) \\
e & = h_z(z) = (z_1, \ldots, z_n)^T 
\end{align*}
\tag{38}
\]

where \( A \in \mathbb{R}^{n \times n} \) is a constant matrix representing the linearized part of the system in terms of \( z \), and \( \psi(z, w) \) represents the higher order terms in \( z \). The higher order term \( \psi(z, w) \) can be further written as

\[
\psi(z, w) = \psi_0(w) + \psi_1(z, w)z
\]

where \( \psi_0(w) \) satisfies \( d\psi_0(w) \not\in \mathcal{W}^* \). It can be shown that \( \psi_0(w) \equiv 0 \) if and only if (37) is satisfied. Therefore, \( \psi(0, 0) = 0 \) and \( (\partial \psi_0(0, w)/\partial w) = 0 \) under condition (37).

Under condition (37) and by the theorem on existence of the center manifold [3, 11], there exists a center manifold of system (38) defined by \( z = \pi(w) \), which passes through \((\pi(0), (\partial \pi/\partial w)(0)) = (0, 0)\). Such a center manifold is actually a zero output invariant manifold in the sense that, for \( c(w) = h_z(\pi(w)) \), if the system state is on the center manifold, then \( c(0) = h_z(\pi(0)) = 0 \) and \( \delta(0) = (\partial h_z/\partial z)\pi(0) = 0 \) are satisfied. It follows that \( \delta^k(0) = 0 \) for any \( k > 1 \). Further, the result on the zero output invariant manifold [11] can be applied to \( c = h_z(\pi(w)) \) to show that the system is locally regulated.

b) Necessity: If condition (37) is not satisfied, then \( \psi(0, w) = \psi_0(w) = 0 \) defines a regular submanifold \( \mathcal{M}_0 \) through the origin. Assume the system is regulated such that \( z(t) \) converges to zero. Then \( z(t) \) converges to zero by Lemma 5.1. It follows from (38) that \( w(t) \) converges to \( \mathcal{M}_0 \) and \( \mathcal{M}_0 \) becomes an area of attraction of the exosystem. However, this contradicts Assumption 5.2 that the exosystem has no areas of attraction. Hence the assumption that the system is regulated is not true.

Remark 5.5: When the exogenous signal is bounded, the necessary and sufficient condition for regulation is relaxed to (37). This condition applies to the intersection of \( \mathcal{O}^* \) and \( \mathcal{W}^* \) only at the specific point \( z = 0 \) and therefore, does not require local regularity of \( \mathcal{E}^* \). Such a result is achieved since the theorem on existence of the center manifold used in the proof is applicable to (38) at \( z = 0 \) for bounded \( w \).

C. Nonlinear System Regulation for Unstable Systems and Unbounded Exogenous Signals

We now consider the problem of nonlinear system regulation when the system can be unstable and the exogenous signal is possibly unbounded. Let \( \mathcal{O}_{s}^* \) and \( \mathcal{O}_{a}^* \) be, respectively, the strongly and absolutely observable dual state spaces of system (1)–(4) with respect to the error output. Proposition 4.3 can be applied to find a state feedback control \( u = \alpha_s(x, w) + u_b \) such that the observable dual state space of the system

\[
\begin{align*}
\dot{x} & = f(x) + g(x)\alpha_s(x, w) + p(x)w + q(x)u_b \\
\dot{w} & = s(w) \\
e & = h(x) + q(w)
\end{align*}
\tag{39}
\]

is identical to \( \mathcal{O}_{s}^* \). Suppose that \( \mathcal{O}_{s}^* \) satisfies \( \mathcal{O}_{s}^* \cap \mathcal{W}^* = 0 \). Then Proposition 4.5 can be applied to transform system (39) into strongly observable canonical form under the new state coordinates \( \{z_2, \ldots, z_{n_e}, z_{n_e+1}, \ldots, z_n, w_1, \ldots, w_r\} \) as

\[
\begin{align*}
z_0 & = \psi_0(z_0) + \phi_{x_0}(z_0)u_0 \\
\dot{z}_u & = \psi_1(z_0, z_u, w) + \phi_w(z_0, w)u_0 + \phi_{uu}(z_0, z_u, w)u_u \\
\dot{w} & = s(u) \\
e & = h_z(z_0) = (z_1, \ldots, z_n)^T
\end{align*}
\tag{40}
\]

where \( z_0 \) is the augmented error output and \( z_u \) is the internal state of the closed loop system (40), and they are in the form of (28) and (31), respectively. For the strongly observable canonical form (40) of the nonlinear system (1)–(4), we make the following assumption.
Assumption 5.3: For $w = 0$, there exists a neighborhood $Z_0 \in \mathbb{R}^{n_x}$ such that the subsystem $z_\alpha = \psi_\alpha(z_0) + \phi_\alpha(z_0)u_\alpha$ is locally asymptotically stabilizable for any $z_\alpha(0) \in Z_0$ and the subsystem

$$z_\alpha = \psi_\alpha(0, z_\alpha, 0) + \phi_\alpha(0, z_\alpha, 0)u_\alpha$$

is globally asymptotically stabilizable for any $z_\alpha(0) \in \mathbb{R}^{n_x}$.

Remark 5.6: The subsystem (41) represents a type of zero dynamics of the nonlinear system (39). Thus Assumption 5.3 requires the zero dynamics of (39) to be globally asymptotically stablizable when $w = 0$. It is noted that the globally asymptotic stabilizability condition in Assumption 5.3 implies that the subspace $\text{span} \{d_{z_{n_x+1}}, \ldots, d_{z_n}\}$ is a globally regular subspace in $\mathcal{D}^\ast$. We have clarified in Remark 5.1b) the meaning of global stabilization with respect to $z_{\alpha}(0) \in \mathcal{R}$.

The following theorem first presents a sufficient condition for nonlinear system regulation subject to a possibly unbounded exogenous signal and then goes on to show that the sufficient condition is also necessary under an additional condition on system observability.

Theorem 5.3 (Unstable Systems and Unbounded Exogenous Signals):

a) Suppose that the nonlinear system (1)–(4) satisfies Assumption 5.1a) and b) and Assumption 5.3. Then the system is locally regulated if the strongly observable dual state space of the system satisfies

$$\mathcal{O}_a^\ast \cap \mathcal{W}^\ast = \emptyset.$$  

b) Suppose that the nonlinear system (1)–(4) further satisfies $\mathcal{O}_a^\ast = \mathcal{O}_a^\ast$ and Assumption 5.1c). Then the system is locally regulated only if (42) is satisfied.

Proof: 

a) Above it is shown that a state feedback control can be applied to system (1)–(4) to obtain the strongly observable canonical form (40) under condition (42). By Assumption 5.3, there exist state feedback controls $u_\alpha(z_\alpha)$ and $u_\alpha(z_\alpha, w)$ and a neighborhood $Z_0 \subset Z_0$ for system (40) such that the following system with $w = 0$ is asymptotically stable for any $z_\alpha(0) \in Z_0 \subset Z_0$ and $z_\alpha(0) \in \mathbb{R}^{n_x}$

$$z_\alpha = \psi_\alpha(z_\alpha) + \phi_\alpha(z_\alpha)u_\alpha(z_\alpha)$$

$$z_\alpha = \psi_\alpha(z_\alpha, z_\alpha, 0) + \phi_\alpha(z_\alpha, z_\alpha, 0)u_\alpha(z_\alpha)$$

$$+ \phi_\alpha(u_\alpha(z_\alpha, z_\alpha, 0), 0)u_\alpha(z_\alpha, 0).$$

Since system (40) is a transformation of system (39) under the new state coordinates, the stabilizing state feedback controls $u_\alpha(z_\alpha)$ and $u_\alpha(z_\alpha, w)$ can be transformed to obtain the corresponding state feedback control $u_\alpha = \alpha_\alpha(z, w)$ for system (39) under the original state coordinates $\{x_1, \ldots, x_{n_x}, u_1, \ldots, u_n\}$. In particular, it is noted that the internal state satisfies $z_{n_x+i} = x_{n_x+i}$ for $i \in n_\alpha$. Thus the closed-loop system can be written as

$$\dot{x} = f(x) + g(x)(\alpha_\alpha(x, w) + \alpha_\alpha(x, w))$$

$$\dot{w} = s(w)$$

$$e = h(x) + q(w).$$

For $w = 0$, the closed-loop system is asymptotically stable for all $x$ such that $z_\alpha(0) \in Z_0$ and $z_\alpha(0) \in \mathbb{R}^{n_x}$.

Let $\mathcal{O}_a^\ast$ be the observable dual state space of the closed-loop system (44) with respect to the error output $e$. Then $\mathcal{O}_a^\ast \subseteq \mathcal{O}_a^\ast$ is satisfied since $u_\alpha(z_\alpha)$ is not a function of $z_\alpha$. It follows from (42) that $\mathcal{O}_a^\ast \cap \mathcal{W}^\ast \neq \emptyset$. Hence, the system is locally regulated following from the sufficiency result of Theorem 5.1.

b) Under the condition that the strongly and absolutely observable dual spaces $\mathcal{O}_a^\ast$ and $\mathcal{O}_a^\ast$ of the nonlinear system (1)–(4) are identical, it follows from Lemma 5.2 that the dual error output space $\mathcal{E}^\ast$ of the closed-loop system (44) is the observable dual state space of the system and satisfies $\mathcal{E}^\ast = \mathcal{O}_a^\ast = \mathcal{O}_a^\ast$.

If (42) is not satisfied such that $\mathcal{E}^\ast \cap \mathcal{W}^\ast = \mathcal{O}_a^\ast \cap \mathcal{W}^\ast \neq \emptyset$, it follows from the necessity result of Theorem 5.1 that the system is not regulated.

Remark 5.7: Theorem 5.3 extends the result of Theorem 5.1 to unstable nonlinear systems. The condition of Theorem 5.3 i) is only sufficient but not necessary since it is possible that the observable dual state space $\mathcal{O}_a^\ast$ of the closed-loop system satisfies $\mathcal{O}_a^\ast \subseteq \mathcal{O}_a^\ast$ and $\mathcal{O}_a^\ast \cap \mathcal{W}^\ast = \emptyset$. In this case, the system is regulated by Theorem 5.1 without satisfying (42). Under the conditions of Theorem 5.3b), the observable dual state space satisfies $\mathcal{O}_a^\ast = \mathcal{O}_a^\ast$ and (42) becomes a necessary and sufficient condition.

D. Nonlinear System Regulation for Unstable Systems and Bounded Exogenous Signals

Our result on nonlinear system regulation for bounded exogenous signals requires the following condition for linearization of the nonlinear system.

Assumption 5.4: For $w = 0$, the linear approximation (5) of the nonlinear plant (1) at the origin $x = 0$ is stabilizable.

The following result is in terms of the absolutely observable dual state space $\mathcal{O}_a^\ast$ of the nonlinear system (1)–(4) with respect to the error output $e$.

Theorem 5.4 (Unstable Systems and Bounded Exogenous Signals): Suppose that the nonlinear system (1)–(4) satisfies Assumptions 5.2 and 5.4. Then the system is locally regulated if and only if

$$\mathcal{O}_a^\ast \cap \mathcal{W}^\ast = \emptyset.$$  

Proof: 

a) Sufficiency: From Proposition 4.4, a state feedback control $u = \alpha_\alpha(x, w) + u_\alpha$ can be found and applied to the nonlinear system (1)–(4), yielding

$$\dot{x} = f(x) + g(x)\alpha_\alpha(x, w) + \alpha_\alpha(x, w)$$

$$\dot{w} = s(w)$$

$$e = h(x) + q(w).$$

For $u_\alpha = 0$, the observable dual state space of the system (46) is $\mathcal{O}_a^\ast$, and the dual error output space of the system is $\mathcal{E}^\ast$ in the form (27). By Lemma 5.2, $\mathcal{E}^\ast = \mathcal{O}_a^\ast$.

Let $z_\alpha = (z_1, \ldots, z_{n_x})^r$ and $z_\alpha = (z_{n_x+1}, \ldots, z_n)^r$. Under (45) and the new state coordinates...
The system (46) can be transformed into
\[
\begin{align*}
\dot{z}_a &= \psi_a(z_a) + \phi_a(z_a, z_b, w)\tilde{u}_a \\
\dot{z}_b &= \psi_b(z_a, z_b, w) + \phi_b(z_a, z_b, w)\tilde{u}_a \\
\dot{w} &= s(w) \\
e &= (z_1, \ldots, z_c)^T
\end{align*}
\]  
(47)
where \(\tilde{u}_a = \beta_a(x, w)u\) is the transformed new control input and \(\beta_a(x)\) is a nonsingular input transformation matrix.

Applying a Taylor series expansion to (47) in terms of \(z_a, z_b,\) and \(w\) at the origin, we get
\[
\begin{align*}
\dot{z}_a &= A_{oa}z_a + B_{oa}\tilde{u}_a + \hat{\psi}_a(z_a) + \hat{\phi}_a(z_a, z_b, w)\tilde{u}_a \\
\dot{z}_b &= A_{ob}z_ao + A_{ob}z_b + A_{ob}w + B_{ob}\tilde{u}_a + \hat{\psi}_b(z_a, z_b, w) \\
\dot{w} &= S(w) + \tilde{s}(w) \\
e &= (z_1, \ldots, z_c)^T
\end{align*}
\]  
(48)
where the linear approximation of the system can be represented by the following constant matrix with appropriate dimension
\[
\begin{pmatrix}
A_{oa} & 0 & B_{oa} \\
A_{ob} & A_{ob} & A_{ob}w + B_{ob}\tilde{u}_a + \hat{\psi}_b(z_a, z_b, w) \\
C & 0 & 0
\end{pmatrix}
\]  
(49)
Since Assumption 5.4 is satisfied

\[
\begin{pmatrix}
A_{oa} & 0 \\
A_{ob} & A_{ob}
\end{pmatrix}
\]  

\[
\begin{pmatrix}
B_{oa} \\
0 & S & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

is a stabilizable pair. It follows that the well-known decoupling technique in [7] can be applied to (48). This technique involves finding a linear feedback control of the form \(\tilde{u}_a = F_a z_a + F_b z_b + \tilde{u}_a\) for (48) and then applying a state transformation
\[
\begin{pmatrix}
z_a \\
z_b
\end{pmatrix} = P \begin{pmatrix}
z_a \\
z_b
\end{pmatrix}
\]
with a nonsingular matrix \(P.\) As a result, the system is transformed into
\[
\begin{align*}
\dot{x} &= \hat{A}x + \hat{B}m_a + \hat{\psi}(x, w) + \hat{\phi}(x, w)m_a \\
\dot{w} &= Sc(w) + \hat{s}(w) \\
e &= (z_1, \ldots, z_c)^T
\end{align*}
\]  
(50)
where \(x\) is the transformed substate, the constant pair \((\hat{A}, \hat{B})\) is stabilizable, and the higher order term \(\hat{\psi}(x, w)\) can be written as
\[
\hat{\psi}(x, w) = \hat{\psi}_a(x) + \hat{\psi}_b(x, w)
\]  
(51)
with \(\hat{\psi}_a(x)\) being a higher order term in \(x\). The linear approximation of (50) can be represented by the following constant matrix:
\[
\begin{pmatrix}
\hat{A} & 0 \\
0 & S & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  
(52)
The matrix (52) shows that the linear approximation of the system state \(x\) is decoupled from the exogenous signal \(w.\)
$\dot{x}_1 = x_1 x_4 + u_2 + u_2$ \hspace{1cm} (54)

with unstable exosystem

$\dot{w}_1 = w_1$
$\dot{w}_2 = w_1$.
(55)

The error output of the system is defined as

$e_1 = x_1 - u_1$
$e_2 = x_2 - u_2$.
(56)

The overall nonlinear system is of sixth order with the full state and error output $e$.

We first apply algorithm SO [20] to obtain the strongly observable canonical form of the system. Introduce a state transformation with the new state vector $z = (z_1, z_2, z_3, z_4)^T = (e_1, e_2, x_3, x_4)^T = (x_1 - w_1, x_2 - w_2, x_3, x_4)^T$. Under this transformation, the system (54)–(56) can be written as

$\begin{pmatrix} x \\ w \end{pmatrix} = (x_1, x_2, x_3, x_4, w_1, w_2)^T$ \hspace{1cm} (57)

Applying the following state feedback control obtained from SO

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -z_4 + w_1 \\ 0 \end{pmatrix} + \begin{pmatrix} u_o \\ u_a \end{pmatrix}$ \hspace{1cm} (58)

to (57) yields the closed-loop system

$\dot{z}_1 = z_3 + (z_3 + 1) u_0$
$\dot{z}_2 = z_1 + u_0$
$\dot{z}_3 = -z_2$
$\dot{z}_4 = z_2 z_1 + z_4 w_1 + w_2 + u_0$
$\dot{w}_1 = w_1$
$\dot{w}_2 = w_1$
$e_1 = z_1$
$e_2 = z_2$. \hspace{1cm} (59)

The system is now in observable canonical form (43). Direct computation of the strongly observable dual state space for (59) yields

$\mathcal{O}^* = \text{span}\{d_21, d_22, d_23\}$

and $\mathcal{O}^*$ satisfies

$\mathcal{O}^* \cap \mathcal{W}^* = 0$. \hspace{1cm} (60)

Thus, the system augmented error output is $z_0 = (z_1, z_2, z_3)^T$, and the internal state is $z_u = z_4$. It is easily verified that system (59) satisfies Assumption 5.3 on system stability. Therefore, the system satisfies the sufficient conditions for regulation in Theorem 5.3 a). It can be further verified that the strongly observable dual state space and the absolutely observable dual state space of the system are identical, i.e., $\mathcal{O}^*_a = \mathcal{O}^*_b$, and the state vector $z$ satisfies Assumption 5.1c). Hence, condition (60) is also the necessary condition for regulation by Theorem 5.3 b).

For $w = 0$, the subsystem consisting of the first three equations in (59) is locally asymptotically stabilizable. For example, $u_o = z_1 - 3z_2 + 3z_3$ stabilizes the subsystem. For $w = 0$ and $z_0 = 0$, the subsystem

$\dot{z}_4 = z_3 z_4 + z_4 u_1 + w_2 + u_2$

representing the system internal state is globally asymptotically stabilizable, for example, by $u_4 = -z_4 - z_4 u_1 - w_2$. Therefore, with $w = 0$, (59) is asymptotically stabilized by

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -z_4 - z_4 u_1 - w_2 \\ -w_4 - x_4 u_1 - w_2 \end{pmatrix}$. \hspace{1cm} (61)

Substituting this control into (58) yields

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 + 3x_3 - x_4 + 3w_2 \\ -x_4 - x_4 u_1 - w_2 \end{pmatrix}$. \hspace{1cm} (61)

This state feedback control asymptotically stabilizes the nonlinear plant (54) for $w = 0$ and regulates the error output of the closed loop system about zero.

**Example 2**: This example was used in [11] to illustrate nonlinear system regulation subject to a bounded exogenous signal. Here we solve the same problem using the differential vector space approach.

Consider the third-order nonlinear plant, the exosystem, and the error output

$\dot{x}_1 = x_2$
$\dot{x}_2 = u$
$\dot{x}_3 = x_1 + x_3 + x_3$
$\dot{w}_1 = w_2$
$\dot{w}_2 = -w_1$
$e = x_1 - w_1$ \hspace{1cm} (61)

where the exosystem generates a sinusoidal reference signal.

Applying algorithm AO [20], the absolutely observable dual state space of system (61) can be formed by taking the derivatives of the error output as follows:

$\dot{c} = x_1 - w_1$
$\dot{c} = x_2 - w_2$
$\dot{c} = w_1 + u$

$\vdots$. \hspace{1cm} (62)

For this system, we let

$u = -w_1 + u_0$

to obtain the absolutely observable dual state space

$\mathcal{O}^*_a = \text{span}_k\{dc, \dot{dc}\}$. \hspace{1cm}
Now introduce a state transformation
\[ z = (z_1, z_2, z_3)^T = (c, \dot{c}, x_3)^T = (x_1 - w_1, x_2 - w_2, x_3)^T \]
to transform the system into the form (48) as
\[ \begin{align*}
\dot{z}_1 &= \dot{z}_2 \\
\dot{z}_2 &= \ddot{u}_a \\
\dot{z}_3 &= z_1 + z_3 + w_1 + (z_2 + w_2)^2 \\
u_1 &= w_2 \\
u_2 &= -w_1 \\
e &= z_1
\end{align*} \]
where \( \ddot{u}_a = u_a \). When \( \ddot{u}_a = 0 \) is satisfied, the absolutely observable dual state space satisfies
\[ \mathcal{O}_a^\perp \cap \mathcal{W}^\dagger = 0 \]
which is the necessary and sufficient condition needed by Theorem 5.4.

In order to eliminate the term \( u_1 \) from \( z_3 = z_1 + z_3 + w_1 + (z_2 + w_2)^2 \) to transform the system into the form (50), we use the decoupling technique in [7] to apply a linear state transformation
\[ \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)^T = (z_1, z_2, z_3 + \frac{w_1}{2} + \frac{w_2}{2})^T. \]
The transformed system in the form of (50) is
\[ \begin{align*}
\dot{\tilde{z}}_1 &= \dot{\tilde{z}}_2 \\
\dot{\tilde{z}}_2 &= \ddot{\tilde{u}}_a \\
\dot{\tilde{z}}_3 &= \tilde{z}_1 + \tilde{z}_3 + \frac{\tilde{z}_2}{2} + 2\tilde{z}_2w_2 + w_2^2 \\
u_1 &= w_2 \\
u_2 &= -w_1 \\
e &= \tilde{z}_1
\end{align*} \tag{63} \]
where \( \ddot{\tilde{u}}_a = \ddot{u}_a \). If the nonlinear term \( u_1^2 \) was absent from (63), regulation could be obtained using linear state feedback. However, since the term \( u_1^2 \) is present, another state transformation is needed in order to exclude it from the system equations. Such a transformation is equivalent to finding a center manifold of system (63).

Let \( \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)^T = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 + \sigma(w))^T, \) where \( \sigma(w) \) satisfies
\[ \frac{\partial \sigma}{\partial w_1} w_2 - \frac{\partial \sigma}{\partial w_2} w_1 = \sigma - w_2^2 \]
and hence
\[ \sigma = \frac{2}{3} w_1^2 + \frac{2}{3} w_2^2 - \frac{2}{3} w_1 w_2. \]
The system under the new state coordinates \( \{\tilde{z}_1, \tilde{z}_2, \tilde{z}_3\} \) becomes
\[ \begin{align*}
\dot{\tilde{z}}_1 &= \tilde{z}_2 \\
\dot{\tilde{z}}_2 &= \ddot{\tilde{u}}_a \\
\dot{\tilde{z}}_3 &= \tilde{z}_1 + \tilde{z}_3 + \frac{\tilde{z}_2^2}{2} + 2\tilde{z}_2w_2 \\
u_1 &= w_2 \\
u_2 &= -w_1 \\
e &= \tilde{z}_1
\end{align*} \tag{64} \]
Thus the nonlinear term \( \tilde{z}_2^2 + 2\tilde{z}_2 w_2 \) and its partial derivative with respect to \( w \) vanish at \( \tilde{z} = 0 \).

Clearly, the linear approximation of the first three equations in (64) about \( \tilde{z} = 0 \) is given by the controllable model
\[ \begin{align*}
\dot{\tilde{z}}_1 &= \tilde{z}_2 \\
\dot{\tilde{z}}_2 &= \ddot{\tilde{u}}_a \\
\dot{\tilde{z}}_3 &= \tilde{z}_1 + \tilde{z}_3.
\end{align*} \]
Thus a linear state feedback control
\[ \tilde{u}_a = -5\tilde{z}_1 - 12\tilde{z}_2 - 116\tilde{z}_3, \tag{65} \]
can be found that locally stabilizes the system for \( w = 0 \), specifically
\[ \tilde{u}_a = -5\tilde{z}_1 - 12\tilde{z}_2 - 116\tilde{z}_3. \tag{66} \]
Since \( \tilde{z} \) is a vector function of the system state \( (\tilde{x}_1, \tilde{x}_2) \) for (61), the control \( \tilde{u}_a \) in (65) can be written in terms of \( (\tilde{x}_1, \tilde{x}_2) \) as \( \tilde{u}_a(x, w) \). This, together with \( u = -u_2 + u_a = -u_2 + \tilde{u}_a = -u_2 + \tilde{u}_a \), leads to
\[ u = \tilde{u}_a(x, w) - w_1. \]
Hence a feedback control \( u(x, w) \) that regulates (61) is
\[ u(x, w) = -5\tilde{z}_1 - 12\tilde{z}_2 - 116\tilde{z}_3 - 116 \left( x_3 + \sigma - \frac{w_1}{2} - \frac{w_2}{2} \right) - w_1 \]
where
\[ \sigma = \frac{2}{3} w_1^2 + \frac{2}{3} w_2^2 - \frac{2}{3} w_1 w_2. \]

VII. Conclusion
This paper has presented solutions to the problem of state feedback nonlinear system regulation for both bounded and unbounded exogenous signals. The results are presented in a differential vector space framework, which differs from the existing approaches to nonlinear systems regulation. Some features of our approach are as follows.

1) The conditions for nonlinear system regulation are presented in terms of the dependence of the dual error output space upon the dual exogenous signal space. Such a presentation connects the problem of regulation to observability properties and consequently provides a clear interpretation of nonlinear system regulation in terms of the “observability” of the exogenous signal from the system error output.

2) The differential vector space approach is applicable to unbounded exogenous signals. This overcomes the limitations of other approaches such as center manifold theory which require boundedness of the system state.

3) In the differential vector space framework, the conditions for nonlinear system regulation are constructive in the sense that the integrability of the dual spaces provides direct computation of the intersection. Thus the conditions and solutions for regulation can be readily computed.

The results in this paper are natural extensions of the linear system results in [7] and [17]. When the results of this paper are
applied to linear systems, they reduce to well-known necessary and sufficient conditions for linear system regulation.

This paper further extends the application of differential linear algebra to control systems and demonstrates that the differential vector space approach can provide a very useful framework for analysis and design of nonlinear control systems. The authors have applied this approach to other nonlinear system problems [22],[ 23].

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REFERENCES


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