Monogamy inequalities for the Einstein-Podolsky-Rosen paradox and quantum steering

M. D. Reid

Centre for Atom Optics and Ultrafast Spectroscopy, Swinburne University of Technology, Melbourne 3122, Australia (Received 17 August 2013; published 23 December 2013)

Monogamy inequalities for the way bipartite Einstein-Podolsky-Rosen (EPR) steering can be distributed among N systems are derived. One set of inequalities is based on witnesses with two measurement settings, and may be used to demonstrate correlation of outcomes between two parties, that cannot be shared with more parties. It is shown that the monogamy for steering is directional. Two parties cannot independently demonstrate steering of a third system, using the same two-setting steering witness, but it is possible for one party to steer two independent systems. This result explains the monogamy of two-setting Bell inequality violations and the sensitivity of the continuous variable (CV) EPR criterion to losses on the steering party. We generalize to m settings. A second type of monogamy relation gives the quantitative amount of sharing possible, when the number of parties is less than or equal to m, and takes a form similar to the Coffman-Kundu-Wootters relation for entanglement. The results enable characterization of the tripartite steering for CV Gaussian systems and qubit Greenberger-Horne-Zeilinger and W states.

DOI: 10.1103/PhysRevA.88.062108

PACS number(s): 03.65.Ud, 42.50.Xa, 03.67.-a

I. INTRODUCTION

Entanglement is a major resource for quantum communication and information processing [1]. An important advantage of quantum communication is the potential for an unprecedented security [2–5]. In quantum information, the security is based on properties, such as the no-cloning theorem [6], that are fundamental to quantum mechanics, but not classical mechanics.

A property closely connected to the no-cloning theorem is the lack of shareability of entanglement between a number of parties. A quantitative formulation for the way entanglement can be shared among three qubit systems was presented by Coffman, Kundu, and Wootters (CKW) [7] and represented an advance in understanding multipartite entanglement. Their formulation was defined in terms of the concurrence measure C_{AB} of bipartite entanglement between two qubits A and B [8]. The CKW monogamy inequality is

$$C_{AB}^2 + C_{AC}^2 \leqslant C_{A\{BC\}}^2, \tag{1}$$

where $C_{A\{BC\}}$ is the concurrence of the bipartition A with the group $\{BC\}$. This relation illustrates that maximum entanglement can be shared between two parties only.

Despite the importance of monogamy relations for quantum information, the knowledge of quantitative relations for other forms of entanglement is so far rather limited. It is known that the CKW relation can be extended to N qubits [9], and that a violation of the two-setting Bell inequalities [10] is completely monogamous [11–13], a property that underpins the extra "device-independent" security provided by quantum cryptography using Bell states [3]. Two-party monogamy does not, however, apply to Bell inequalities involving three measurement settings per site [14]. There is also a relative lack of quantitative knowledge about the shareability of nonlocality in the more complex continuous variable (CV) systems, although there have been new investigations for Svetlichny's nonlocality [15] and much progress has been

made for the entanglement of CV Gaussian states [16], for which quantitative monogamy relations have been worked out [17]. Recent work [18,19] analyzes the reasons for the difficulty in developing monogamy relations using more general measures of entanglement, such as entanglement of formation. In Gaussian CV cryptography [4,20], because Bell inequalities are not directly violated [21], the monogamy of other forms of nonlocality is likely to be especially useful.

The objective of this paper is to understand more about the monogamy associated with the Einstein-Podolsky-Rosen (EPR) paradox [21–23]. This is the subclass of entanglement called "quantum steering" [24] that was first formalized as a distinct type of nonlocality by Wiseman, Jones, and Doherty [25–30]. Comparatively little is known about the shareability of this nonlocality, which we refer to as "EPR steering" [31].

Here, we will derive monogamy relations that quantify the amount of bipartite EPR steering that can be shared by a number of parties. As might be expected, we find the lack of shareability is greater than for entanglement. An important feature of EPR steering monogamy is its *directionality*. While entanglement is defined symmetrically with respect to both parties, this is not true of steering or the EPR paradox [25]. "One-way" steering has been realized [32–34]: that party A may "steer" another system B does not imply the converse. This property has implications for the way EPR steering can be used to achieve secure quantum communication [4]. In this paper, we identify the directionality associated with steering monogamy.

Like the CKW result, the relations derived here are expressed in terms of inequalities. The monogamy relations are specific to particular EPR steering witnesses. We introduce two- and three-setting "steering parameters" $S_{B|A}^{(2)}$ and $S_{B|A}^{(3)}$, that involve the variances of Pauli spin matrices, and prove monogamy relations that apply to three qubits *A*, *B*, and *C*:

$$S_{B|A}^{(2)} + S_{B|C}^{(2)} \ge 2 \max\left\{1, S_{B|\{AC\}}^2\right\}$$
(2)

and $S_{B|A}^{(3)} + S_{B|C}^{(3)} + S_{B|D}^{(3)} \ge 3 \max\{1, S_{B|\{AC\}}^2\}$, and

$$S_{B|A}^{(3)} + S_{B|C}^{(3)} \ge 2S_{B|\{AC\}}^{(3)}.$$
(3)

1050-2947/2013/88(6)/062108(10)

Here, $S_{B|A}^{(2)}, S_{B|A}^{(3)} < 1$ are criteria sufficient to demonstrate an EPR steering of system *B*, by measurements made on system *A*. $S_{B|A}^{(2)}, S_{B|A}^{(3)} \rightarrow 0$ implies maximum steering. Similar results are derived for *m*-setting steering inequalities. These relations imply a tight monogamy: steering of a system *B* can only be confirmed by m - 1 other parties, using the same *m*-setting inequality. The monogamy of Bell inequality violations is explained because Bell inequalities are also steering inequalities. Using a graphical representation based on that of Plesch and Buzek [35], we apply these results to depict the distribution of bipartite steering associated with the tripartite Greenberger-Horne-Zeilinger (GHZ) and *W* states.

More fundamental are monogamy relations based on criteria for EPR steering that are *necessary and sufficient* for detecting steering. Restricting to CV Gaussian systems, we find that such monogamy relations are possible. EPR steering of *B* by *A* exists iff one can show that a parameter $E_{B|A}$ involving conditional variances for Bob's system (given measurements by Alice) satisfies $E_{B|A} < 1$ [25,26,36]. This is the EPR paradox criterion in which "elements of reality" deduced for Bob's system show incompatibility between local realism and the completeness of quantum mechanics [23,36]. For any three parties *A*, *B*, and *C*, we will see that

$$E_{B|A}E_{B|C} \ge \max\left\{1, E_{B|\{AC\}}^2\right\} \tag{4}$$

and $E_{B|A} + E_{B|C} \ge 2 \max\{1, E_{B|\{AC\}}^2\}$. If steering is shared between more than two Gaussian sites, then it becomes directional. Two systems *A* and *C* cannot both (Gaussian) steer a third system *B*, but we will show by example that the converse is not true.

A lack of robustness of the EPR criterion to losses on the steering party, but not on the party being steered, has been noted in experiments [37,38]. This effect is now explained in terms of the monogamy relation, and is seen to be a fundamental one, independent of the mechanism of generation of the EPR fields, or the way in which losses are implemented. This very tight form of monogamy comes about because the witness $E_{B|A}$ is based only on two observables: position and momentum.

We conclude the paper with a brief discussion. The steering monogamy inequalities are likely to be useful in establishing threshold efficiency bounds [39] and one-sided device-independent quantum communication security [4].

II. MONOGAMY OF TWO-SETTING AND CV GAUSSIAN STEERING

A. CV EPR steering

Consider the situation of three distinct and separated systems or parties, labeled *A*, *B*, and *C*. For each system, quadratures *X*, *P* are defined: X_A , P_A for system *A*, and similarly for *B* and *C*. We now examine the monogamy result for the two-observable EPR criterion used to verify the CV EPR paradox [21–23].

We begin by defining a "steering parameter" that enables confirmation of the EPR paradox between two systems A and B [36]. The steering parameter is

where $(\Delta_{\inf} X_{B|A})^2$ is the variance of the conditional distribution for the measurement X_B , given a measurement at A. We normally assume that the measurement at A has been optimized to minimize the conditional variance value. Here, X and P are scaled position and momentum quadratures, so that the Heisenberg relation for system B is given by $\Delta X_B \Delta P_B \ge 1$. A confirmation of the EPR paradox [36], and quantum steering [25,26], is given when

$$E_{B|A} < 1. \tag{6}$$

This type of inequality is called an "EPR steering" or "steering" inequality. EPR steering inequalities based on entropic uncertainty relations have also been derived [40].

Following and summarizing the work of Refs. [25,26,31], we will use the terminology that measurements of the system A (of Alice) "steer" another system B (of Bob), if some conditions are satisfied, that imply the ensemble for B has been affected by those measurements. This relates closely to EPR's notion of "spooky action at a distance," and steering is illustrated by an EPR paradox, when Alice's inferences about Bob's system cannot be reconciled for consistency between local realism premises and the completeness of quantum mechanics. Steering manifests as a failure of a local hidden variable (LHV) theory that additionally constrains Bob's local hidden variable system to be describable as a quantum state. If the conditional variances for B are reduced, so that $E_{B|A} < 1$, then this implies a directional EPR paradox, whereby the measurements of A steer the system B. Throughout this paper, we abbreviate this last phrase, to say that "A steers B," or "A EPR steers B."

Now, we come to the monogamy result for the CV EPR steering.

Result (1). If A steers B, in such a way that (6) holds, then it is certain that $\Delta_{\inf} X_{B|C} \Delta_{\inf} P_{B|C} > 1$, i.e., C cannot be shown to steer B by the EPR criterion. This result is expressed as the monogamy relation

$$E_{B|A}E_{B|C} \ge 1. \tag{7}$$

The relation has been stated and then proved in a previous paper [39]. We present the full details here again because the nature of the proof is central to the results that follow.

Proof. The observer (Alice) at *A* can make a local measurement O_A to infer a result for an outcome of X_B at *B*. Denoting the outcomes of Alice's measurement by A_i , we can evaluate the variance of each conditional distribution $P(X_B|A_i)$ and then take the average to define the inference variance $(\Delta_{inf} X_{B|A})^2$. That is, Alice's measurement is a measurement of Bob's X_B , where the uncertainty is given by $\Delta_{inf} X_{B|A}$. Similarly, she can make a measurement of Bob's P_B , with accuracy $\Delta_{inf} P_{B|A}$. Now, observer *C* ("Charlie") can also make inference measurements, with uncertainty $\Delta_{inf} X_{B|C}$ and $\Delta_{inf} P_{B|C}$. Since Alice and Charlie can make the measurements *simultaneously*, it is guaranteed by the Heisenberg uncertainty relation that

$$\Delta_{\inf} X_{B|A} \Delta_{\inf} P_{B|C} \ge 1. \tag{8}$$

The relations are ensured because a conditional quantum density operator $\rho_{B|\{A_iC_i\}}$ for system *B*, given the outcomes A_i and C_i for Alice and Charlie's measurements, can be defined, and correctly predicts the results of all measurements.



FIG. 1. (Color online) Depiction of ways in which bipartite steering can be shared among three Gaussian CV systems. The results (a) and (c) also hold for steering detected by two-setting inequalities. (a) Two parties *cannot* steer the same system. Here we depict the monogamy relation $E_{A|B}E_{A|C} \ge 1$. (b) The GHZ state has no bipartite steering between individual parties, but there is two-way steering between any one party and the group of the other two. (c) The dual steering by one party of two systems can be realized. (d) The monogamy relation of (a) prevents the "passing on" of steering. If A can steer B, and B can steer C, then we know that A cannot steer C.

Then, the result follows straightforwardly, on using the Cauchy-Schwarz inequality and the definition of the inference variances given in Ref. [23]. Similarly, Alice can measure to infer P_B and Charlie can measure to infer X_B , and it must also be true that

$$\Delta_{\inf} P_{B|A} \Delta_{\inf} X_{B|C} \ge 1. \tag{9}$$

Hence, it follows that $E_{B|A}E_{B|C} \ge 1$.

The monogamy Result (1) is depicted schematically in Fig. 1(a) using a generalization of the "entangled-graph" representation developed by Plesch and Buzek [35]. The representation depicts the distribution of bipartite entanglement in multipartite systems. The circles or nodes represent distinct physical systems, and a line connecting two systems represents the bipartite entanglement between them. We generalize the depiction in the obvious way to denote the bipartite steering of *A* by *B* by an arrow from *B* pointing toward *A*. We note the distinction from the graph-state representation of Hein *et al.* [41], in which lines between nodes represent interactions.

CV Gaussian systems are defined as those whereby the quantum states have a positive Gaussian Wigner function and the measurements are restricted to be Gaussian [16]. For such systems, the Result (1) is particularly useful since in this case the optimized EPR steering inequality (6) is *necessary and sufficient* to detect bipartite EPR steering of *B* by *A* [25,26]. (The optimized inequality is that which optimizes the measurement at *A*, to minimize the conditional variances.) Thus, in the Gaussian case, we can make the stronger statement that a system can be steered by *only one other* system, i.e., two distinct systems cannot independently steer the same third system [Fig. 1(a)]. We will see in Sec. II C that this sort

of monogamy is one-way only. The properties of tripartite Gaussian steering are therefore *directional*. Also, we note that the monogamy inequality (7) can be saturated: $E_{B|A} = 1$ was measured by Bowen *et al.* [37] and Buono *et al.* [38] for a CV Gaussian state, with 50% loss on the mode *A*, which implies a second mode *C* satisfying $E_{B|A} = E_{B|C} = 1$.

B. Monogamy of two-observable EPR steering

The crucial aspect to the proof of the monogamy relation Result (1) is that the steering inequality involves only two observables (measurement "settings") at each site, e.g., position and momentum (X and P). Similar monogamy relations can therefore be established for other two-observable steering inequalities.

Let us consider three systems A, B, and C of a fixed dimension corresponding to a spin J. We define the "steering parameter"

$$S_{B|A}^{(2)} = \left(\left(\Delta_{\inf} J_{B|A}^X \right)^2 + \left(\Delta_{\inf} J_{B|A}^Y \right)^2 \right) / C_J$$
(10)

using the notation explained for (5). Here, J^X , J^Y , and J^Z are the spin components, and the constant C_J is defined by the uncertainty relation $(\Delta J^X)^2 + (\Delta J^Y)^2 \ge C_J$ [42,43]. EPR steering of *B* by *A* is confirmed if $S_{B|A}^{(2)} < 1$ [44–46]. This inequality detects what we will refer to as "two-observable EPR steering" since the inequality involves only two measurement settings, J^X and J^Y , at each site.

Result (2). The monogamy relation

$$S_{B|A}^{(2)} + S_{B|C}^{(2)} \ge 2 \tag{11}$$

holds. The proof follows as a straightforward extension of the proofs given for Results (1) and (3) (see below).

The relation has the same consequences for monogamy as Result (1). If EPR steering of *B* by *A* is confirmed by $S_{B|A}^{(2)} < 1$, then it follows that $S_{B|C}^{(2)} > 1$, i.e., the system *C* cannot be shown to steer *A* by using the same steering inequality.

The case $J = \frac{1}{2}$ is especially important since it relates to the original Bell states on which many experiments and quantum information protocols are based. In terms of Pauli spin matrices σ^X and σ^Y , we find that $S_{B|A}^{(2)} = (\Delta_{inf}\sigma_{B|A}^X)^2 + (\Delta_{inf}\sigma_{B|A}^Y)^2$. If bipartite EPR steering of *B* by *A* is observed as $\sigma_{B|A}^{(2)} = (\Delta_{inf}\sigma_{B|A}^X)^2 + (\Delta_{inf}\sigma_{B|A}^Y)^2 < 1$, then we know that for any third site *C*, there is no such steering: that is, $\sigma_{B|C}^{(2)} = (\Delta_{inf}\sigma_{B|C}^X)^2 + (\Delta_{inf}\sigma_{B|C}^Y)^2 \ge 1$. The inequality (11) gives us information about the minimum noise levels for Bob's qubit values as inferred by any third "eavesdropper" observer at *C*, given that we know the noise levels for Bob's qubit values as inferred by Alice at *A*.

C. Categories of tripartite Gaussian and two-observable EPR steering

Figures 1 and 2 show possible distributions of bipartite steering for a tripartite CV Gaussian system [16,17,47]. The restrictions and possibilities apply also to steering detected by a two-observable steering inequality. These depictions are useful because the steering, or lack of steering, for a specific inequality can give important information about the



FIG. 2. (Color online) Other configurations for tripartite steering. Whether (a) and (c) are possible for a Gaussian system is not established in this paper, but the monogamy Result (1) immediately tells us that the configurations of (b) are impossible, for Gaussian systems, and where steering is detected via two-observable inequalities. The configurations (d) are likely to be achieved by adding thermal noise to the single sites.

way correlations or noise inference levels are shared among three parties.

Before discussing possible bipartite distributions, we recall several properties of steering [25,26]. First, steering requires entanglement. We say the EPR steering is "maximum" if the EPR conditional variances go to zero, i.e., $E_{B|A}$, $S_{B|A}^{(2)} \rightarrow 0$. For some pure bipartite systems, the EPR steering can achieve the maximum value and this corresponds also to the "strongest" entanglement, as measured either by concurrence [8] or logarithmic negativity [48]. This is true for the two-mode squeezed state ($E_{B|A} \rightarrow 0$) [49] and for the qubit Bell-Bohm EPR state ($S_{B|A}^{(2)} \rightarrow 0$) [10,45]. As not all entanglement will show EPR steering, two systems can be entangled even if there is no EPR steering between them.

The possibilities for steering shared between three systems are therefore limited by the possibilities for entanglement. Two distinct types of pure tripartite entangled qubit states exist [50]. These are the Greenberger-Horne-Zeilinger (GHZ) [51] and W states. Similar states have been defined for the CV case [52–54]. Here, we discuss the bipartite distribution for specific CV Gaussian states only, leaving the qubit case until Sec. VI, since for qubits it is important to also consider steering detected by three-observable inequalities.

The tripartite GHZ state allows no pairwise bipartite entanglement between any of the three systems A, B, and C [7]. The same will be true for the EPR steering of a GHZ state (i.e., $E_{B|A} = E_{B|C} \ge 1$) since steering is a special sort of entanglement. The GHZ state, however, has bipartite entanglement between A and the composite system B - C. A tripartite CV GHZ state is a simultaneous eigenstate of $X_i - X_j$ ($i, j = A, B, C, i \ne j$) and $P_A + P_B + P_C$ with eigenvalues 0 [52]. Party A can choose to predict either of two noncommuting observables (a single position or the sum of the momenta) of the combined system BC, and the parties BC can



FIG. 3. (Color online) Schematic of the generation and EPR steering of the CV GHZ state, which shows the tripartite steering of Fig. 1(c). The strong bipartite steering and entanglement of the two-mode squeezed state can be generated by interfering two squeezed modes at a beam splitter (BS1).

choose to predict either the position or momentum of system A [39,55]. Thus, there is a (maximum) "two-way" steering, i.e., the system A can steer the composite system BC (e.g., $E_{A|\{BC\}} = 0$) and vice versa (e.g., $E_{\{BC\}|A} = 0$). This situation is depicted in Figs. 1(b) and 3.

Bipartite steering between two individual sites is possible for other sorts of tripartite CV Gaussian states. However, we deduce that this bipartite steering, in order to be consistent with the monogamy relation Result (1), must be "one-way" only. We find that the outward "dual" steering, where A steers both B and C, is possible [Fig. 1(c)]. This type of tripartite steering can be created between modes A, B, C as in Fig. 4. We argue as follows. The final bipartite steering between the pair A and B (and similarly between A and C) is equivalent to that between a mode A with no loss and a second mode B that has been



FIG. 4. Schematic of the generation of the "dual" EPR steering as depicted in Fig. 2(c). Strong bipartite two-way EPR steering is first created between A and B'. The tripartite steering of Fig. 2(c) is generated using the second BS2 with vacuum input and efficiency of transmission $\eta = 0.5$.

The monogamy rule (Result (1)) negates the possibility of the steering "the other way:" that the lossy Gaussian system B (of Fig. 4) steers the "lossless" Gaussian system A. The monogamy rule tells us that steering of A by both B and C is ruled out. With 50% loss on the original B' channel, there will be symmetry of the correlation between A and C, and A and B, in which case if B can steer A, then so can C. This would lead to a contradiction of Result (1). That the EPR paradox cannot be demonstrated with 50% loss on the steering channel was noted experimentally [37,38].

There are some open questions. The monogamy Result (1) tells us that if A can steer B, and B can steer C, then A cannot steer C, so that two-observable steering cannot be "passed on" [Figs. 1(d) and 2(a)]. It is left unaddressed, however, whether the scenario of Figs. 2(a) and 2(c) is possible, although for qubits, the state discussed by Plesch and Buzek [35] will give this possibility. The arrangements of Fig. 2(d) are not ruled out, and are likely to be realized by adding noise to specific sites, based on results that indicate steering of a system B by another (A) is lost if thermal noise is added to B [56]. Another unaddressed question concerns how the one-way dual steering of Fig. 1(c) can be shared. We might expect that "once split" the degree of steering would be reduced, in accordance with a monogamy rule like that of CKW.

III. MULTIOBSERVABLE QUBIT AND QUDIT STEERING MONOGAMY RELATIONS

More monogamy relations may be derived for EPR steering inequalities that involve m observables, i.e., m measurement settings, at each site. We show that no more than m-1independent parties can demonstrate "steering" of a system B, using the same *m*-observable steering inequality.

A. Bohm's EPR paradox monogamy

We consider a bipartite EPR steering inequality that involves three observables: J^X, J^Y, J^Z . We define the steering parameter

$$S_{B|A}^{(3)} = \left(\left(\Delta_{\inf} J_{B|A}^X \right)^2 + \left(\Delta_{\inf} J_{B|A}^Y \right)^2 + \left(\Delta_{\inf} J_{B|A}^Z \right)^2 \right) / J.$$
(12)

EPR steering of system A by B is obtained when $S_{B|A}^{(3)} < 1$, which confirms Bohm's EPR paradox for spins when J = $\frac{1}{2}$ [31,45]. This steering inequality was derived from the uncertainty relation $(\Delta J^X)^2 + (\Delta J^Y)^2 + (\Delta J^Z)^2 \ge J$ that applies to all quantum states of fixed spin J, i.e., to qudit systems of dimension d = 2J + 1 [43]. For two qubit systems, $S_{B|A}^{(3)} = \left[(\Delta_{\inf} \sigma_{B|A}^X)^2 + (\Delta_{\inf} \sigma_{B|A}^Y)^2 + (\Delta_{\inf} \sigma_{B|A}^Z)^2 \right] / 2.$ *Result (3).* We can apply the method of proof of Result (1)

to derive the monogamy steering relation

$$S_{B|A}^{(3)} + S_{B|C}^{(3)} + S_{B|D}^{(3)} \ge 3.$$
(13)

Proof. The observer at A (Alice) can make the measurement that gives her the value of Bob's observable J_B^X with uncertainty $\Delta_{\inf} J_B^X$. The observer at C (Charlie) can make the measurement that gives the result for Bob's J_R^Y

with uncertainty $\Delta_{\inf} J_B^Y$, and the observer at D can make the measurement that gives the result for Bob's J_B^Z with uncertainty $\Delta_{inf} J_B^Z$. Since the three observers can measure simultaneously, we see that the quantum uncertainty relations for spins (as given above) constrains the variances to satisfy

$$\left(\Delta_{\inf}J_{B|A}^{X}\right)^{2} + \left(\Delta_{\inf}J_{B|C}^{Y}\right)^{2} + \left(\Delta_{\inf}J_{B|D}^{Z}\right)^{2} \ge J.$$
(14)

Similarly,

$$\left(\Delta_{\inf}J_{B|A}^{Y}\right)^{2} + \left(\Delta_{\inf}J_{B|C}^{Z}\right)^{2} + \left(\Delta_{\inf}J_{B|D}^{X}\right)^{2} \ge J \qquad (15)$$

and also

$$\left(\Delta_{\inf} J_{B|A}^{Z}\right)^{2} + \left(\Delta_{\inf} J_{B|C}^{X}\right)^{2} + \left(\Delta_{\inf} J_{B|D}^{Y}\right)^{2} \ge J.$$
(16)

We then see that the monogamy relation (13) follows, upon adding the three inequalities.

The monogamy Result (3) does not exclude two observers from being able to steer B. However, the relation certainly prevents all three observers from being able to demonstrate steering of the same system B via the violation of the three-observable steering inequalities (i.e. we cannot attain $S_{B|A}^{(3)} < 1, S_{B|C}^{(3)} < 1$ and $S_{B|D}^{(3)} < 1$). We can extend the proof of Result (3) to derive similar results involving *m*-observable steering inequalities.

B. Steering inequalities with *m* observables

Steering inequalities for two qubit systems have been derived and analyzed in Refs. [27–31]. The multiobservable steering inequalities derived by Saunders et al. [27] and Bennet et al. [28] have been used in experiments that confirm steering without fair sampling assumptions [28-30]. Expressed in terms of correlation rather than as a noise reduction, these steering inequalities, similar to Bell inequalities, take the general form $\tilde{S}_{B|A}^{(m)} \leq 1$, where

$$\tilde{S}_{B|A}^{(m)} = \frac{1}{\mathcal{C}_m} \sum_{j=1}^m c_j \langle \sigma_B^j \sigma_A^{P_j} \rangle.$$
(17)

Here, $\sigma_A^{p_j}$, σ_B^j is the Pauli spin component at angle θ_{p_j} , θ_j for system A/B respectively (where p_j is a function of j), $|c_i| = 1, C_m$ is a constant, and m is the number of measurement

settings at each site. Steering is obtained when $\tilde{S}_{B|A}^{(m)} > 1$. Result (4). The two-observable monogamy relation is $\tilde{S}_{B|A}^{(2)} + \tilde{S}_{B|C}^{(2)} \leqslant 2$, which generalizes to

$$\sum_{k=1}^{m} \tilde{S}_{B|A_{K}}^{(m)} \leqslant m, \tag{18}$$

where the different parties (distinct from *B*) are labeled A_k . The result also applies to the two-observable Bell-Clauser-Horne-Shimony-Holt (CHSH) inequality

$$\begin{split} \tilde{S}_{B|A}^{\text{Bell}} &= \left\langle \sigma_B^X \sigma_A^{X'} \right\rangle - \left\langle \sigma_B^Y \sigma_A^{Y'} \right\rangle + \left\langle \sigma_B^X \sigma_A^{Y'} \right\rangle + \left\langle \sigma_B^Y \sigma_A^{X'} \right\rangle \\ &\leqslant 2 \end{split}$$

which is also an EPR steering inequality [25]. EPR steering is observed when $\tilde{S}_{B|A}^{\text{Bell}} > 2$, and the monogamy relation is $\tilde{S}_{B|A}^{\text{Bell}} + \tilde{S}_{B|C}^{\text{Bell}} \leq 4.$ *Proof.* To prove (18), we recall that steering is a failure of

a special type of separable model, called a Local Hidden State

(LHS) model [25,26]. For any LHS model,

$$\langle \sigma_A^X \sigma_B^Y \rangle = \int \rho(\lambda) \langle \sigma_A^X \rangle_{\lambda} \langle \sigma_B^Y \rangle_{\lambda} d\lambda,$$
 (19)

where $\langle \sigma_{A/B}^{X/Y} \rangle_{\lambda}$ is the predicted average of the measurement $\sigma_{A/B}^{X/Y}$ for the local state λ , and the local state for the system *B* is to be consistent with a *local quantum state* (LQS). If the LHS model is valid, the steering parameter can be written as

$$\tilde{S}_{B|A}^{(m)} = \int \rho(\lambda) \tilde{S}_{B|A}^{(m)}(\lambda) d\lambda, \qquad (20)$$

where $\tilde{S}_{B|A}^{(m)}(\lambda) = \frac{1}{C_m} \sum_{j=1}^m c_j \langle \sigma_B^j \rangle_{\lambda} \langle \sigma_A^{P_j} \rangle_{\lambda}$. The steering inequality $\tilde{S}_{B|A}^{(m)} \leq 1$ follows from this assumption. A similar result holds for the Bell-CHSH inequality.

Consider an experiment where the *m* parties $A_1, \ldots A_k, \ldots$ measure simultaneously $\sigma_{A_1}^{p_1}, \ldots, \sigma_{A_k}^{p_k}, \ldots$ respectively, and the party at *B* measures σ_B^j . We denote the outcomes of the measurements by the symbols $\sigma_{A_k}^{p_k}$ but note they are in fact numbers, and will be identified as a "hidden" variable set $\{\lambda_1, \ldots, \lambda_m\} \equiv \{\sigma_{A_1}^{p_1}, \ldots, \sigma_{A_m}^{p_m}\}$. The state at *B* conditioned on these outcomes is definable by a quantum density matrix $\rho_{B|\lambda}$, and has an expectation value for σ_B^j which we symbolize by $\langle \sigma_B^j | \{\sigma_{A_k}^{p_k}\} \rangle$. The linear combination $\sum_{k=1}^m \tilde{S}_{B|A_k}^{(m)}$ can be written in the form of an LHS model, where the probability $\rho(\lambda)$ is established as the probability *P* of obtaining the outcomes $\{\sigma_{A_k}^{p_k}\}$ of the simultaneous measurements. Explicitly, we can write

$$\sum_{j=1}^{m} c_j \langle \sigma_B^j \sigma_{A_j}^{p_j} \rangle = \sum_j \sum_{\sigma_{A_k}^{p_k}} P(\{\sigma_{A_k}^{p_k}\}) c_j \langle \sigma_B^j | \{\sigma_{A_k}^{p_k}\} \rangle \sigma_{A_j}^{p_j}, \quad (21)$$

which becomes

$$\sum_{j=1}^{m} c_j \langle \sigma_B^j \sigma_{A_j}^{p_j} \rangle = \sum_j c_j \int \rho(\lambda) \langle \sigma_B^j \rangle_{\lambda} \langle \sigma_{A_j}^{p_j} \rangle_{\lambda} d\lambda, \qquad (22)$$

where we see that the moments $\langle \sigma_{A_j}^j \rangle_{\lambda}$ are those of the quantum state $\rho_{B|\lambda}$, and that $\langle \sigma_{A_j}^{p_j} \rangle_{\lambda} = \sigma_{A_j}^{p_j} = \lambda_j$. The last line satisfies the LHS model (20) and hence must be less than or equal to C_m . This is true regardless of the choice of p_j . The $\sum_{k=1}^m \tilde{S}_{B|A_k}^{(m)}$ contains *m* groups of *m* terms like (22), but where different choices of simultaneous measurements are used for a given *j*. In this way, the result follows.

To prove the Bell-CHSH result, we consider an experiment where the parties at *A* and *C* measure simultaneously $\sigma^{X'}$ and $\sigma^{Y'}$, and the party at *B* measures σ^X or σ^Y . We denote the outcomes of the measurements at *A* and *C* by the symbols $\sigma_A^{X'}$ and $\sigma_C^{Y'}$ but note they are in fact numbers, and will be identified as a "hidden" variable set $\{\lambda\} \equiv \{\lambda, \lambda_2\} \equiv \{\sigma_A^{X'}, \sigma_C^{Y'}\}$. The state at *B* conditioned on these outcomes is definable by a quantum density matrix $\rho_{B|\lambda}$, and has moments which we once again denote by $\langle \sigma_B^X | \sigma_A^{X'}, \sigma_C^{Y'} \rangle$ and $\langle \sigma_B^Y | \sigma_A^{X'}, \sigma_C^{Y'} \rangle$ (we drop the parentheses for convenience of notation). Now, we see that the linear combination $\tilde{S}_{B|A}^{(Bell)} + \tilde{S}_{B|C}^{(Bell)}$, namely

$$\begin{split} & \left\langle \sigma_{B}^{X} \sigma_{A}^{X'} \right\rangle + \left\langle \sigma_{B}^{X} \sigma_{C}^{X'} \right\rangle + \left\langle \sigma_{B}^{Y} \sigma_{A}^{X'} \right\rangle + \left\langle \sigma_{B}^{Y} \sigma_{C}^{X'} \right\rangle \\ & + \left\langle \sigma_{B}^{X} \sigma_{A}^{Y'} \right\rangle + \left\langle \sigma_{B}^{X} \sigma_{C}^{Y'} \right\rangle - \left\langle \sigma_{B}^{Y} \sigma_{A}^{Y'} \right\rangle - \left\langle \sigma_{B}^{Y} \sigma_{C}^{Y'} \right\rangle, \end{split}$$

can be written consistent with a LHS model since the probability $\rho(\lambda)$ can be established as the probability of obtaining the outcomes $\sigma_A^{X'}$ and $\sigma_C^{Y'}$ of the simultaneous measurements. Explicitly, we can write

$$\langle \sigma_B^X \sigma_A^{X'} \rangle + \langle \sigma_B^X \sigma_C^{Y'} \rangle$$

$$= \sum_{\sigma_A^{X'}, \sigma_C^{Y'}} P\left(\sigma_A^{X'}, \sigma_C^{Y'}\right) \{ \langle \sigma_B^X | \sigma_A^{X'}, \sigma_C^{Y'} \rangle \sigma_A^{X'} + \langle \sigma_B^X | \sigma_A^{X'}, \sigma_C^{Y'} \rangle \sigma_C^{Y'} \},$$

which takes the form

$$\begin{aligned} &\sigma_B^X \sigma_A^{X'} \rangle + \langle \sigma_B^X \sigma_C^{Y'} \rangle \\ &= \int \rho(\lambda) \{ \langle \sigma_B^X \rangle_\lambda \langle \sigma_A^{X'} \rangle_\lambda + \langle \sigma_B^X \rangle_\lambda \langle \sigma_C^{Y'} \rangle_\lambda \} d\lambda, \end{aligned}$$

and similarly

where we see that the moments $\langle \sigma_B^X \rangle_{\lambda}$, $\langle \sigma_B^Y \rangle_{\lambda}$ are those of the *quantum* state $\rho_{B|\lambda}$, and $\langle \sigma_A^{X'} \rangle_{\lambda} = \sigma_A^{X'} = \lambda_1$ and $\langle \sigma_C^{Y'} \rangle_{\lambda} = \sigma_C^{Y'} = \lambda_2$. In this way, we can write

$$\langle \sigma_B^X \sigma_A^{X'} \rangle + \langle \sigma_B^X \sigma_C^{Y'} \rangle + \langle \sigma_B^Y \sigma_A^{X'} \rangle - \langle \sigma_B^Y \sigma_C^{Y'} \rangle$$

$$= \int \rho(\lambda) \{ \langle \sigma_B^X \rangle_{\lambda} \langle \sigma_A^{X'} \rangle_{\lambda} + \langle \sigma_B^X \rangle_{\lambda} \langle \sigma_C^{Y'} \rangle_{\lambda}$$

$$+ \langle \sigma_B^Y \rangle_{\lambda} \langle \sigma_A^{X'} \rangle_{\lambda} - \langle \sigma_B^Y \rangle_{\lambda} \langle \sigma_C^{Y'} \rangle_{\lambda} \} d\lambda$$

$$= \int \rho(\lambda_1, \lambda_2) \{ \langle \sigma_B^X \rangle_{\lambda} \lambda_1 + \langle \sigma_B^X \rangle_{\lambda} \lambda_2$$

$$+ \langle \sigma_B^Y \rangle_{\lambda} \lambda_1 - \langle \sigma_B^Y \rangle_{\lambda} \lambda_2 \} d\lambda.$$

$$(24)$$

The last line satisfies the LHS model (20), on letting $\langle \sigma_A^{X'} \rangle_{\lambda} = \lambda_1$ and $\langle \sigma_C^{Y'} \rangle_{\lambda} = \lambda_2$, and hence must be less than or equal to 1. By the same argument, we can show $\langle \sigma_B^X \sigma_A^{Y'} \rangle + \langle \sigma_B^X \sigma_C^{X'} \rangle - \langle \sigma_B^Y \sigma_A^{Y'} \rangle + \langle \sigma_B^X \sigma_C^{X'} \rangle \leq 1$. Hence, $\tilde{S}_{B|A}^{(\text{Bell})} + \tilde{S}_{B|C}^{(\text{Bell})} \leq 4$.

C. Monogamy of steering using Bell-CHSH moments

Two useful EPR steering inequalities that apply to the Bell– Clauser-Horne-Shimony-Holt (CHSH) state and experiment are

$$\left\langle \sigma_{B}^{X}\sigma_{A}^{X'}\right\rangle - \left\langle \sigma_{B}^{Y}\sigma_{A}^{Y'}\right\rangle \leqslant \sqrt{2}$$
 (25)

and $\langle \sigma_B^X \sigma_A^{Y'} \rangle + \langle \sigma_B^Y \sigma_A^{X'} \rangle \leq \sqrt{2}$ [27,31,46]. If either of these inequalities is violated, steering is confirmed. Result (4) allows us to immediately write monogamy relations associated with these steering inequalities:

$$\left\langle \sigma_{B}^{X} \sigma_{A}^{X'} \right\rangle - \left\langle \sigma_{B}^{Y} \sigma_{A}^{Y'} \right\rangle + \left\langle \sigma_{B}^{X} \sigma_{C}^{X'} \right\rangle - \left\langle \sigma_{B}^{Y} \sigma_{C}^{Y'} \right\rangle \leqslant 2\sqrt{2}$$
(26)

and $\langle \sigma_B^X \sigma_A^{Y'} \rangle + \langle \sigma_B^Y \sigma_A^{X'} \rangle + \langle \sigma_B^X \sigma_C^{Y'} \rangle + \langle \sigma_B^Y \sigma_C^{X'} \rangle \leq 2\sqrt{2}.$

IV. CHSH-BELL NONLOCALITY MONOGAMY

Since the Bell inequalities are also steering inequalities, the monogamy of steering implies the monogamy of the twosetting CHSH–Bell inequalities. The CHSH–Bell inequalities are

$$\widetilde{S}_{B|A}^{\text{Bell}} = \left\langle \sigma_B^X \sigma_A^{X'} \right\rangle - \left\langle \sigma_B^Y \sigma_A^{Y'} \right\rangle + \left\langle \sigma_B^X \sigma_A^{Y'} \right\rangle + \left\langle \sigma_B^Y \sigma_A^{X'} \right\rangle \\
\leqslant 2.$$
(27)

Any Bell inequality is also an EPR steering inequality [25]. Using Result (4), we can therefore deduce the monogamy relation for the CHSH-Bell inequality

$$\tilde{S}_{B|A}^{\text{Bell}} + \tilde{S}_{B|C}^{\text{Bell}} \leqslant 4.$$
(28)

The symmetry of the Bell-CHSH inequalities implies that in an experiment (where there is a fixed choice of measurement settings at each location) $\tilde{S}_{B|A}^{\text{Bell}} > 2$ is equivalent to $\tilde{S}_{A|B}^{\text{Bell}} > 2$. That is, as indeed must be true generally given the definition of Local Hidden Variable theories [25], the violation of a Bell inequality implies "two-way" steering. The monogamy relations $\tilde{S}_{A|B}^{\text{Bell}} + \tilde{S}_{A|C}^{\text{Bell}} \leqslant 4$ and $\tilde{S}_{C|A}^{\text{Bell}} + \tilde{S}_{C|B}^{\text{Bell}} \leqslant 4$ also hold. If two parties *A*-*B* can violate the Bell–CHSH inequality, then the pairs A-C, and C-B cannot.

This result for the monogamy of Bell-CHSH violations is not new [11-13]. What we have discovered from our analysis is that the monogamy follows as a result of steering monogamy. All two-observable (setting) steering inequalities possess oneway monogamy. Since the Bell-CHSH violations imply twoway steering, this is enough to explain Bell-CHSH monogamy.

Our results explain the shareability, with respect to three sites, of the three-observable Bell inequality violation of Collins and Gisin [14]. Being three-observable steering inequalities, we can expect, however, using Result (4), that these violations cannot be shared among *four* sites.

V. SHARING OF BIPARTITE STEERING

We have seen that a very tight monogamy arises for the correlations of a steering witness when the number of parties equals or exceeds m + 1, where m is the number of observables that need to be measured at each site. Now, we examine the constraints on the distribution of bipartite steering, when the number of systems is less than m + 1.

In this section, we therefore derive relations for steering monogamy that are similar to the CKW inequalities, for particular witnesses. We quantify how the "total amount of steering" is shared among the subgroups. Similar to the result for sharing of entanglement with qubits, we find that the strongest steering exists only when all the steering is shared between two parties. Once steering is distributed over a series of systems, the pairwise steering will diminish. In this paper, we prove such a rule for steering in one direction only.

A. CV bipartite sharing

We begin with the CV EPR steering relation (5). Given the definition of the steering parameter $E_{B|A}$, it must be true that

$$E_{B|\{AC\}} \leqslant E_{B|A}.\tag{29}$$

This simple result follows because $E_{B|A}$ is the lowest variance product possible that arises from the best inference of Bob's X_B or P_B by the group A of Alice. Alice can use any local observable, defined as a measurement performed on the system A. The inference of Bob's measurement by the group AC, which includes both A and C, must be at least as good as that of A alone since the observables of system A are a subset of those of the combined system AC. The steering of B by a combined group cannot be less effective than that of a subset. It is also true that $E_{B|\{AC\}} \leq E_{B|C}$. On multiplying the two inequalities together, we can easily derive several new monogamy relations.

Result (5). For the three systems A, B, and C, it follows that

$$E_{B|A}E_{B|C} \geqslant E_{B|\{AC\}}^2. \tag{30}$$

We can express the product relation in terms of a sum relation, similar to CKW, by using the simple identity $x^2 + y^2 \ge 2xy$. *Result* (6). It is also true that

$$E_{B|A} + E_{B|C} \ge 2E_{B|\{AC\}}.\tag{31}$$

This follows since we can let $x = \sqrt{E_{B|A}}$ and $y = \sqrt{E_{B|C}}$, and use that $E_{B|\{AC\}} \leq \sqrt{E_{B|A}} \sqrt{E_{B|C}}$. Since the maximal steering is defined when $E_{B|A} = 0$, and $E_{B|A} = 1$ is the threshold for steering, the direction of the inequality is reversed as compared to that for the CKW relation. We note also that we could choose $x = E_{B|A}$ and $y = E_{B|C}$ from which we derive the monogamy result

$$E_{B|A}^2 + E_{B|C}^2 \ge 2E_{B|\{AC\}}^2.$$
(32)

The relations (30)–(32) express a type of conservation law for steering. If there is steering of B by a group AC that has components A and C, so that $E_{B|\{AC\}} < 1$, then the steering is shared among the components. The individual steering of Bby A, or B by C, is reduced and bounded by the monogamy relations.

If the property (29) is specified as a condition for a witness for EPR steering, then the relation holds for all such witnesses. The monogamy relations (30) and (31) would then become fundamental results for steering monogamy, which are nonspecific to a particular steering witness or uncertainty relation.

The monogamy relation of Result (1) is stronger than Result (5) when steering is present since steering requires $E_{B|A} < 1$. We thus write the monogamy relation for the CV EPR witness (5) as

$$E_{B|A}E_{B|C} \ge \max\left\{1, E_{B|\{AC\}}^2\right\}.$$
(33)

One could test this relation experimentally by adding noise to mode B so that $E_{B|\{AC\}} > 1$. We have not demonstrated saturation of the inequality, except where $E_{B|\{AC\}} = 1$, which was discussed in Sec. II.

B. Qubits and qudits

The qubit case is more interesting. Following the same approach, we can deduce that $S_{B|A}^{(2)} \ge S_{B|\{AC\}}^{(2)}$ and $S_{B|C}^{(2)} \ge$ $S_{B|\{AC\}}^{(2)}$, which implies

$$S_{B|A}^{(2)} + S_{B|C}^{(2)} \ge \max\left\{2, 2S_{B|\{AC\}}^{(2)}\right\}$$
(34)

and similarly

$$S_{B|A}^{(3)} + S_{B|C}^{(3)} + S_{B|D}^{(3)} \ge \max\left\{3,3S_{B|\{ACD\}}^{(3)}\right\}.$$
 (35)

Also,

$$S_{B|A}^{(3)} + S_{B|C}^{(3)} \ge 2S_{B|\{AC\}}^{(3)}.$$
(36)

The relation (36) for sharing of steering is significant for qubit systems since it will apply to limit the steering detected using three-observable steering inequalities, for tripartite systems (here, the number of sites is less than m + 1). This relation resembles the CKW relation for entanglement. We use the relation (36) in the next section to derive the steering properties of the tripartite qubit *W* state.

VI. STEERING MONOGAMY OF TRIPARTITE GHZ AND W STATES

We are now in a position to analyze the distribution of bipartite steering for the tripartite qubit GHZ and W states. Consider the GHZ state for three-qubit (spin- $\frac{1}{2}$) systems:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \{|\uparrow\rangle_A|\uparrow\rangle_B|\uparrow\rangle_C - |\downarrow\rangle_A|\downarrow\rangle_B|\downarrow\rangle_C\}.$$
 (37)

The spins can be measured for each system by measurements performed by Alice, Bob, and Charlie, respectively. By selecting appropriate measurements, any two parties can predict precisely the value of any spin component (J^X, J^Y) , or J^{Z}) of the remaining spin system [51]. It was explained in Refs. [31,39] how this implies the two- and three-observable steering of any one party (e.g., B) by the remaining group (e.g., AC), i.e., $S_{B|\{AC\}}^{(3)} = S_{B|\{AC\}}^{(2)} = 0$. It is also true that the measurement of the single spin system B allows perfect inference of the orthogonal spin components of the collective system AC. This implies a Bohm's EPR paradox and hence steering since two spin components cannot both be specified simultaneously in a quantum state description [22,31,46]. Such two-way collective EPR steering for the GHZ state is depicted in Fig. 5. The bipartite steering between the individual systems is evaluated, by tracing over one system, to obtain the reduced quantum state of the other two. As is well known [7,35,50], the



FIG. 5. (Color online) Composition of bipartite EPR steering for tripartite qubit *W* and GHZ states. (a) No bipartite EPR steering can be detected for the *W* state using two-setting inequalities. Bipartite entanglement exists, as illustrated by the dashed lines. (b) The GHZ state shows no bipartite steering. Collective steering of one site by the group of two can be detected using three- and two-setting inequalities. (c) Bipartite two-way EPR steering exists for the *W* state, and can be detected by the three-setting inequality.

reduced system is a mixture of product states, and is therefore not entangled. Hence, there can be no bipartite steering.

The W state [50]

$$|\psi\rangle = \frac{1}{\sqrt{3}} \{|\uparrow\rangle_A|\downarrow\rangle_B|\downarrow\rangle_C + |\downarrow\rangle_A|\uparrow\rangle_B|\downarrow\rangle_C + |\downarrow\rangle_A|\downarrow\rangle_B|\uparrow\rangle_C\}$$
(38)

gives a different sort of steering entanglement. It can be shown that there is steering of *B* by the group *AC*, but the steering is reduced so that $0 < S_{B|\{AC\}}^{(3)} < 1$ [39]. The reduced state for *BA* after tracing over *C* is

$$\rho_{AB} = \frac{1}{3} \{ 2|\psi\rangle\langle\psi| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \},\tag{39}$$

where $|\psi\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ (we use the shortened notation $|\uparrow\uparrow\rangle \equiv |\uparrow\rangle_A|\uparrow\rangle_B$). Conditional variances for Alice inferring Bob's results of measurement of spin are calculated in the Appendix. If Alice measures σ_Z^A , then the average conditional variance is $(\Delta\sigma_{B|A}^Z)^2 = \frac{2}{3}$. If she measures either σ_A^X or σ_A^Y , then, respectively, $(\Delta\sigma_{B|A}^X)^2 = \frac{5}{9}$ and $(\Delta\sigma_{B|A}^Y)^2 = \frac{5}{9}$. Although no steering can be deduced from the two-observable inequalities of Sec. II B, the values are enough to confirm three-observable bipartite steering since (using the expression from Sec. III A) $S_{B|A}^{(3)} \leq \frac{8}{9} < 1$. From the symmetry of the *W* state, we can deduce that this steering must be two way (Fig. 5). We note that the values are consistent with the monogamy relation (36) $[S_{B|A}^{(3)} + S_{B|C}^{(3)} \geq S_{B|\{AC\}}^{(3)}]$ that applies in this case.

The two-observable steering behavior is different. Here, the stricter monogamy inequality (34) applies: $S_{B|A}^{(2)} + S_{B|C}^{(2)} \ge 2$. For the *W* state, we deduce that *nosteering* is detectable via two-observable inequalities. The *W* state has complete symmetry with respect to the three sites. Hence, if there is steering of *B* by *A*, then there must be steering of *B* by *C*, which we have seen is impossible for two-observable inequalities (Results (2) and (4)).

VII. DISCUSSION AND CONCLUSION

The monogamy inequalities for EPR steering are likely to be useful. For example, in order to observe EPR steering with twosetting inequalities, we understand why it is necessary for the steering party to have greater than 50% efficiency for detection of data [37,38]. Otherwise, an eavesdropper could detect the steering also, which is forbidden by the two-setting monogamy relation. The argument extends to the *m*-setting inequalities, where the bound for efficiency η is $\eta > 1/m$ [28,39].

Monogamy relations give a simple way to understand security in quantum communication. If it can be shown that A steers B via a two-observable inequality, so that $E_{B|A}$ or $S_{B|A}^{(2)} < 1$, then it is guaranteed that for a third (eavesdropper) observer C, $E_{B|A}$ or $S_{B|A}^{(2)} \ge 1$. Where the steering witness is directly related to the variance of the conditional inference for Bob's values of qubits or amplitudes, given Alice's measurements, the monogamy relations quantify the minimum noise levels for an eavesdropper to infer Bob's values. This aids our understanding of QKD schemes based on a shared quadrature amplitude value or a shared qubit value.

The new feature associated with quantum steering is the potential to implement *one-sided device-independent* cryptographic security [4,25]. The noise levels for the eavesdropper are quantified based on the uncertainty relation only, and do not depend on the details of a particular protocol. The device-independent security is one way since it is Alice's inference of Bob's amplitudes or spin values that are secured by the steering monogamy relations.

The monogamy with respect to steering witnesses has explained the monogamy of violations of Bell inequalities. Bell monogamy arises because Bell inequalities are also steering inequalities. As such, the degree of monogamy will depend on the number of observables (settings) of the Bell inequality.

Finally, the results presented here have enabled a characterization of the bipartite sharing for the tripartite CV Gaussian states, and for qubit GHZ and W states, and several experimental tests and realizations have been proposed. Open questions remain. For example, the monogamy results given in this paper give a quantification of how the steering of a single system by a group is shared, but the nature of the reverse monogamy has not been examined.

ACKNOWLEDGMENTS

This research was supported by an Australian Research Council Discovery grant. I thank Q. He, S. Armstrong, P. K. Lam, P. Drummond, A. Zeilinger, and B. Wittmann for stimulating discussions on steering and related topics.

APPENDIX

From (39), if Alice measures σ_Z^A , then the average conditional variance is

$$\left(\Delta\sigma_{B|A}^{Z}\right)^{2} = \sum_{i} P\left(\sigma_{A}^{Z}=i\right) \left[\Delta\left(\sigma_{B}^{Z}|\sigma_{A}^{Z}\right)\right]^{2}$$
$$= \frac{1}{3} \times 0 + \frac{2}{3} \times 1 = \frac{2}{3}.$$

The joint probabilities for measurement are as follows: $\frac{1}{3}$ for both Alice and Bob with spins down; $\frac{1}{3}$ for Alice's spin down and Bob's up; and $\frac{1}{3}$ for Alice's spin up and Bob's down. If Alice measures spin +1, then Bob's state is $|\downarrow\rangle$ and the conditional variance is 0. If Alice measures -1, then Bob's spin is up and down with probability $\frac{1}{2}$, and the conditional variance is 1. We can rewrite in the basis of spin *X*:

$$\rho_{AB} = \frac{1}{3} \{ 2 |\psi_X\rangle \langle \psi_X| + |\psi_{mX}\rangle \langle \psi_{mX}| \}.$$

Here, $|\psi_X\rangle = (|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/\sqrt{2}$ and $|\psi_{mX}\rangle = \frac{1}{2}\{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle - |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$. If Alice measures σ_A^X spin +1 (with probability $\frac{1}{2}$), then the probability is $\frac{5}{6}$ for Bob's up and $\frac{1}{6}$ down, for which the mean is $\frac{2}{3}$ and the conditional variance is $1 - \frac{4}{9} = \frac{5}{9}$. The same variance is obtained for outcome -1. Thus, $(\Delta\sigma_{B|A}^X)^2 = \frac{5}{9}$. Rewriting in the basis of *Y*, we obtain the same conditional variance $(\Delta\sigma_{B|A}^Y)^2 = \frac{5}{9}$ as for spin *X*.

- [1] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
- [2] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
- [3] A. Acin, N. Gisin, and L. Masanes, Phys. Rev. Lett. 97, 120405 (2006); A. Acin, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, *ibid.* 98, 230501 (2007).
- [4] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, Phys. Rev. A 85, 010301(R) (2012).
- [5] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing*, Bangalore, India (IEEE, New York, 1984), p. 175.
- [6] W. Wootters and W. Zurek, Nature (London) 299, 802 (1982).
- [7] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
- [8] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
- [9] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006).
- [10] J. S. Bell, Physics 1, 195 (1964); J. F. Clauser, M. A. Horne,
 A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
- [11] Ll. Masanes, A. Acin, and N. Gisin, Phys. Rev. A 73, 012112 (2006).
- [12] B. Toner, Proc. R. Soc. A 465, 59 (2009).
- [13] V. Scarani and N. Gisin, Phys. Rev. Lett. 87, 117901 (2001).
- [14] D. Collins and N. Gisin, J. Phys. A: Math. Gen. 37, 1775 (2004).
- [15] G. Adesso and S. Piano, arXiv:1307.3288.
- [16] C. Weedbrook et al., Rev. Mod. Phys. 84, 621 (2012).
- [17] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A 73, 032345 (2006); Phys. Rev. Lett. 93, 220504 (2004); A. Serafini, G. Adesso, and F. Illuminati, Phys. Rev. A 71, 032349 (2005);

G. Adesso and F. Illuminati, New J. Phys. 8, 15 (2006);
T. Hiroshima, G. Adesso, and F. Illuminati, Phys. Rev. Lett. 98, 050503 (2007);
G. Adesso, D. Girolami, and A. Serafini, *ibid.* 109, 190502 (2012).

- [18] F. F. Fanchini, M. C. de Oliveira, L. K. Castelano, and M. F. Cornelio, Phys. Rev. A 87, 032317 (2013).
- [19] M. F. Cornelio and M. C. de Oliveira, Phys. Rev. A 81, 032332 (2010).
- [20] T. C. Ralph, Phys. Rev. A 61, 010303(R) (1999); M. Hillery, *ibid.* 61, 022309 (2000); M. D. Reid, *ibid.* 62, 062308 (2000);
 C. Silberhorn, N. Korolkova, and G. Leuchs, Phys. Rev. Lett. 88, 167902 (2002); F. Grosshans *et al.*, Nature (London) 421, 238 (2003); V. Scarani *et al.*, Rev. Mod. Phys. 81 1301 (2009);
 Lars S. Madsen *et al.*, Nat. Commun. 3, 1083 (2012); J. Leach *et al.*, Phys. Rev. A 85, 060304 (2012); F. Furrer *et al.*, Phys. Rev. Lett. 109, 100502 (2012).
- [21] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, Phys. Rev. Lett. 68, 3663 (1992).
- [22] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
- [23] M. D. Reid et al., Rev. Mod. Phys. 81, 1727 (2009).
- [24] E. Schroedinger, Naturwiss. 23, 807 (1935); Proc. Cambridge Philos. Soc. 31, 555 (1935); 32, 446 (1936).
- [25] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. 98, 140402 (2007).
- [26] S. J. Jones, H. M. Wiseman, and A. C. Doherty, Phys. Rev. A 76, 052116 (2007).
- [27] D. J. Saunders et al., Nat. Phys. 6, 845 (2010).

- [28] A. J. Bennet, D. A. Evans, D. J. Saunders, C. Branciard, E. G. Cavalcanti, H. M. Wiseman, and G. J. Pryde, Phys. Rev. X 2, 031003 (2012); D. A. Evans, E. G. Cavalcanti, and H. M. Wiseman, Phys. Rev. A 88, 022106 (2013).
- [29] D. Smith et al., Nat. Commun. 3, 625 (2012).
- [30] B. Wittmann et al., New J. Phys. 14, 053030 (2012).
- [31] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Phys. Rev. A **80**, 032112 (2009).
- [32] S. L. W Midgley, A. J. Ferris, and M. K. Olsen, Phys. Rev. A 81, 022101 (2010).
- [33] K. Wagner et al., arXiv:1203.1980.
- [34] V. Handchen et al., Nat. Photonics 6, 596 (2012).
- [35] M. Plesch and Vladmir Buzek, Phys. Rev. A 67, 012322 (2003);
 68, 012313 (2003).
- [36] M. D. Reid, Phys. Rev. A 40, 913 (1989).
- [37] W. Bowen, R. Schnabel, P. K. Lam, and T. C. Ralph, Phys. Rev. Lett. 90, 043601 (2003).
- [38] D. Buono, G. Nocerino, A. Porzio, and S. Solimeno, Phys. Rev. A 86, 042308 (2012).
- [39] Q. Y. He and M. D. Reid, Phys. Rev. Lett. **111**, 250403 (2013).
- [40] S. Walborn *et al.*, Phys. Rev. Lett. **106**, 130402 (2011);
 J. Schneeloch *et al.*, Phys. Rev. A **87**, 062103 (2013).
- [41] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A 69, 062311 (2004).
- [42] Q. Y. He, Shi-Guo Peng, P. D. Drummond, and M. D. Reid, Phys. Rev. A 84, 022107 (2011).

- [43] H. F. Hofmann and S. Takeuchi, Phys. Rev. A 68, 032103 (2003).
- [44] Q. Y. He, P. D. Drummond, and M. D. Reid, Phys. Rev. A 83, 032120 (2011).
- [45] E. G. Cavalcanti et al., Opt. Express 17, 18693 (2009).
- [46] E. G. Cavalcanti, Q. Y. He, M. D. Reid, and H. M. Wiseman, Phys. Rev. A 84, 032115 (2011).
- [47] G. Giedke *et al.*, Phys. Rev. A **64**, 052303 (2001); Phys. Rev. Lett. **87**, 167904 (2001).
- [48] J. Eisert and M. Plenio, Int. J. Quantum Inf. 1, 479 (2003).
- [49] B. L. Schumaker and C. M. Caves, Phys. Rev. A 31, 3093 (1985);
 R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Metz, and J. F. Valley, Phys. Rev. Lett. 55, 2409 (1985).
- [50] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000); R. Horodecki *et al.*, Rev. Mod. Phys. 81, 865 (2009);
 O. Guhne and M. Seevinck, New J. Phys. 12, 053002 (2010).
- [51] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. 58, 1131 (1990).
- [52] P. van Loock and S. L. Braunstein, Phys. Rev. Lett. 84, 3482 (2000); Phys. Rev. A 63, 022106 (2001).
- [53] T. Aoki, N. Takei, H. Yonezawa, K. Wakui, T. Hiraoka, A. Furusawa, and P. van Loock, Phys. Rev. Lett. 91, 080404 (2003).
- [54] S. Armstrong et al., Nat. Commun. 3, 1026 (2012).
- [55] M. K. Olsen *et al.*, J. Phys. B: At. Mol. Opt. Phys. **39**, 2515 (2006).
- [56] Q. Y. He and M. D. Reid, Phys. Rev. A 88, 052121 (2013).