Quantum Anomaly and Collective Oscillations in 2D Fermi gases

Tyson Peppler
Center for Quantum and Optical Sciences
Swinburne University of Technology

December 3, 2018
Supervised by Chris Vale
Declaration:

I, Tyson Peppler, declare that this thesis entitled:

“Quantum Anomaly and Collective Oscillations in 2D Fermi gases”

contains no material which has been accepted for the award to the candidate of any other degree or diploma, except where the due reference is made in the text of the examinable outcome; to the best of my knowledge, contains now material perviously published or written by another person except where due reference is made in the text of the examinable outcome; and where work is based on joint research or publications, discloses the relative contributions of the respective workers or authors.

Tyson Peppler

Centre for Quantum and Optical Science (CQOS)
Faculty of Science, Engineering and Technology
Swinburne University of Technology
Melbourne, Australia
For his Nobel Prize money, Niels Bohr bought a summer residence in Tisvilde in 1924. A horse shoe was nailed above its front door. A visitor once asked in surprise: `You don’t believe that this will bring good fortune?’. `No, no, of course not,’ answered Niels Bohr, `but I have been told it will bring good fortune whether you believe in it or not.’

Acknowledgements

The work presented here is the sum of many hours of toil, against the forces of fate and luck as well as all the mundane physical ones: the battle against which I have learned is the hallmark of all experimental physics. That I have had the pleasure and privilege to undertake such a work, and for his incredible patience and understanding, I would like to primarily thank my supervisor Chris Vale - if sailors have honed their skills at piloting the seas, then Chris has honed his skills piloting a rowdy and temperamental sea of physics labs, and I have only gained to study under him.

I would secondly like to thank Paul Dyke. To say this work would exist in its present form without the incredible work ethic and perseverance of Paul is to speak false, and it was awe inspiring to witness such industriousness over the course of many years. In addition I thank the rest of the immediate group, including Sascha Hoinka - whose pursuit of precision and excellence in all tasks I could only hope to match - as well as my former candidature members, Kristian Fenech and Marcus Lingham, for keeping me grounded in the office. Speaking of the office, I would thank profusely Peter Hannaford, for his many wisdoms on the walk to our weekly meetings. Also Marta Zamorano, whose work while she visited was greatly appreciated. Lengthy discussions and banter with Brendan Mulkerin were always welcome, as his knowledge of theoretical matters and ability to discuss were very stimulating.

Outside of the University I would thank all my friends for being there throughout, always confused at my frustration when they asked how it was going. To my family, especially Mum and Dad, who are now very excited about the prospect of me entering the less academic world - for all their patience and running around over the years. And to Emma my older sister, who has (today) produced the first of the next generation, as well as Karra my younger sister.

Lastly I get to the most important acknowledgement, which is of course to Oscar Mike coffee barista cafe fantastic... to Clare Due, my partner and closest friend, I am eternally grateful we weathered the long-distance years and double PhD and we only became stonger for it.

In closing I will acknowledge all those who perform in academia but also deal with mental illness, as it is a pernicious thing, and difficult.
Abstract:
The experimental study of ultracold Fermi gases allows a highly tunable benchtop system to explore the intersection between fundamental quantum, atomic and condensed matter physics. Tuning parameters such as atom number, interaction strength, temperature and trap geometry we examine the nature of collective oscillations that can be excited in reduced dimensional atomic clouds - specifically in the 2D and quasi-2D dimensional regimes. It turns out the collective dynamics of these atom clouds are closely tied to their fundamental properties, such as their equation of state and interaction parameters, making collective oscillations a powerful probe to analyse such systems. One especially interesting phenomena that can be accessed by collective oscillations in a 2D Fermi gas is a probing of the ‘Quantum Anomaly’: an anomalous symmetry breaking which arises due to the need to renormalize the interaction Hamiltonian present in the trapped 2D gas. I present our findings on experiments concerning the anomaly as well as the dimensional crossover from 2D to quasi-2D using collective oscillations in this thesis.
## Contents

**List of Figures**

**List of Publications**

1. **Introduction**

2. **Theory**
   2.1 Thermodynamics of Fermi gases
   2.2 Scattering Physics
      2.2.1 Scattering in 3D, 2D and quasi-2D
      2.2.2 Fano-Feshbach Resonances
      2.2.3 Bound states and the BEC-BCS crossover
   2.3 Collective Dynamics
      2.3.1 Polytropic Equation of State
      2.3.2 Hydrodynamics
   2.4 Scale Invariance and the Anomaly

3. **Experimental Methods**
   3.1 General apparatus
   3.2 Cooling the gas
   3.3 2D Trap schematics
      3.3.1 Transverse trapping potential: TEM$_{01}$ mode dipole
      3.3.2 Radial trapping potential: Magnetic field
   3.4 Imaging Techniques

4. **Analytic Methods**
   4.1 Thermometry
      4.1.1 Density Profiles and Equation of State
      4.1.2 Virial Expansion
      4.1.3 The GG theory
      4.1.4 Fitting Routine
   4.2 Principal Component Analysis (PCA)
5 The Dimensional Crossover

5.1 2D Criteria ................................................................. 79
5.2 Transverse width experiment ......................................... 81
5.3 Collective oscillation experiment .................................. 84
  5.3.1 Experimental method: Breathing mode ...................... 84
  5.3.2 Experimental method: Dipole mode ....................... 89
  5.3.3 Results discussion ............................................. 93
5.4 The Quantum Anomaly ................................................ 96
  5.4.1 Experimental method ......................................... 96
  5.4.2 Results discussion ............................................. 97

6 Conclusion & Outlook .................................................. 101

A Nano-Fabrication of Phase Plate .................................. 103
  A.1 Plasma Etching .................................................. 105
  A.2 Evaporative Coating of MgF\textsubscript{2} ................. 107

Bibliography ............................................................... 109
List of Figures

2.1 Pictorial representation of the energy states in a harmonic oscillator. On the left is shown the behaviour for Bose-Einstein statistics (many particles can occupy the same state) for zero temperature (bottom image) and greater than zero temperature (top image) with similar on the right for for Fermi-Dirac statistics (only one particle may occupy the same state). The blue and red represent two different components, which are not identical and can interact. ............................. 7

2.2 Results for the thermodynamic variables for 2D and 3D, where \( \overline{\omega} = \prod_i \omega_i \) .......................................................... 9

2.3 Density profile of a 2D cloud for an ideal gas at zero temperature, in a harmonic trap. Generated with \( N = 1 \times 10^4 \text{atoms} \) atoms, the vertical axis is density \( n \) in \( \text{atoms/m}^2 \) The horizontal axis are pixels, simulated to match the parameters of our experiment, where 1pixel = 2.84\( \mu \)m. . . . . 12

2.4 Feshbach resonance of the \( |1\rangle - |2\rangle \) and \( |1\rangle - |3\rangle \) hyperfine states showing magnetic field vs scattering length, where \( |1\rangle = |F = 1/2, m_F = +1/2 \rangle \), \( |2\rangle = |F = 1/2, m_F = -1/2 \rangle \) and \( |3\rangle = |F = -1/2, m_F = -1 \rangle \). . . . . . . 19

2.5 In 3D, varying pairing mechanism exist as the scattering length is altered. For positive scattering lengths the existence of the bound state gives rise to tightly bound dimers. Where the scattering length diverges - at unitarity - we have a region of strong interactions where the physics is universal and the only relevant length scale is the mean spacing between particles. The BCS side of the resonance exists for negative scattering lengths, where pairs are formed by long-range Cooper pairing on the surface of the Fermi sea. ............................. 21
2.6 Difference in scattering to bound state energy plotted versus the harmonic oscillator length $l_z$ divided by the 3D scattering length $a_s$. There exists a bound state for all values of the scattering length in 2D and quasi-2D, where in 3D there only exists a (dimer) bound state for positive scattering lengths. The quasi-2D behaviour is shown to approach the limit of both 2D and 3D bound state energies in the relevant limits. This is because as the binding energy $E_b$ becomes larger relative to the energy of the transverse oscillator ground state ($\omega_z$), the system takes on a 3D character. Plot taken from discussion by Levinsen et al.[49].

3.1 This is a photo of the apparatus as it stands in the lab. You can see the glass window of the science cell in the center of the image, with one of the MOT coils above it. The relevant schematics of the 2D trap will be shown below, this image here is shown to give a feel of what the apparatus looks like in reality - in fact much of the optics in the photo are used for experiments on Bragg spectroscopy[74].

3.2 Shown here is the electronic energy spectrum of the lithium atoms[76][69]. The cooling and repump beams are used in the Zeeman slower and MOT systems, where we use the transitions labeled. It is important to have the repump beam as any atoms decaying from the $|F' = 5/2\rangle$ state into the $|F = 1/2\rangle$ state can be re-cycled into the $|F = 3/2\rangle$ state to access the cooling beam once again.

3.3 The convention for the dimensions when talking about the 2D laser trapping potential. The transverse direction is also referred to as the 'tight' direction - it is the dimension which is constrained so that the atom cloud exists in the $x,z$ plane. Where this convention changes (sometimes we use $z$ for the transverse direction when doing analysis) I will take care to outline it.

3.4 Visualization of the intensity for the TEM$_{01}$ mode curve as given by equation (90). The scales here are in arbitrary units for visualization. The vertical axis is intensity.
3.5 Here we see images taken of the TEM$_{01}$ mode. These images are at the focal point of the mode, where a sweep of many images each 50$\mu$m apart is taken, through the focus. We can use these images to calibrate the mode as shown below. Width of the valley ($1/e^2$) in the transverse direction $W_y \approx 10\mu$m. The $z$-dimension is the direction of laser beam propagation. Camera used to take these images was BASLER acA2500 - 14um with attached confocal lens of numerical aperture NA = 0.25 placed at the end of a lens tube.

3.6 This is a scan-through of the intensity profile of the 2D trap. It is made up of a series of images taken through the focal point of the trap from the final lens, showing how the intensity distribution varies over this focal length and how the intensity of the 'lobes' - the two peaks - varies. We use images like this to calibrate the potential of the trap, as we can calculate the waist sizes and thus frequencies from this in our MATLAB program. The $1/e^2$ waist here is $W_y = 10\mu$m. The numbers shown on the axes correspond to camera pixels - the camera had pixel size 6.45$\mu$m, and we use a magnification of $\times$9.3 to our confocal imaging lens, so this gave us a final conversion of 1unit = 0.65$\mu$m along the $x$-axis of this image. The $y$-axis is how far the stage is wound between successive images, which here is 2.5$\mu$m per unit. In the bottom image the wave-like imperfections are due to slight misalignment as the camera stage is wound by hand between slices.

3.7 Vertical displacement of the centre of mass in pixels vs time in ms. This measurement is of the center of mass measured from the side of the 2D trap at differing time steps. Each measured time step (blue) is the average of 3 images. The fit (red) is of an exponentially damped sine wave $ae^{-bx}\sin(cx+d) - ex + f$, with $a$-$f$ the fitting variables and $x$ the timestep. This exponential captures the reduction in amplitude over the range of the data, with the sine wave capturing the oscillation itself. The fit results in a tight trapping frequency of $\omega_z = 2\pi \times 5.65$kHz.

3.8 Schematic of the 2D trapping optical setup. Lengths are not to scale.

A detailed description of the setup is provided in the text.
3.9 The Feshbach coils in Helmholtz configuration from which the magnetic trapping potential is produced. Since the coils have radius 4cm and are placed 5cm apart, this results in the residual curvature. ........................................ 55

3.10 Here we have a slice of the magnetic field intensity distribution, for 111 Amps modeling our in-house Feshbach fields in Helmholtz configuration. You can see that for the center slice (left), there is an inverted bowl potential for high-field seeking atoms, which peaks at around 832G for input current of 111 Amps. For the slice through of the height (right) we have a small dip in the magnetic potential as our coils are displaced slightly more than a regulation Helmholtz configuration. ........................................ 57

3.11 A contour surface slice-through of the magnetic field for the 111 Amps as in the figure above. Here we see the magnetic field intensity variation at all positions around the coils (black dots - they are infinitely thin in this model), with the saddle that exists in between the coils. The axes here are in meters, with the field intensity colours labeled on the contours themselves. ................................................................. 57

3.12 Two images of the cloud at different exposure lengths and intensities. The optical density of the clouds is shown for each as the false-colour bar. Pixel length here is 2.84μm with the magnification of the top camera (×4.58) already factored in. ......................................................... 61

3.13 Here we can see an absorption image from the side camera setup, where this shot is taken after a time-of-flight of 600μs: in-situ images of the 2D cloud from the side are so thin as to be beyond the diffraction limit. We use side images such as these to calibrate the tight trapping frequency. 62

4.1 Results of the equation of state for ideal density profiles at T=0 (blue) and T=25nK (red). The plots are numerical simulation, at discrete points (hence the scatter). The value $\tilde{p}$ is the reduced pressure and the $\tilde{\kappa}$ the reduced compressibility. This theory could be used to calculate the temperatures, by fitting to our measured experimental density profiles, however it does not take into account interactions, and so we consider more complex theories in the next section. These curves model functions present in equations (110) on. ...................................................... 66
4.2 This graph shows the comparison between the GG and the 2nd and 3rd order virial function for the choice of $0.2\beta E_B$. You can see they converge for high $T$ or negative chemical potential $\mu$. ........................................ 68

4.3 Surface showing the GG data that was generated for different interaction strengths ($\beta E_B$) across the various $\beta \mu$. From the surface we can choose a fidelity that we require by interpolating where needed. Note that we don’t use the part of the surface where there are no generated points from the GG theorem. .................................................. 69

4.4 Here is an example of the temperature fitting result, where we have on the y-axis the equation of state have used data at 725G and an average over 20 absorption images which are then radially averaged to produce the density profile. The green circles are the data points chosen to fit to the composite function (in brown), where the 2nd and 3rd order virial functions (dashed lines) are shown as well. The GG function is also show - but is identical to the composite function over this range. You can see in the negative $\mu$ or high $T$ limit the virial functions overlap the composite function. Note that the temperature from the fit $T_{fit}$ is very close to the temperature to the initial guess - $T_{\beta E_B}$ - showing the converge via iteration over this value. This example also illustrates data that does not have the best background signal/noise, and this method of the composite GG function allows us to fit well up to around $\beta \mu$ of 0.7. ................................................................. 73

4.5 Table reporting the results of the temperature fits for various magnetic fields for the $|1\rangle$ $|3\rangle$ mixture, where $|1\rangle = |F = 1/2, m_F = +1/2\rangle$ and $|3\rangle = |F = -1/2, m_F = -1\rangle$. .................................................. 74

4.6 The result of the PCA algorithm for a set of 35 images, taken at 725G for a $|1\rangle, |3\rangle$ mixture and $N \approx 45K$ atoms. The first image is the ‘mean’ image which is the average of all the input images. The next three reconstructed images show eigenmodes of greatest variance, in this case a breathing mode, dipole mode and weak quadropole mode. The rest of the images show very weak to arbitrary noise signals which seem not to indicate any real systematic cause. Noise sources that could be indentified include imaging beam diffraction fringes, camera position (mechanical) variation and stray light which varies from shot to shot. .............................. 77
4.7 Here we see the first two principal components from the previous figure, when projected back on to the mean image. The top image shows the breathing mode oscillation, with the bottom showing a dipole mode. The right hand graph gives a measure of the proportion of this component contained in the mean image vs time (ms), which allows us to fit and gain the frequency of these modes. ........................................ 78

5.1 Diagram of the zero temperature picture. Here we see the radial and transverse harmonic energy levels with the condition that the first excited state of the transverse harmonic oscillator not be occupied. The two spin states are denoted by the red and blue balls. On the left the transverse direction is the z-axis. ........................................ 80

5.2 The results from our analysis[36]. On top the three plots display graphs of the transverse width vs atom number for three different interaction strengths. In those plots, (b) and (c) show an elbow point where the trend goes from no increase in transverse width to a linear increase with atom number - this being indicative of a change from below 2D to quasi-2D dimensional behaviour. The bottom surface plot is of atom number on the vertical axis (represented by $E_F/\hbar\omega_z$), with interaction strength the horizontal axis (represented by the harmonic oscillator length divided by the 3D scattering length $l_z/a$). The colour is then the transverse width. The elbow points of the top graphs are plotted as the large grey dots in the bottom surface, showing a trend toward the ideal criterion of $E_F/\hbar\omega_z$ as the interaction strength shifts away from the Feshbach resonance on the BCS side. ........................................ 82

5.3 ........................................ 85
5.4 Result from a breathing mode oscillation measurement at a magnetic field of 700 G, with a cloud of atom number $N \approx 10 \times 10^3$ per spin state. Top left is the absorption image of the cloud from above, where the pixel scale has conversion $1 \text{px} = 2.84 \mu m$. The false colour scale is the column density measured in atoms/pixel area, where the raw data has had an optical density correction of 1.28 applied (see 3.4). Top right is the same cloud after radial averaging about the centre, with a gaussian fit $f_{\text{gauss, fit}}(x) = ae^{-x^2/b^2}$, where $(a, b)$ are fit variables, in green. We take the $1/e^2$ radius of the fit, which is equal to the fit variable $b$. The bottom is the breathing mode data for this particular choice of magnetic field and atom number, where each data point is the average of three hold times $t_{\text{hold}}$. The fit function is an exponentially damped sine wave $f_{\text{sine, fit}}(x) = ae^{-bx}\sin(cx + d) + e$ which gives the breathing mode frequency $\omega_B$. In this case $\omega_B = 2\pi \times 44.8(44.63, 44.98)$, where the error values in brackets are determined from the fitting function and represent the 95% confidence interval. The error on each data point is the standard deviation from summing over the three radius measurements taken at each $t_{\text{hold}}$. 

5.5 Variation of the centre of mass of the atom cloud versus time for 1D profiles X and Y, where the profiles were created by averaging the cloud over the X and Y directions. This data was for a magnetic field of 700 G and atom number of $N \approx 17000$. Each point is an average of two images at that hold time, where the error is the standard deviation of the average of the center pixel of those two images. For the top camera $1 \text{ px} = 2.84 \mu m$. The fit function used here is an exponentially damped sine wave, $f_{\text{sine, fit}}(x) = ae^{-bx}\sin(cx + d) + e$, with the letters (a-e) being the fit variables. We see that the top image for Y has frequency $\omega_y = 2\pi \times 21.59(21.41, 21.76)$ where the error is the 95% confidence interval on the fit variable. For X we have $\omega_x = 2\pi \times 22.39(22.29, 22.49)$. This gives us the radial trapping frequency for this magnetic field, which is the geometric mean of the X and Y, $\omega_r = \sqrt{\omega_x\omega_y}$, here $\omega_r = 21.99(21.85, 22.12)$. 
5.6 Results for the geometric mean radial frequencies $\omega_r$ for various magnetic fields. The blue points are $\omega_r$ taken on different days, their scatter we attribute to different lab conditions which could affect the anti-trapping caused by slight misalignments of the 2D laser trap, affecting the measurement of the dipole oscillation. We took many points and then fit the function $f(B) = \alpha \sqrt{B} + \beta$ to the averages at each magnetic field (represented by the green crosses).

5.7 Results for the breathing mode oscillation across the dimensional boundary, quantified by the atom number relative to the 2D ideal criterion atom number, $N/N_{2D}^{id}$. The vertical axis is the breathing mode frequency proportional to the radial trapping frequency, $\omega_B/\omega_r$. The four different interactions we explored are the different colours, with one on the near-BEC side of the resonance ($l_z/a_{3D} = 0.53$ blue), which shows a similar trend to that of the resonance ($l_z/a_{3D} = 0$ green) and the near-BCS side of the resonance ($l_z/a_{3D} = -0.48$ red). The $\sqrt{10/3}\omega_r$ result is the breathing mode frequency we would expect for a ‘pancake’ quasi-2D Bose gas at $T = 0$. As we travel further from the resonance on the BCS side ($l_z/a_{3D} = -2.12$ purple) we see the cloud remain closer to the expected $2\omega_r$ of the 2D breathing mode behaviour for atom numbers $N/N_{2D}^{id} \leq 1$ relative to the 2D criterion, which still appears to converge to the $\sqrt{3}\omega_r$ of the quasi-2D regime as the 2D atom number criterion is exceeded.

5.8 Table of breathing mode frequencies and polytropic co-efficients as in [37]

5.9 The results of our breathing mode oscillation frequency divided by the radial trapping frequency $\omega_B/\omega_r$ plotted versus interaction strength log($k_F a_{2D}$). We see a deviation of around 2.5% across the strong interacting regime up to around log($k_F a_{2D}$) = 3, where we see a return to the 2D breathing mode frequency of twice the radial trapping frequency. This data was taken at temperature of around $T/T_F = 0.15$, and at an atom number $N = 0.3 N_{2D}^{id}$, well below the 2D ideal atom number criteria. The error on the points is due to the error in the frequencies for the breathing and radial modes, which stems from the uncertainty in the fitting functions.
5.10 Theory curves from the literature. In the left panel is a graph from the review article by Levinsen and Parish review[49] which displays the theory curve from Hofmann[44] in blue. The vertical axis on the left is the frequency shift $\delta \omega$ from the breathing mode $\omega_B$, divided by the dipole mode frequency $\omega_0$. Both graphs plot versus the interaction strength in terms of $\ln(k_F a_{2D})$. The data points in the left panel are the results from the experiment of Vogt[46], which deviate from both the theory and our results for values of $\ln(k_F a_{2D}) < 3$. This is likely due to the temperature they were taken at, which was around $T/T_F = 0.37$, contrasted with our results at $T/T_F = 0.15$. The right hand graph is the theory from the work of Gao and Yu[43], where the vertical axis is the same as the left hand graph but for a factor of two. A combined graph is presented in figure 5.11.

5.11 Plot of our data points (in red) for the anomaly overlaid on the theory curves from the literature. They include the Hofmann results in blue[44], Gao & Yu as the purple points[43] and the results of Mulkerin et al in green[117]. This is the plot presented in our paper[115] and displays the deviation of the breathing mode frequency $\delta \omega$ from the dipole mode $\omega_c$. The scale invariant prediction of $2\omega_c$ is the dashed black line. Again the $x$-axis is the interaction strength quantified by $\ln(k_F a_{2D})$.

A.1 Phase plate schematic. The top diagram shows the optic with magnesium fluoride deposited on the surface, which should be of thickness $x_{\text{MgF}_2} = 702\text{nm}$ to create the $\pi$-phase shift. The bottom is the method to etch away the fused silica quartz flat. For the phase shift this would be of thickness $x_{\text{air}} = 577.4\text{nm}$.

A.2 AFM images of plasma etching. On the left we see the cross-section corresponding to the line in the right image. It shows the transition from un-etched half of the optic to the etched surface. The step height is around 520 nm for this sample. It can be seen that the edge is not as sharp as is desirable. The etched surface also displays a roughness which is exaggerated from the AFM cantilever sticking as it passes across the surface but indicates a real roughness of the surface. These images were analysed in the software program Gwyddion.

A.3 The Intlvac Nanochrome II thin film e-beam evaporative coater.
List of Publications


1 Introduction

The research question pursued in my thesis is the examination of collective oscillations in two-dimensional and quasi-two-dimensional interacting ultra-cold Fermi gases. This may provide answers to at least two fascinating physical questions. The first is a question of dimensionality - a determination of where the boundaries between being in a quasi-2D, 2D and even 3D regime lie - along with a description of what conditions must be met for a physical system to fully satisfy the conditions for achieving such dimensionalities. The second question is whether there is evidence of the breakdown of a classical scale invariance of the Hamiltonian with the subsequent appearance of the so-called ‘quantum anomaly’. In this chapter I will present a brief history, with motivation, for the field of ultra-cold atomic gases, with specific treatment of 2D and quasi-2D Fermi gases as well as collective oscillations in such gases, followed by a discussion of the quantum anomaly.

There are many fascinating and beautiful phenomena to be found in nature - especially if one delves into the quantum world. An illustrative example of such a phenomenon, one which if could be brought in our everyday macroscopic world, would bring about a revolution in technology, is that of superfluidity. Superfluidity, first discovered by Pyotr Kapitsa, John F. Allen and Donald Misener in 1938[1] when examining the behaviour of supercooled Helium-4 is the tendency of matter to flow without friction - that is - for the fluid to have zero viscosity. I begin this thesis with a brief discussion of superfluidity because (as we shall see) it is through the methods of experimental atom physics research - the advent of the manipulation of many atoms to be cooled and trapped by lasers - that we can shed light on the underlying fundamental quantum mechanisms that give rise to such exotic phases of matter.

The initial discovery of superfluidity was made in Helium-4. This integer spin (bosonic) species of Helium was cooled below its critical temperature (λ-point) of 2.17 K where it would display novel effects like scaling the walls of beakers[2] and persistent flow in annular containers. Explanation of these effects centered on its bosonic counting statistics - at low enough temperature all the particles will occupy the same energy level and combine to form a Bose-Einstein condensate (BEC). This BEC forms a single macroscopic wave-function on which disturbances can induce a collective re-
response - exciting many particles in unison. This excitation relates to fluctuations of the order parameter (the parameter quantifying the first order correlations between particles at different locations), rather than addressing individual particles - superfluid rather than normal fluid behaviour. This explanation is satisfactory for bosons but for superfluidity in particles with half-integer spin (fermions) it is more complex. Here the Pauli exclusion principle prohibits such macroscopic occupation of a single state and would seem to rule out superfluidity in a fermionic system. Such a superfluid Fermi system was however discovered in 1972 by Osheroff et al[3] in Helium-3, with a critical temperature of around 2mK. A microscopic explanation of how this could occur was found to exist in the closely related phenomena of superconductivity, put forth in 1957 by Bardeen, Cooper and Schrieffer in two seminal papers[4, 5]. This theory states that weak attractive interactions between electrons mediated by phonon exchange in a lattice lead to long-range Cooper pairing of electrons, which in turn can condense at low enough temperatures and give rise to a gap in the conductance band and hence superconductivity. Similar Cooper-type pairing between spin species can occur for Helium-3 atoms atop a filled Fermi sea which leads to condensation of pairs and a gap in the excitation spectrum: superfluidity.

Written as it is this sounds very straightforward. But what if we begin to question the parameters laid out, such as, why does the $\lambda$-point exist at 2.17 K for the bosonic Helium-4 and not another temperature? What role does the strength of the interaction between particles have? Why do we recover the same phenomena of superfluidity in the Fermi species (Helium-3) but with a pairing mechanism (that of Cooper pairs)? How does the system depend on the number of particles? Does it persist when there are only two, or three particles present? If there is a cutoff due to the number of particles, at, say, $n$ - why is it $n$? What of different species of atoms?

Such a wealth of questions stayed largely outside the reach of bench-top experiments until the advent of modern laser cooling and trapping technologies. Indeed the first realization of Bose-Einstein condensation (BEC) in a gas of atoms was achieved at JILA in 1995[6] followed by the production of a degenerate gas of fermionic atoms in 1999[7]. Since then many studies have been performed with two-component interacting Fermi gases focusing on topics such as the smooth crossover between BEC and Bardeen-Cooper-Schrieffer (BCS) pairing behaviours underpinning the superfluid
mechanisms present in 3D\cite{8, 9, 10, 11, 12, 13} and the equation of state (EOS)\cite{14}. In these experiments tunability of interactions is facilitated by a magnetic Feshbach resonance\cite{15} which allows control over the interactions between fermions in two different spin states. Lower dimensional systems have also been a keen topic of interest: in one-dimension exactly solvable theoretical models exist including the 1D Hubbard model\cite{16}; whereas in two-dimensions theoretical models are very difficult owing to the enhanced role of thermal and quantum fluctuations (not to say these fluctuations are not important in 1D, but there they can be solved using the Bethe ansatz\cite{17}) which rule out long-range-order at finite temperatures in non-trapped gases\cite{18}. Due to experimental challenges 2D Fermi gases were only first explored in 2010\cite{19, 20} but since then much interest has occurred in both the theoretical and experimental communities due to a succession of experiments\cite{21, 19, 20, 22, 23, 24, 25, 26}.

Collective oscillations provide a probe of ultracold Fermi gases which connects the few-body properties of the interactions between particles of the gas with the many-body collective behaviour of the whole gas. In that sense, collective oscillations are sensitive to the interactions between particles, to the temperature of the particles, as well as the nature of the trapping potential in a non-homogeneous case. Looking at the collective oscillations, measuring their frequency and character, has been a method of inquiry utilized from the beginning in quantum gas research, in theory\cite{27} and in experiment\cite{28, 29}. In 3D Fermi gases collective oscillations played a role in searching for superfluidity in early BEC-BCS crossover measurements\cite{12} as well as the transition from hydrodynamic to collisionless regimes\cite{30}.

Concepts central to the understanding of collective oscillations are that of the hydrodynamic regime: where, as the name suggests, the system behaves as a liquid, with the excitation wavelength much greater than the mean-free path for collisions between atoms, leading to first sound waves which are only weakly damped by these collisions\cite{31}. This is in counterpart to the collisionless regime, where the behaviour is more like an ideal gas - the collision rate is much smaller than the excitation frequency and particles traverse the gas essentially without colliding with other atoms\cite{32}. As well as there being differing regimes of collective oscillations there are also various modes the oscillations can take on. In 2D the simplest of these are the dipole mode (sloshing mode), the breathing mode and the quadrupole mode. The dipole mode is the
most elementary, involving simply center-of-mass oscillations. The *breathing mode* is more exotic, corresponding to a radially symmetric increase and decrease in the radius of the cloud; increasing and decreasing the density as a result. This breathing mode is typically used as a probe of the compressibility of the gas and is also connected to the thermodynamic equation of state (EOS). The *quadrupole mode* is a surface mode where a decrease in radius in one direction is counterbalanced by a corresponding increase in radius in another - thereby not changing the overall volume. This mode can be used to ascertain the shear viscosity of the gas[33, 34], and theoretically predictions of it exist for an imbalanced trapped Fermi gas in 2D[35]. For our exploration we focus primarily on the breathing mode for data and the dipole mode for calibration purposes.

One question of interest when studying 2D gases is that of dimensionality. By dimensionality, I mean the difference in the physics which occurs between a gas of three dimensions (3D), two dimensions (2D) and the in-between region known as quasi 2D. Various behaviours, such as the thermodynamic relations governing an equation of state, and the criteria for the onset of exotic phases such as superfluidity and Bose-Einstein condensation, vary with dimensionality. Much work has been done to map the transition from a 2D to a quasi-2D Fermi gas[20][36]. The breathing mode measurement allows another window into this question, as the hydrodynamical equations - of which the breathing mode frequency can be found as a solution - coupled with the exponent of the polytropic equation of state, give differing solutions to the ratio of the breathing mode to dipole mode frequencies $\frac{\omega_B}{\omega_0}$ as the dimensionality changes[37][38][39].

What then, is a quantum anomaly? To wit, the Hamiltonian for 2D harmonically trapped gas with $\delta$-potential interactions displays a SO(2,1) classical scaling symmetry. This, however, is broken at a quantum level[40] - a consequence of the interatomic interaction potential - and that renormalization of the contact interaction necessarily introduces a length scale characterized by $a_{2D}$ which gives rise to the ‘quantum anomaly’. There are other ways for symmetry breaking to occur, such as explicit or spontaneous symmetry breaking, but for the case where the symmetry breaking occurs when the system is quantized is known as anomalous[41]. That the quantum anomaly which we can interrogate in our ultra-cold gas experiment is of similar form to other anomalous symmetry breaking phenomena (such as the trace and chiral anomalies in
quantum chromodynamics[42]) is a good example of the ability of these bench-top experiments to serve as surrogates for experimentally inaccessible phenomena, as I contended at the beginning of this introduction. To probe the anomaly we seek the value of the deviation from the scale invariant classical prediction. The breathing mode of the gas is dependent on the interactions between the atoms and gives us a way to access this parameter. Theoretical predictions exist for this value in Fermi[43][44] - that it vary from the classically predicted twice the weak trapping frequency by up to around 10% - and Bose[45] gases.

It should be noted that there has previously been an exploration of such collective modes (breathing, hydrodynamic) in a Fermi gas, presented by Vogt et al in 2012[46]. Their results showed no deviation from the scale invariant result, but the parameter space they were working in was quite different to that which we explore in our experiment, and is discussed more in the later chapter on the quantum anomaly result.
2 Theory

In this chapter we go over the theoretical formalism associated with our experiment. Specifically we look at the statistical mechanics underlying the condensed matter physics of our experiment, followed by scattering properties and Feshbach resonances. We then go to theory of collective dynamics, including the hydrodynamics of the gas and the role of the polytropic equations of state. We end the chapter with a theoretical description of the quantum anomaly. The treatments in this section are completed in three, two and quasi-two dimensions where possible. Where I follow working and proofs laid out in literature relevant citations are provided.

2.1 Thermodynamics of Fermi gases

The ensemble we study in our laboratory consists of a two-component gas of the isotope lithium-6, cooled to near absolute zero, at around absolute temperatures of 10-100 nK. Particles of each component of the gas (corresponding to a particular spin state) is indistinguishable to other particles of the same spin state. This is important from a statistical mechanics viewpoint - for an ideal gas the state occupation of $N$ particles is governed by a probabilistic function which has different form if the particles are distinguishable. The notion of phase space density - $n \lambda_{dB}^3$, where $n$ is the density of the gas and $\lambda_{dB} = \sqrt{\frac{2\pi k_B^2}{mk_B T}}$ the thermal de Broglie wavelength at a given temperature $T$ (where $k_B$ is Boltzmann’s constant, $m$ is the mass of the particle and $h$ the reduced Planck’s constant) - is illustrative. Where $n \lambda_{dB}^3 < 1$, we have more classical behaviour, with the statistics being essentially Maxwell-Boltzmann. As $n \lambda_{dB}^3$ increases to 1 and above, we have the wavelength becoming larger than the interparticle spacing $n^{-1/3}$, requiring a quantum treatment. The dominant statistics in the quantum regime are either Fermi-Dirac or Bose-Einstein, depending on the symmetry of the wavefunction: bosonic wavefunctions are symmetric under the exchange of two particles - corresponding to particles with integer spin - and can therefore occupy the same single-particle state; fermions are conversely anti-symmetric, corresponding to particles with half-integer spin, and cannot occupy the same state (the Pauli exclusion principle).

It is the case that the average occupation of a state with energy $E_i$ is given by equation (1), with the sign of the denominator $+$ for Fermi-Dirac statistics and $-$ for
Figure 2.1: Pictorial representation of the energy states in a harmonic oscillator. On the left is shown the behaviour for Bose-Einstein statistics (many particles can occupy the same state) for zero temperature (bottom image) and greater than zero temperature (top image) with similar on the right for Fermi-Dirac statistics (only one particle may occupy the same state). The blue and red represent two different components, which are not identical and can interact.
Bose-Einstein. Here $k_B$ is Boltzmann’s constant, $T$ is the temperature as before and $\mu$ the chemical potential.

$$\langle n(E_i) \rangle = \frac{1}{e^{\frac{E_i - \mu}{k_B T}} \pm 1}$$

For the total number of particles we simply sum over each state:

$$N = \sum_{i=1}^{\infty} \langle n(E_i) \rangle$$

We now define the Fermi energy $E_F$ as the energy of the highest occupied state at $T = 0$, as the state occupation at zero temperature is full up to this point. We also define the chemical potential $\mu$ as the energy required to add another particle to the system - at $T = 0, \mu = E_F$. This can be calculated if we know the total number of atoms. Physically we have a statistical distribution of energy across finite energy states. We can generalize, bringing in a geometric description of the density of states $g(\epsilon)$ which depends on the confining potential, as well as taking the states to a continuous limit:

$$N = \int g(\epsilon)\langle n(\epsilon_i) \rangle d\epsilon$$

where we have restated the Fermi-Dirac distribution in terms of a phase space taking into account the harmonic oscillator Hamiltonian for the system, with potential:

$$V(r) = \frac{1}{2} m \sum_{i=1}^{D} \omega_i^2 x_i^2$$

we now work in the Thomas-Fermi approximation, valid when $k_B T$ is much greater than the energy level spacing[47]. The semi-classical state occupation function can be written as a function of the D-dimensional position and momentum $\{r, p\}$ and the confining potential stated above $V(r)$:

$$f(r, p) = \frac{1}{e^{(\frac{p^2}{2m} + V(r) - \mu)/k_B T} + 1}$$

The density of states $g(\epsilon)$ is set by the energy dependence of the allowed quantum states in the potential $V(r)$, as well as the dimension $D$, and can be found by counting
<table>
<thead>
<tr>
<th>Potential</th>
<th>Quantity</th>
<th>3D</th>
<th>2D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Homogeneous</strong></td>
<td>$g(\epsilon)$</td>
<td>$\sqrt{\frac{m^2 \epsilon}{2 \pi^2}} \frac{V}{\hbar^2}$</td>
<td>$\frac{m \cdot A}{2 \pi \hbar^2}$</td>
</tr>
<tr>
<td></td>
<td>$E_F$</td>
<td>$\frac{\hbar^2}{2m} (n3\pi^2)^{\frac{2}{3}}$</td>
<td>$\frac{n2\pi \hbar^2}{m}$</td>
</tr>
<tr>
<td></td>
<td>$k_F$</td>
<td>$\left(\frac{m^2 \epsilon}{2 \pi^2}\right)^{\frac{1}{3}}$</td>
<td>$\frac{\sqrt{2 \pi \hbar^2}}{\sqrt{2N \hbar \omega_r}}$</td>
</tr>
<tr>
<td><strong>Harmonically Trapped</strong></td>
<td>$g(\epsilon)$</td>
<td>$\frac{\epsilon^2}{2\hbar^2 \pi^2}$</td>
<td>$\frac{\epsilon}{\sqrt{2R \hbar \omega_r}}$</td>
</tr>
<tr>
<td></td>
<td>$E_F$</td>
<td>$\hbar \omega_r (6N)^{\frac{1}{3}}$</td>
<td>$\sqrt{2N \hbar \omega_r}$</td>
</tr>
<tr>
<td></td>
<td>$k_F$</td>
<td>$\sqrt{\frac{2m \pi}{\hbar} (6N)^{\frac{1}{3}}}$</td>
<td>$\sqrt{\frac{2m \hbar \omega_r}{\hbar} (2N)^{\frac{1}{3}}}$</td>
</tr>
</tbody>
</table>

Figure 2.2: Results for the thermodynamic variables for 2D and 3D, where $\omega = \Pi D_i \omega_i$.

the states from position $\epsilon$ to $\epsilon + \delta \epsilon$. For 3D, in a homogeneous system, this results in counting states over a sphere of uniform density in phase space. As you can see from table (2.2), in 3D the density of states is dependent on the energy $\epsilon$. In 2D, again for a homogeneous system, counting over a disc results in a density of states not dependent on $\epsilon$. To calculate the density of states for a harmonically trapped system in D-dimensions, we can solve the following equation over the phase space of the harmonic system[48]:

$$g(\epsilon) = \int \frac{d^D r d^D p}{(2\pi \hbar)^D} \delta(\epsilon - \epsilon(r, p)) \quad (6)$$

Where the $\epsilon(r, p) = \frac{p^2}{2m} + V(r)$. Once we have the density of states we can solve the following equation at zero temperature to find the Fermi energy:

$$N = \int_0^{E_F} g(\epsilon) d\epsilon \quad (7)$$

$$N = \frac{1}{\hbar^D} \int d^D r d^D p f(r, p) \quad (8)$$

The density distribution in the Thomas-Fermi approximation can be calculated from the following integral over momentum space in $D$ dimensions:

$$n(r) = \int \frac{d^D p}{(2\pi \hbar)^D} f(r, p) \quad (9)$$

Using equation 5, this gives us the following analytical results for the density profiles
in 2D and 3D:

\[
2D : n(x, y) = -\frac{1}{\lambda_{dB}^2} \text{Li}_1 \left( -e^{(\mu - \frac{1}{2}m\omega_y^2x^2 - \frac{1}{2} m\omega_x^2 y^2)/(k_B T)} \right) \quad (10)
\]

\[
3D : n(x, y, z) = -\frac{1}{\lambda_{dB}^3} \text{Li}_\frac{3}{2} \left( -e^{(\mu - \frac{1}{2}m\omega_z^2z^2 - \frac{1}{2} m\omega_y^2 y^2 - \frac{1}{2} m\omega_x^2 x^2)/(k_B T)} \right) \quad (11)
\]

These results contain functions known as ‘polylogarithms’ (the Li\(_n\) functions). To integrate over these functions - a process which is useful to know as it comes up later when computational analysis is applied to the images of the Fermi gas - requires knowledge of the properties of these polylogs. One identity of theirs is that they have a series representation:

\[
\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (12)
\]

and also an integration identity (which can be derived from the preceding series decomposition):

\[
\int_{-\infty}^{\infty} \text{Li}_n \left( ze^{-\alpha x^2} \right) \, dx = \sqrt{\frac{\pi}{\alpha}} \text{Li}_{n+\frac{1}{2}}(z) \quad (13)
\]

Utilizing these identities we can integrate over the dimensions of the trapped density profiles. As we deal with images and line profiles of our 2D Fermi gas, having these equations is useful for modelling and comparison to the experimental data. First is the single integral over one dimension, here dimension \( y \):

\[
\int_{-\infty}^{\infty} n(x, y) \, dy = -\frac{1}{\lambda_{dB}^2} \sqrt{\frac{\pi k_B T}{\frac{1}{2} m\omega_y^2}} \text{Li}_{\frac{3}{2}} \left( -e^{(\mu - \frac{1}{2}m\omega_x^2 x^2)/(k_B T)} \right) \quad (14)
\]

Following we have the double integral over both dimensions, giving the analytical result:

\[
N = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n(x, y) \, dx \, dy = -\frac{1}{\lambda_{dB}^2} \sqrt{\frac{\pi k_B T}{\frac{1}{2} m\omega_y^2}} \sqrt{\frac{\pi k_B T}{\frac{1}{2} m\omega_x^2}} \text{Li}_2 \left( -e^{\mu/k_B T} \right) \quad (15)
\]

where to compute the double integral we have made use of another identity, the integral
of the square of the density profile:

\[
\iint_{-\infty}^{\infty} n(x, y)^2 \, dx \, dy = -\frac{1}{\lambda_B^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Li}_2^2\left( -e^{(\mu - \frac{1}{2}m\omega_x^2x^2 - \frac{1}{2}m\omega_y^2y^2)/(k_B T)} \right) \, dx \, dy \tag{16}
\]

These equations were very useful to me in my analysis, for the simplest way to fit to a density to find the temperature, for example. A more in-depth discussion of this is found in the chapter on thermometry.

This concludes the analysis of the thermodynamics of the Fermi gas in 2D and 3D, trapped and homogeneous environments.
Figure 2.3: Density profile of a 2D cloud for an ideal gas at zero temperature, in a harmonic trap. Generated with $N = 1 \times 10^4$ atoms, the vertical axis is density $n$ in atoms/m$^2$. The horizontal axis are pixels, simulated to match the parameters of our experiment, where 1 pixel = 2.84$\mu$m.
2.2 Scattering Physics

A feature of our experiments is that we probe two hyperfine states of lithium-6. These states are able to interact via scattering, where this is suppressed for only a single component gas due to Pauli exclusion. As we will see, the scattering process in our quantum system can be quantified largely by the scattering length, and its relationship to the other lengths scales inherent; such as the interparticle spacing, the effective range of the potentials, the de Broglie wavelength of the colliding atoms and the properties of the trap.

From elementary physics, we have two main types of scattering: elastic and inelastic. Both occur in our system to differing result. Elastic scattering, where the internal energy of the particles is unchanged, is desirable as it allows for a rethermalization of the gas when we cool using the evaporative cooling technique. Inelastic scattering usually includes three particles, for example in three-body recombination - where three particles collide - with two of those particles forming a low-lying bound state and giving up that binding energy to the third, which (in the case it does not immediately escape the trap) interacts thermally with other particles in the trap. This results in a net heating effect. Since it is our desire to stay as cold as possible, this is a hindrance. Understanding how scattering works for our system allows us to better control for inelastic scattering and allow elastic scattering when required. It will also allow us to manipulate the scattering length using a Fano-Feshbach resonance, giving us control of interparticle interactions. The formalism of scattering theory proceeds below, where we follow various treatments[49][50][51].

2.2.1 Scattering in 3D, 2D and quasi-2D

For two particles colliding, we can select a stationary centre-of-mass frame around which to construct the problem. At the centre of our coordinate system we then have a description of the potential which will be impacted by a single particle of reduced mass \( m_r = \frac{m_1 m_2}{m_1 + m_2} \). A description of the system with relative coordinate \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \) denoting the distance from the scattering potential and \( \mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2 \) the initial relative wave-vector, we can construct the Hamiltonian with scattering states \( \Psi_k \) as follows:

\[
-\frac{\hbar^2}{2m_r} \nabla^2 \Psi_k(\mathbf{r}) + V(\mathbf{r})\Psi_k(\mathbf{r}) = E\Psi_k(\mathbf{r})
\]  

(17)
Solutions to this can be formed by a superposition of an incoming plane wave with an outgoing wave, in the region far from the potential $r \gg R_0$ where $R_0$ is the characteristic length scale of the potential, which in ultracold atom experiments is typically also much smaller than the de Broglie wavelength $\lambda_{dB}$. The form of the scattered, outgoing wave depends on the dimensionality. For 3D we have:

$$\Psi_{3D,k}(r) \propto e^{ikr} + f_{3D}(k,k') \frac{e^{ikr}}{r}$$  \hspace{1cm} (18)

Where the amplitude of the scattered wave is given by the function $f_{3D}(k,k')$ where $k$ is the wave vector of the incoming wave and $k' = k r / r$ is the wave vector of the scattered wave. Given energy conservation we have $k' = k$. The solution in 2D is similar, but the form of the outgoing wave is cylindrical, so we have:

$$\Psi_{2D,k}(r) \propto e^{ikr} - f_{2D}(k,\phi) \sqrt{\frac{i}{8\pi kr}} e^{ikr}$$  \hspace{1cm} (19)

Where the scattering amplitude depends now on the wave vector and the angle between incoming and scattered wave, $f_{2D}(k,\phi)$. Following an expansion via partial waves (after expressing the Hamiltonian in a one-dimensional radial coordinates form with spherical/cylindrical potential) we have solutions in the asymptotic limit ($r \to \infty$) for the scattering amplitudes of:

$$f_{3D}(k,\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1)(e^{2i\delta_l(k)} - 1) P_l(\cos\theta)$$  \hspace{1cm} (20)

$$f_{2D}(k,\theta) = \frac{2}{i} \sum_{l=0}^{\infty} \epsilon_l(1 - e^{2i\delta_l}) \cos(l\theta)$$  \hspace{1cm} (21)

Here the $\theta$ is the one-dimensional angle between incoming and scattered waves, $P_l(\cos\theta)$ are the Legendre polynomials for angular momenta $l$. Because of the centrifugal barrier, any higher angular momentum scattering (such as $p$-wave or $d$-wave scattering) is usually energetically suppressed for ultracold atoms, so we set $l = 0$ in the following, corresponding to s-wave scattering only. Note that $\delta_l$ is the phase shift between the
incoming and scattered waves. The s-wave scattering amplitudes are then:

\[ f_{3D}(k) = \frac{1}{k \cot \delta_0(k) - ik} \]  

\[ f_{2D}(k) = \frac{4}{i - \cot \delta_0(k)} \]  

It is useful to also express the scattering in terms of the total scattering cross-section \((\sigma_{tot})\) and differential cross section \((\frac{d\sigma}{d\Omega})\), where the total scattering cross section expressed in this way is known as the optical theorem.

\[ \sigma_{tot}^{3D} = \frac{4\pi}{k} \text{Im}[f(k)], \quad \frac{d\sigma_{3D}}{d\Omega} = |f_{3D}(k', k)|^2 \]  

\[ \sigma_{tot}^{2D} = -\frac{1}{k} \text{Im}[f(k)], \quad \frac{d\sigma_{2D}}{d\phi} = \left| f_{2D}(k, \phi) \right|^2 \]  

At this point we note that the phase shift \(\delta_0\) depends on the relative momentum \(k\).

Let us focus for a moment on the 3D case. In 3D we can introduce the scattering length \(a\) by [47] noting the \(k \cot \delta_0(k)\) is an even function of \(k\), due to time-reversal symmetry of the scattering process. We can thus expand it to second order \((k^2)\), valid for low momenta \(k \ll \frac{1}{\delta_0}\):

\[ k \cot \delta_0(k) \approx -\frac{1}{a_{3D}} + \text{Re} \frac{k^2}{2} \]  

With \(\text{Re} \) the effective range of the scattering potential. Thus we have the scattering potential in 3D:

\[ a_{3D} = -\frac{\tan \delta_0}{k} \]  

and now the 3D scattering amplitude:

\[ f_{3D}(k) = \frac{1}{-\frac{1}{a_{3D}} + \text{Re} \frac{k^2}{2} - ik} \]  

This indicates that the scattering behaviour is different depending on the relative values of the momentum, effective potential range and scattering length. We encapsulate
them, for $R_{\text{eff}} \ll k$:

$$f_{3D}(k) = \begin{cases} -a_{3D} & \text{where } k|a_{3D}| \ll 1 \text{ (weakly interacting regime)} \\ \frac{i}{k} & \text{where } k|a_{3D}| \gg 1 \text{ (unitary regime)} \end{cases}$$

Here the weakly interacting regime is where the amplitude of the scattered wave becomes independent of the momentum, and is equal to just the negative scattering length. The unitary regime is where the scattering length diverges, and the amplitude of the scattering is proportional to the collision wavevector and the cross section for collisions is $\sigma = \frac{4\pi}{k^2}$. This unitary regime may come about when the scattering phase shift is equal to $\frac{\pi}{2}$ and usually corresponds to a new bound state being supported by the potential. Such a bound state is accessible using to a Feshbach resonance, as discussed below.

Let us return to the 2D case, where the scattering amplitude was $f_{2D}(k) = \frac{4}{i - \cot\delta_0(k)}$. We gain a similar expansion at low energies:

$$\cot\delta_0(k) = -\frac{2}{\pi} \ln\left(\frac{1}{ka_{2D}}\right)$$

where we have introduced the 2D scattering length $a_{2D}$. Thus the scattering amplitude in 2D:

$$f_{2D}(k) = -\frac{4}{i + \frac{2}{\pi} \ln\left(\frac{1}{ka_{2D}}\right)}$$

Since the amplitude is dependent on a logarithm, it goes to zero as the collision energy goes to zero, in the weakly interacting regime. We can see that this is different to the 3D case, where the scattering amplitude is equal to the negative 3D scattering length.

The quasi-2D scattering case has a different formulation to both 2D and 3D[52]. In quasi-2D, the density of states and the scattering amplitude is the same as in 2D. Along with this, the effective range of the interaction potential $R_{\text{eff}}$ is much smaller than that of the confining (harmonic) potential: $R_{\text{eff}} \ll l_z$ (here $l_z = \sqrt{\hbar/m\omega_z}$ is the characteristic length of the harmonic potential). However effects on the scattering due to the third dimension can not be neglected. A relation between the two scattering
lengths is given by \([49]\):

\[
a_{2D} = l_z \sqrt{\frac{\pi}{B}} \exp \left(-\sqrt{\frac{\pi}{2a_{3D}}}l_z\right)
\]

(31)

with \(B \approx 0.905\). This result is from a scattering calculation using a T-matrix approach in \([49]\) and is valid as long as the scattering energy is small - compared with the tight confining potential.

This form is fine for the case where the momentum of the scattered particles \(k_0\) and the harmonic oscillator length \(l_z\) satisfy \(k_0 l_z \ll 1\), however where this is not the case (where \(k_0 l_z\) is not small - tight confinement and/or small scattering momenta) we must make a correction to the effective 2D scattering length here corresponding to (see supplement of \([53]\)):

\[
a_{2D} = l_z \sqrt{\frac{\pi}{B}} \exp \left(-\sqrt{\frac{\pi}{2a_{3D}}}l_z\right)e^{-\frac{1}{2} \Delta w(k_0^2l_z^2/2)}
\]

(32)

where \(\Delta w(\xi) = w(\xi) - w_{\text{lim}}(\xi) \geq 0\) for the function

\[
w(\xi) = \lim_{J \to \infty} \left[\sqrt{\frac{4J}{\pi}} \ln \left(\frac{J}{e^2}\right) - \sum_{j=0}^{J} \frac{(2j - 1)!}{(2j)!} \ln(j - \xi - i0)\right]
\]

(33)

In all dimensional cases, where the de Broglie wavelength and the interparticle spacing are large compared to the interatomic potential, this potential can be reasonably approximated by a more convenient mathematical form, known as the contact or delta function-potential \(\delta(r)\). This is the case in our experimental setup due to the very low temperatures of the particles compared to the dominant van der Waals interaction potential.

2.2.2 Fano-Feshbach Resonances

In our experiments we utilize a Fano-Feshbach resonance between particles in differing spins states to vary the scattering length \(a\). This scattering resonance was discovered theoretically in nuclear scattering processes by Feshbach \([54]\) in 1958 and by Fano for
atomic physics [55] in 1961, building on work by Hans Beutler and Fano in the thirties. As in the majority of the literature I will drop the Fano from here and just refer to it as the Feshbach resonance.

In our species $^6\text{Li}$ we cool usually two of the lower lying hyperfine states, for example the two lowest: $|1\rangle = |F = 1/2, m_F = +1/2\rangle$ and $|2\rangle = |F = 1/2, m_F = -1/2\rangle$. These hyperfine states can couple together to form a bound state. We also make use of a third hyperfine state, $|3\rangle = |F = -1/2, m_F = -1\rangle$, and utilize a $|1\rangle - |3\rangle$ coupling for some experiments (it will be made explicit when we use this mixture). These bound state closed channels are singlet channels and therefore have a different magnetic moment than the open, triplet, channel scattering states - meaning that application of a magnetic field can tune the relative energies of these two states. When the energy of a bound state in the closed channel is equal to the threshold energy in the open channel, we have a resonance which is at threshold (and may also exist above threshold [56]), resulting in an enhancement of scattering and a divergence of the scattering length. This resonance is a Feshbach resonance. It allows tuning of the scattering length so that a bound state is available at the kinetic energy corresponding to the atom-atom collision. The creation of a bound molecular state by sweeping of the magnetic field is reversible and does not produce any heat.

The resonance occurs at a certain magnetic field $B_0$. For $^6\text{Li}$ as used in our experiment is at 832.2G for the $|1\rangle - |2\rangle$ scattering, and 690G for $|1\rangle - |3\rangle$[54][57]. The equation governing the resonance in terms of scattering length vs magnetic field is, near to the resonance[47][58]:

$$a = a_{bg}(1 - \frac{\Delta B}{B-B_0})(1 - \alpha(B-B_0))$$

(34)

With $a_{bg}$ corresponding to background scattering length - the scattering length away from any scattering resonances. The $\alpha$ term is a leading-order correction parameter (see [58]). Here there is a characteristic width to the resonance, given by $\Delta B$, which has relation, where $\Delta\mu$ is the difference in magnetic moment between the scattering and bound states, and $E_0$ is a constant associated with the coupling constant, $g_0$[57]:

$$\Delta B = \sqrt{\frac{2\hbar^2 E_0}{m} \frac{1}{\Delta\mu a_{bg}}}, \quad E_0 = \left(\frac{g_0^2}{2\pi} \frac{m}{2\hbar^2}\right)^{3/2}$$

(35)
Figure 2.4: Feshbach resonance of the $|1\rangle - |2\rangle$ and $|1\rangle - |3\rangle$ hyperfine states showing magnetic field vs scattering length, where $|1\rangle = |F = 1/2, m_F = +1/2\rangle$, $|2\rangle = |F = 1/2, m_F = -1/2\rangle$ and $|3\rangle = |F = -1/2, m_F = -1\rangle$. 
2.2.3 Bound states and the BEC-BCS crossover

A natural consequence of the existence of a bound state with a lower energy than the colliding particles is the ability to form molecular dimers. This is one of two pairing mechanisms available to the system - the other being the formation of Cooper pairs[4]. In 3D the smooth transition between pairing due to the bound state - forming molecular dimers - and pairing due to the availability of scattering states on the surface of a Fermi sea (Cooper pairing) - is known as the BEC-BCS crossover. The BEC is for Bose-Einstein condensation, where the superfluid is made up from a condensate of these dimers, which are bosonic in nature (due to the addition of fractional fermion spins). This is contrasted by the Bardeen-Cooper-Schrieffer (BCS) side of the resonance, where Cooper pairs are formed. This crossover is smoothly connected by varying the scattering length, going through the unitarity regime, where the scattering length diverges at the resonance. In 2D, the situation is different, as there always exists a dimer bound state across the relevant parameter of ln(k_Fa_{2D}). Due to thermal fluctuations destroying long range order in 2D (that is, thermal fluctuations prevent the formation of phase coherence over large distances in the fluid), superfluidity relies on a different mechanism involving the pairing of vortices to establish quasi-long-range order, and is known as a BKT (Berezinskii-Kosterlitz-Thouless) superfluid.

In 3D, bound dimers only exist for a positive scattering length, a_{3D} > 0. They have the energy:

\[ \epsilon_{B,3D} = \frac{\hbar^2}{ma_{3D}^2} \]  

This is contrasted to the situation in 2D, where the energy of the bound state is similar, but this bound state exists for all values of the scattering length - as a_{2D} is always positive for all values of ln(k_Fa_{2D}).

\[ \epsilon_{B,2D} = \frac{\hbar^2}{2ma_{2D}^2} \]  

In quasi-2D the situation is different again, as the length scale associated with the confining potential in the tight direction, l_z = \sqrt{\hbar/m\omega_z}, can be much greater than the scattering length of the particles (a_{3D} \ll l_z), so the third dimension can play a role in the microscopic physics of the gas. This then is influenced by the confinement - and
Figure 2.5: In 3D, varying pairing mechanism exist as the scattering length is altered. For positive scattering lengths the existence of the bound state gives rise to tightly bound dimers. Where the scattering length diverges - at unitarity - we have a region of strong interactions where the physics is universal and the only relevant length scale is the mean spacing between particles. The BCS side of the resonance exists for negative scattering lengths, where pairs are formed by long-range Cooper pairing on the surface of the Fermi sea.

can be thought of as confinement induced. In the limit of small scattering lengths (for example when $a$ is much less than the harmonic oscillator length in the tightly confined direction, $a \ll l_z$, the size of the molecule is smaller than the transverse ground state) the bound state in quasi-2D behaves as in the 3D case, and in the opposite limit as the 2D bound state. The following equation describes the energy in quasi-2D for scattering length small and positive:

$$\epsilon_{B,\text{quasi-2D}} \approx \frac{\omega_z}{\pi} B \exp\left(\sqrt{2\pi} \frac{l_z}{a_{3D}}\right), \quad \frac{l_z}{a_{3D}} \ll -1 \quad (38)$$

With $B \approx 0.905$ a constant.
Figure 2.6: Difference in scattering to bound state energy plotted versus the harmonic oscillator length $l_z$ divided by the 3D scattering length $a_s$. There exists a bound state for all values of the scattering length in 2D and quasi-2D, where in 3D there only exists a (dimer) bound state for positive scattering lengths. The quasi-2D behaviour is shown to approach the limit of both 2D and 3D bound state energies in the relevant limits. This is because as the binding energy $E_b$ becomes larger relative to the energy of the transverse oscillator ground state ($\omega_z$), the system takes on a 3D character. Plot taken from discussion by Levinsen et al[49]
2.3 Collective Dynamics

Collective dynamics of degenerate gases are a powerful lens with which to examine the physical properties of these gases. A focus on the breathing mode behaviour, as we looked at in our experiments, in quasi-2D and 2D regimes has been theoretically reviewed in a recent paper by Giulia De Rosi and Sandro Stringari\cite{37}. In this section I will look at the theory underpinning this treatment, as it provides analytic results and predictions for breathing mode frequencies in regimes where the equation of state can be expressed in a polytropic form. Explanations of this and of the collective dynamics follows.

2.3.1 Polytropic Equation of State

Collective excitations, and especially the breathing mode of a gas, are sensitive to the thermodynamic equation of state. Here we are interested in excitations that have a wavelength long compared to the transverse cloud size, on the scale of the radial cloud size. In this case we can integrate over the transverse direction even when the cloud is thermodynamically 3D. It is shown in\cite{37} that there are many applicable configurations of a gas corresponding to differing dimensionality, temperature and interaction parameters where the pressure may be more simply expressed as a function of the density to an exponent $P \propto n^q$, with $q$ the so-called ‘polytropic coefficient’. This is a simplified form of the equation of state. In general the equation of state is a function which connects the thermodynamic quantities of the gas together, the most common being the ideal gas law $PV = Nk_B T$. The general polytropic form follows\cite{37}:

$$P(n, s) = n^q p(s)$$ (39)

where $P(n, s)$ is the pressure of the gas in terms of density $n$ and entropy per particle $s$, with $q$ the polytropic coefficient. Here $p(s)$ is a function fixed by the thermodynamic behaviour at zero and finite temperature. This equation (39) can also be written (at $T = 0$) as

$$\mu(n, T = 0) = \frac{q}{q - 1} n^{q-1} p(s = 0)$$ (40)

where $\mu$ the chemical potential has been related to the pressure through the Gibbs-Duhem thermodynamic relation $(dP = nd\mu + sdT)$. We can then gain the sound velocity at zero temperature, with $c$ the velocity of sound in the medium (in the
hydrodynamic regime), as

$$\left( \frac{dP}{dn} \right)_s = mc^2 = n(\frac{\partial \mu}{\partial n})_s = (q - 1)\mu$$

(41)

Determination of the polytropic index requires formulation of the equation of state in a form that satisfies (39) above. As mentioned there are a few situations where this does arise. In the case of a unitary Fermi gas in 3D the chemical potential at $T = 0$ is[48]

$$\mu = \xi_B E_F$$

(42)

where $E_F \propto n^{2/3}$ and $\xi_B$ is the Bertsch parameter, a dimensionless parameter which quantifies the energy shift due to the interparticle interactions at unitarity. We can see from (40) a polytropic coefficient of 5/3.

Further regimes, still at zero temperature and in an ideal gas - but now with trapping potential - make use of the local density approximation (LDA). The local density approximation can be employed where the gas behaves locally as a uniform gas, even though it is under a trapping potential that generally is not uniform. For a 2D environment this requires that the chemical potential be much larger than the potential along the axis which the approximation is to be made, $\mu \gg \hbar \omega_z$ where $z$ is the ‘tight’ axis of a pancake trap. Where this LDA applies, the chemical potential ‘follows’ the shape of the potential according to:

$$\mu_0 = \mu[n(r)] + V_{ho}(r)$$

(43)

where $\mu_0$ is the chemical potential calculated at $r = 0$ and tied to the total atom number by $N = \int n(r)\,dr$. Returning to the pancake trap, with tight trapping axis $z$ in a harmonic oscillator potential we have:

$$\mu_0 = \mu(n) + \frac{m}{2} \omega_z^2 z^2$$

(44)

This is the case for a ‘pancake’ 2D trap - confined in the radial $(x, y)$ directions with trapping frequency less than the axial, $z$ direction - but not so tightly that the LDA can still be used in all 3 dimensions. For this pancake regime, with the Fermi gas at unitarity, the 3D equation of state is given by[59] $\mu(n) = \xi_B (\hbar^2/2m)(3\pi^2 n)^{2/3}$, where
\( \xi_B \) is the Bertsch parameter. With the density distribution:
\[
n(z) = n(z = 0)(1 - \frac{z^2}{R^2})^{3/2}
\]  
(45) 

integrating along \( z \), and substituting gives the following for the 2D equation of state:
\[
\mu = \xi_B \frac{\hbar^2}{2m} (3\pi^2 n(0))^{2/3} = (2\pi \xi_B^{3/2} \frac{\hbar^3}{m} n_{2D})^{1/2}
\]  
(46) 

From this result and the form of (40) we find that the polytropic coefficient in this pancake shaped but thermodynamically 3D regime has the value \( q = 3/2 \).

There is another regime where a polytropic coefficient may be extracted for a strictly 2D gas, where the presence of ‘tight trapping’ exists. At zero temperature the condition for this regime is \( \mu \ll \hbar \omega_z \), and at temperature greater than zero we require \( k_B T \ll \hbar \omega_z \), where the first excited state of the harmonic oscillator potential is energetically inaccessible. We can no longer use the previous method of modification to the 3D equation of state via the local density approximation in this regime, as the LDA is no longer valid. Instead we identify for this tightly trapped region that the chemical potential varies directly with density \( \mu \propto n \), at least far from the resonance on both the BCS and BEC sides. This results in a polytropic coefficient of \( q = 2 \), which is true for zero and higher temperatures, and which scale invariance predicts should also be true for all interactions. Whether this coefficient holds for the resonance in 2D is not clear, and is the subject of the experiment performed in this thesis, covered by the discussion of the anomaly in the later section.

Putting these results together we get the following table outlining the polytropic coefficients in the various regimes[37]:

<table>
<thead>
<tr>
<th>Regime</th>
<th>Polytropic Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unitary 3D</td>
<td>5/3</td>
</tr>
<tr>
<td>Unitary Pancake (LDA)</td>
<td>3/2</td>
</tr>
<tr>
<td>2D Mean Field (T=0)</td>
<td>2</td>
</tr>
<tr>
<td>2D Mean Field (high T)</td>
<td>2</td>
</tr>
</tbody>
</table>
2.3.2 Hydrodynamics

To quantify the theory of collective oscillations we employ a hydrodynamic theory, again following the treatment by Stringari et al. [37],[59]. Our goal in this subsection is to express the hydrodynamic equations in such a way that we may make use of the previous sections’ results for the polytropic coefficients. The treatment to follow produces the equations of linearized hydrodynamics for a trapped gas. Collective oscillations may also be understood using kinetic theory, where the Boltzmann equation is solved under certain approximations. It is not clear that this is a more general approach, and in any case, the limiting regimes provided by the polytropic coefficients combined with the following hydrodynamic formalism provides us with testable predictions that are examined via experiment later in this thesis. The Boltzmann approach can be found in reference [60].

As mentioned in the introductory chapter, there are two relevant regions into which the study of the gas can be separated, due to different behaviour with respect to collective oscillations: the hydrodynamic and the collisionless regimes.

In the collisionless regime, within a trapped gas the collision time (the mean time between collisions) is large compared the period of the inverse trapping frequency, $\tau \gg \frac{1}{\omega_{ho}}$, (where $\tau$ is the inverse collision time and $\omega_{ho}$ the trapping frequency) indicating that the dynamics are dominated by the trapping potential rather than particle collisions. Factors that can influence the collision time in a gas include the scattering length, as well as Pauli blocking - where the gas is cold enough that the Fermi sea is filled such that the final state of a scattering event is already occupied - suppressing collisions. Another factor is the density - the probability of scattering events goes down as the density decreases. The excitations in such a collisionless regime are thus decoupled from the environment in that the excitation wavelength is typically small and the particles undergo independent oscillations rather than collective ones.

Conversely, the hydrodynamic regime is characterized by the wavelength of excitations being much greater than the mean-free path for collisions between particles. In a trapped gas this corresponds to the inverse collision time being now much less than the inverse trapping frequency: $\tau \ll \frac{1}{\omega_{ho}}$.

The starting point for the hydrodynamic treatment are the hydrodynamic equa-
tions, where we have assumed a gas with equal density of spin states \( n_\uparrow(\mathbf{r}, t) = n_\downarrow(\mathbf{r}, t) \), and we work at the limit of small velocities and amplitude oscillations. First is the continuity equation:

\[
\frac{\partial n(\mathbf{r}, t)}{\partial t} + \nabla \cdot [n_0(\mathbf{r}) \mathbf{v}(\mathbf{r}, t)] = 0 \tag{47}
\]

where we have the time derivative of the density conserved with the spatial derivative of the massless current (the current being \( \mathbf{j}(\mathbf{r}, t) = mn_0(\mathbf{r}) \mathbf{v}(\mathbf{r}, t) \)). Where \( \mathbf{v}(\mathbf{r}, t) \) is the velocity field, and \( n_0 \) refers to the equilibrium density and entropy respectively. We also have for the entropy density, \( s \):

\[
\frac{\partial s(\mathbf{r}, t)}{\partial t} = -\nabla \cdot [s_0(\mathbf{r}) \mathbf{v}(\mathbf{r}, t)] \tag{48}
\]

This entropy density equation vanishes at \( T = 0 \), but as we will see there are some results at higher temperatures where this equation becomes important. Finally the Euler equation for the current density:

\[
mn_0(\mathbf{r}) \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} = -[\nabla P(\mathbf{r}, t) + n(\mathbf{r}, t) \nabla V_{\text{ext}}(\mathbf{r})] \tag{49}
\]

which is an expression of how the current varies over time with respect to the spatial derivative of the pressure and external potential \( V_{\text{ext}}(\mathbf{r}) \). To advance from here to a general formula which expresses the hydrodynamics in terms of the equation of state we do as follows. Taking the time derivative of (49):

\[
mn_0 \frac{\partial^2 \mathbf{v}(\mathbf{r}, t)}{\partial t^2} = -\nabla \frac{\partial}{\partial t} P - \nabla V_{\text{ext}}(\frac{\partial}{\partial t} n) \tag{50}
\]

where the time derivative of pressure is expanded in the chain rule:

\[
\frac{\partial P}{\partial t} = \left( \frac{\partial P}{\partial n} \right)_T \frac{\partial n}{\partial t} + \left( \frac{\partial P}{\partial T} \right)_n \frac{\partial T}{\partial t} \tag{51}
\]

which contains the time derivative of the temperature \( \frac{\partial T}{\partial t} \). Using a thermodynamic relation (see Appendix B of [37]) the time derivative of temperature can be written in the form:

\[
\frac{\partial T}{\partial t} = -\frac{T}{c_v} \left( \frac{\partial P}{\partial T} \right)_n \frac{\nabla \cdot \mathbf{v}}{n_0} \tag{52}
\]

27
where $c_v$, the specific heat at constant volume, can be expressed as:

$$c_v = \frac{T}{n_0} \left( \frac{\partial P}{\partial T} \right)_n \left( \frac{\partial n}{\partial T} \right)_s$$  \hspace{1cm} (53)$$

where in these equations the $\bar{s}$ refers to the entropy per particle $\bar{s} = \frac{s}{n}$. This then leads to a re-expression of the time derivative of the Euler equation, so a re-expression of (50), where we are working at equilibrium and with velocity fields with solutions of the form $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r})e^{-i\omega t}$:

$$m\omega^2 \mathbf{v} = -\nabla \left[ \left( \frac{\partial P}{\partial n} \right)_{\bar{s}} \left( \mathbf{\hat{n}} \cdot \mathbf{v} \right) \right] + (\gamma - 1)(\nabla \mathbf{V}_{ext}) \left( \mathbf{\hat{n}} \cdot \mathbf{v} \right) + \nabla \left( \mathbf{v} \cdot \nabla \mathbf{V}_{ext} \right)$$  \hspace{1cm} (54)$$

where $\gamma = \left( \frac{\partial P}{\partial n} \right)_{\bar{s}} / \left( \frac{\partial P}{\partial n} \right)_T$ is the adiabatic coefficient. What we have now, is an expression which we can proceed to formulate in terms of the polytropic coefficients. We do this first by looking back at the expression for the time derivative of temperature (52). This equation can be reformulated using the thermodynamic relation $c_v = \left( \frac{\partial U}{\partial T} \right)_n$ and the following expression for energy per particle in terms of pressure:

$$U = \frac{1}{q - 1} \frac{P}{n}$$  \hspace{1cm} (55)$$

where we are again using $q$ to represent the polytropic coefficient. The temperature time derivative becomes:

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} = -(q - 1) T \nabla \cdot \mathbf{v}(\mathbf{r}, t)$$  \hspace{1cm} (56)$$

We can now get equation (54) in terms where the equation of state in polytropic form is valid. This form is:

$$m\omega^2 \mathbf{v} = -\left( \frac{\partial P}{\partial n} \right)_{\bar{s}} \nabla (\mathbf{\hat{n}} \cdot \mathbf{v}) + (q - 1)(\nabla \mathbf{V}_{ext}) \nabla \cdot \mathbf{v} + \nabla (\mathbf{v} \cdot \nabla \mathbf{V}_{ext})$$  \hspace{1cm} (57)$$

which reduces to the following at $T = 0$ and $\gamma = 1$:

$$m\omega^2 \mathbf{v} = -(q - 1) \nabla [\mu(n) \nabla \cdot \mathbf{v}] + \nabla [\mathbf{v} \cdot \nabla \mathbf{V}_{ext}]$$  \hspace{1cm} (58)$$

We have now - with these two equations - forms of the hydrodynamic equations governing the behaviour of the velocity fields which we can utilize where the polytropic
equation is valid and the coefficients given in the last subsection apply.

For 2D traps the external potential for the harmonic oscillator is \( V_{\text{ext}} = \frac{1}{2} m (\omega^2_x x^2 + \omega^2_y y^2) \). Substituting this and using the irrotational ansatz to the velocity field:

\[
\mathbf{v}(r, \psi) \propto \nabla [(r^{2n} + \ldots) r^{m} e^{\pm im\psi}]
\]  

(59)

where \( n \) counts the radial nodes in the density shifts and \( m \) is the axial or \( z \) component of the angular momentum. Applying this ansatz to (58) above we find a solution for the eigenfrequencies:

\[
\omega^2(n, m, q) = \omega^2_\perp (2n + |m| + 2n(q - 1)(n + |m|))
\]  

(60)

Before we get to breathing modes, we can study more elementary excitations known as surface modes - modes where the divergence of the velocity field is zero \( \nabla \cdot \mathbf{v} = 0 \). In this case the above equation has \( n = 0 \) and reduces to:

\[
\omega^2 = \omega^2_\perp [m]
\]  

(61)

where this accounts for notable surface modes such as the dipole mode \(|m| = 1\) and the quadrupole mode \(|m| = 2\). It is clear here that these do not depend on the equation of state of the system, as there is no dependence on the polytropic coefficient. Applying the same substitutions for the breathing mode where we have \( n = 1, m = 0 \) and the general eigenfrequency equation above reduces to:

\[
\omega^2_B = \omega^2_\perp 2q
\]  

(62)

which does depend on the polytropic coefficient. Utilizing the values for the polytropic coefficients found in the last subsection, we get for ‘pancake’ trapping that the breathing mode is expected to be \( \omega_B = \sqrt{3} \omega_\perp \), where of course the \( \omega_\perp \) is the trapping potential in the radial direction. For tighter trapping regimes where the thermodynamic behaviours cannot be derived from the 3D EOS the expectation of this theory is that the breathing mode \( \omega_B = 2 \omega_\perp \). A table including the coefficients is below, where the 3D isotropic behaviour has also been included. The last three listed are the most relevant for the experiments detailed in this thesis.
<table>
<thead>
<tr>
<th>Regime</th>
<th>Polytropic Coefficient</th>
<th>Breathing Mode Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unitary 3D</td>
<td>5/3</td>
<td>$2\omega_{ho}$</td>
</tr>
<tr>
<td>Unitary Pancake (LDA)</td>
<td>3/2</td>
<td>$\sqrt{3}\omega_{\perp}$</td>
</tr>
<tr>
<td>2D Mean Field (T=0)</td>
<td>2</td>
<td>$2\omega_{\perp}$</td>
</tr>
<tr>
<td>2D Mean Field (high T)</td>
<td>2</td>
<td>$2\omega_{\perp}$</td>
</tr>
</tbody>
</table>
2.4 Scale Invariance and the Anomaly

Symmetry is an increasingly portentous topic in physics. In many ways this shouldn’t be surprising, as symmetry is a mathematical description of how things are similar, and under what conditions they differ. Given a major goal of physics is to describe how natural quantities change, in time and space, and to delineate the regimes under which similarities exist, studying symmetry seems an obvious way to gain insight into the physical laws. A quintessential example of this connection is that of Noether’s theorem[61], which relates how the laws of conservation can be tied to the symmetries of the Lagrangian governing the system.

Just as important as where symmetries exist, is when and how they break. In the case where the symmetry breaking is due to a quantization, this is known as an anomalous symmetry breaking. We shall see that for an attractive delta function potential in 2D, the classical Hamiltonian displays scale invariance which is only broken upon quantization - a needed quantization as the scale invariance itself prohibits the evaluation of the bound state energy: the energy scale is dimensionless, necessitating the introduction of a new length scale. In the scale invariant case, the breathing mode collective excitation frequency of a 2D gas is simply twice the dipole mode(2\omega_0), however, due to the existence of a finite energy bound state - due to the quantum anomaly - the breathing mode is theoretically predicted to vary from this value, by up to around 10% for the strongest interaction region. [62, 63, 43, 45, 44, 41].

We begin with a discussion of ‘scale invariance’. Scale invariance is an example of a symmetry which exists if the Hamiltonian of the system is invariant under the scale transformation \( r \rightarrow \lambda r \). This is a translational symmetry. The mathematical definition of a function which is invariant is that:

\[
  f(x) \rightarrow f(\lambda x) = \lambda^{\Delta} f(x)
\]

where the \( \Delta \) here is the scaling dimension, an exponent of the dilation \( \lambda \). Consider the 2D Hamiltonian:

\[
  H = \left( -\frac{\hbar^2}{2m}\nabla^2 - g\delta^2(r) \right)
\]

where \( g \) is the coupling constant and the delta function potential is attractive. Incorporating the following property of the delta function of dimension \( D \) under dilation \( \lambda \):
\[ \delta^D(\lambda \mathbf{r}) = |\lambda|^{-D} \delta^D(\mathbf{r}) \] (65)
gives
\[ H_r \rightarrow H_{\lambda r} = \lambda^{-2} H_r \] (66)
So the Hamiltonian displays the required scale invariance (note: we will come to what happens when harmonic trapping potential is added later). Unexpected things start to happen when we begin to solve for bound states, and indeed scattering solutions, with this 2D Hamiltonian. To begin we will obtain the scattering amplitudes using a method found in[64] using Green’s functions. For positive energies \( E = k^2 \) the Schrödinger equation can be formed as the Lippmann-Schwinger equation:
\[
\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int d^D r' G^R_0(\mathbf{r}, \mathbf{r}') \lambda \delta(\mathbf{r}') \psi(\mathbf{r}')
\] (67)
where here \( \psi_0(\mathbf{r}) \) is the solution to the free particle Schrödinger equation, and \( G^R_0(\mathbf{r}, \mathbf{r}') \) is the free particle retarded Green’s function. This equation reduces to:
\[
\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + g G^R_0(\mathbf{r}, 0) \psi(0)
\] (68)
We can identify here that the free particle solution is an incoming plane wave of momentum \( k \):
\[
\psi_k(\mathbf{r}) = e^{ik \cdot \mathbf{r}} + g \psi_k(0) G^R_0(\mathbf{r}, 0)
\] (69)
Where the 2D Green’s function is[63]:
\[
G(\mathbf{r}) = \int \frac{d^2 p}{(2\pi)^2} \frac{e^{i p \cdot \mathbf{r}}}{p^2 - k^2 - i\delta}
\] (70)
is integrated over momentum space and can be re-expressed using zeroth order Hankel functions. The \( \delta \) in the denominator is used to give the integral a small imaginary part. We now look at the consistency condition, setting \( \mathbf{r} = 0 \) in (69) and solving for \( \psi_k(0) \):
\[
\psi_k(0) = \frac{1}{1 - g G(0)}
\] (71)
Where we see an issue arise, as \( G(0) \) is logarithmically divergent. To counter this, we apply techniques of renormalization, to bound the problem and stop the divergence.
occurring. This requires that we introduce a cutoff, at $|p| \leq \epsilon$, into the Green’s function integral (70) above. This hard momentum-space cutoff acts as a scale, and breaks the original scale invariance of the problem - we have traded the dimensionless quantity of the coupling constant $g$ for a quantity with dimension - the cutoff. This is the heart of the quantum anomaly, which is also known as dimensional transmutation when performed in field theory. After introducing this cutoff, we can solve the integral at the origin to get:

$$G(0) = \frac{1}{4\pi} \ln\left(\frac{\epsilon^2}{-k^2}\right)$$ (72)

Using the previous two results and substituting into the scattering wavefunction equation, we find the scattering amplitude equal to (again, see [63]):

$$f(\theta) = \sqrt{\frac{2}{\pi k}} \left[ \frac{1}{g} - \frac{1}{4\pi} \ln\left(\frac{\epsilon^2}{k^2}\right) - \frac{i}{4} \right]^{-1}$$ (73)

There is a general result in quantum mechanics that states the poles of the scattering amplitude at imaginary values of the momentum ($k$) occur at bound state energies. Replacing $k \rightarrow ik$ in the above we find a pole at (where we have implemented a change of units):

$$k^2 = \epsilon^2 \exp\left(-\frac{4\pi}{g}\right) = \frac{2m|E|}{\hbar^2}$$ (74)

With the right hand side being equal to the bound state energy.

We have now shown that the scale invariant $\delta$-function potential can be renormalized to give a finite solution for $E_b$ (the bound state energy) - but what of the real experimental situation with a two spin state system ($|\uparrow\rangle, |\downarrow\rangle$) of fermions in a harmonic trap? And how do we relate this to the breathing mode of the gas?

To proceed to answer these questions we look at the results of Hofmann[44]. The Hamiltonian for the experimental system may be written as:

$$H = \int d^2 r \left[ \psi_\uparrow \frac{\nabla^2}{2m} \psi_\downarrow(r) + \frac{g}{m} \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow(r) \right]$$ (75)

where we have creation and annihilation operators for the fermion fields $\psi_\uparrow^\dagger(r), \psi_\downarrow(r)$ for the two spin states $\sigma = \uparrow, \downarrow$. Here the coupling constant $g$ is dimensionless, and
as before a momentum space cutoff $\epsilon$ must be introduced, giving the binding energy:

$$E_B = -\frac{\epsilon^2}{m} \exp\left(-\frac{4\pi}{g}\right) = -\frac{4}{ma^2_{2D}e^{2\gamma_E}}$$  \hspace{1cm} (76)

with $\gamma_E$ being Euler’s constant and $\hbar$ having been set equal to 1. Experimentally our gas is trapped by a potential that tightly confines it in one direction $(z)$, relative to the other directions $(x, y)$. This can be included in the Hamiltonian:

$$H_{ho} = H + \int d^2 r \frac{m \omega_0^2 r^2}{2} \psi^\dagger \psi(r)$$ \hspace{1cm} (77)

with $\omega_0$ the trapping frequency in the azimuthal direction. This trapping potential itself breaks the scale invariance, however, it is shown in [40] that the gas trapped in a cylindrically symmetric harmonic oscillator potential exhibits a SO(2,1) Lorentz symmetry in the Hamiltonian. The algebra that exhibits this symmetry is constructed from the following functions:

$$D = \int d^2 r m j_i(r), \quad j_i(r) = -i(\psi^\dagger \partial_i \psi - \partial_i \psi^\dagger \psi)/2m$$ \hspace{1cm} (78)

$$C = \int d^2 r \frac{mr^2}{2} \psi^\dagger \psi_\sigma(r)$$ \hspace{1cm} (79)

where we can express the Hamiltonian including the harmonic trap as $H_{ho} = H + \omega_0^2 C$

Then, we have the commutation relations as:

$$[D, H] = 2iH, \quad [D, C] = -2iC \quad \text{and} \quad [H, C] = -iD$$ \hspace{1cm} (80)

Which form the SO(2,1) symmetry group algebra, which holds on the classical level, when combined in the following way:

$$L^\pm = (L_+ \pm L_-)/4, \quad L_\pm = (H - \omega_0^2 C \pm i\omega_0 D)/\omega_0, \quad L_3 = H_{ho}/2\omega_0$$ \hspace{1cm} (81)

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = -L_1, \quad [L_3, L_1] = L_2$$ \hspace{1cm} (82)

This algebra contains $L_\pm$, which can be shown to constitute ladder operators on the
Hamiltonian, generating an infinite series of eigenstates separated by energy spacing of $\pm 2\omega_0$. It is shown in [40] that these can then be associated with the breathing mode of a weakly interacting 2D gas, resulting in the lowest breathing mode frequency of exactly $2\omega_0$. This is not the end of the story, as we know for the case above, there is a quantum correction which breaks scale invariance, and indeed there is also such a correction that breaks the SO(2,1) Lorentz symmetry - recovering the quantum anomaly. This then shifts the $2\omega_0$ breathing mode value reported for the uncorrected system.

There have been various works [43][44][62] which present this theoretical correction to the breathing mode value, and of course the results of this thesis show an experimental result also concerning this shift.

In fact this correction can be related to Tan’s contact parameter via the pressure $\mathcal{P}$ [65]. Where the pressure is proportional to the energy density $\mathcal{E}$, we have

$$\mathcal{P} = \mathcal{E} + \frac{\hbar^2}{4\pi m} C$$

with $C$ the contact density $C = \frac{\hbar^2}{2\pi m \partial \mathcal{E} / \partial (a_{2D})}$. For scale invariance to hold, the contact must go to zero, and this only happens in the very weakly-interacting (to non-interacting) limit[66][44][67][68].

Discussion of these theoretical predictions and how closely they relate to our found experimental values can be seen in the chapter on the quantum anomaly experiment.
3 Experimental Methods

The experimental apparatus we utilized to achieve our goal of measuring the collective oscillations in an ultra-cold Fermi gas consist of many stages and devices. In this chapter I will outline each of these and the methods by which we create and measure the dynamics of the gas. The two spin states of atoms we trap and cool are the lowest hyperfine states of $^6\text{Li}$, $|F = 1/2, m_F = \pm 1/2\rangle$[69]. We trap these states using both laser and magnetic trapping. These are high-field seeking states, that is, their energy is lowered in regions of greater magnetic field. While we cool exclusively in these two hyperfine states, we can access others by applying radio-frequency excitation to drive atoms from one of these two spin states to the desired state. It should be noted that the general apparatus used to cool and trap our Fermi atoms has existed for several years and other discussions of the apparatus can also be found in prior theses[70][71][72]. As I go through I will spend more space detailing the changes and updates made during my tenure, which is centered around the 2D trapping potential. I have set out the following sections to follow the sequential process of the experiment, from the initial cooling of the atoms to the many stages of trapping, and then on to the absorption imaging. The next chapter follows on from there, starting from the imaging and detailing the analysis methods we use to extract physical information.

3.1 General apparatus

In this section I will begin describing the equipment and procedures we apply to trap and cool our atoms. Broadly speaking we combine both magnetic and optical traps[73] - projected onto atoms kept isolated from the lab environment at ultra high vacuum - to both confine and cool large numbers of atoms. Many steps are involved in this process, which we sequentially control via a computer loaded with a custom LabVIEW program, which interfaces with the various mechanical and electronic components of the apparatus using a data acquisition card (DAQ), both sending and receiving digital and analogue signals through an array of coaxial cables.

Our experimental setup begins with the lithium oven, which is a cylinder kept under vacuum of around $10^{-5}$ mbar, filled with a few grams of 95% enriched $^6\text{Li}$. In solid state the lithium is a white-ish soft metal, with melting point of 181° C. Heating wire covered with ceramic beads is wound around the outside of the oven, which keeps the
temperature at around 360° C during operation. This ensures that the lithium is kept in a liquid state with a high vapour pressure. The oven has a 4mm diameter tube opening in one of the walls, where a ceramic-bearing atomic shutter (model VPS38 from UHVdesign.com) is situated to turn on and off the flow of atoms. This shutter is controlled by pressurised air and uses magnets to connect it from outside to inside the vacuum compartment (the pneumatic valve has model EVK3120-5DO-01F-Q and is from SMC Corporation).

The atoms fly ballistically out of the oven and toward the science cell with mean thermal velocity around 1700 m/s - the length of the chamber is a series of ultra-high vacuum (UHV) stages which get progressively higher in vacuum until this science cell is reached, where the pressure is held at less than $10^{-11}$ mbar. We use a combination of ion pump and titanium-sublimation pump to keep the desired UHV at the point where it does not vary. The science cell is a bespoke glass chamber measuring $120 \times 30.3 \times 30.3$ mm with 2.7 mm thick Vycor quartz walls. It was produced by Hellma and is the chamber through which all our optical fields reach the atoms. It is anti-reflection coated on both sides for wavelengths near the electronic transitions we access - at 671 nm - and also for a far-detuned laser at 1030 nm (although we no longer use this wavelength).

### 3.2 Cooling the gas

Once the gas is released from the oven it undergoes a series of cooling stages to reach the desired temperatures of around $10-100\text{nK}$. I will describe the cooling process generally, and again note that detailed treatments are available in previous theses\cite{72,70}.

#### Zeeman Slower

The first element of cooling we apply to the atoms is that of a Zeeman slower. This works as follows. Once exiting the oven the atoms are all traveling in a direction with wave vector $\mathbf{k}$, down the length of a long tube. We shine a resonant laser opposite the direction of motion of these atoms, whereby they absorb photons from this beam and gain a momentum kick of $\hbar \mathbf{k}_{\text{photon}}$, slowing them by this momenta while they stay in the excited state\cite{75}. As is natural, the atoms then spontaneously emit a photon and
Figure 3.1: This is a photo of the apparatus as it stands in the lab. You can see the glass window of the science cell in the center of the image, with one of the MOT coils above it. The relevant schematics of the 2D trap will be shown below, this image here is shown to give a feel of what the apparatus looks like in reality - in fact much of the optics in the photo are used for experiments on Bragg spectroscopy[74].
return to their ground state. This emission process is, however, in a random direction - meaning that they retain, on average, the momentum kick opposite their direction of travel. Their maximum deceleration via this method is given by

\[ a = \frac{\hbar k_{\text{photon}}}{2m\tau} \]  

where \( \tau \) is related to the average lifetime of the atoms and the spontaneous emission rate, \( \tau = 1/\Gamma \) and \( m \) is the mass of the lithium atom. This process works until the thermal motion of the atoms is slowed to the point where their resonance frequency changes due to their velocity change. One way to keep the laser resonant with the atoms in this case is to apply a varying magnetic field along the length of the tube, to balance out this resonance shift due to the velocity change. This ‘Zeeman field’ is the heart of what is known as the Zeeman slower. Our Zeeman slower consists of a 30cm long tube wound with a single tapered coil of approximately 2400 turns, which is cooled by a second copper wire with hollow core through which cooling water flows. The specifications allows us to decelerate an atom traveling at around 650 m/s to around 50 m/s as it enters the science cell. The laser light we use to enable this slowing is \( \sigma^- \) polarized light with detuning of around 920 MHz relative to the \( |F = 3/2\rangle \rightarrow |F' = 5/2\rangle \) transition (an energy level splitting spectrum for \(^6\)Li is shown in figure (3.2)). Added to this are sidebands (120 MHz laser diode current modulation) which enhance the number of atoms caught by the resultant MOT by approximately a factor of two.

**Magneto-Optical Trap (MOT)**

As the atoms drift out of the end of our Zeeman slower they have been slowed to milli-Kelvin temperatures. From here we catch them in a magneto-optical trap (MOT) located approx 18cm from the end of the slower to provide further confinement and cooling. A MOT, as the name suggests, is a combination of magnetic and laser fields to cool and confine the atom cloud in all directions. Similar to the way the Zeeman slower provided a momentum kick opposite the direction of travel in one dimension, the MOT uses a six-way cross of lasers overlapping in each spatial dimension to create an optical molasses and slow the movement of the atoms in whichever direction they move. This optical molasses part of the trap can slow the atoms down to the Doppler limit of 140\( \mu \)K for \(^6\)Li, and sets the lower limit for this cooling method. To keep the
Figure 3.2: Shown here is the electronic energy spectrum of the lithium atoms\cite{76}\cite{69}. The cooling and repump beams are used in the Zeeman slower and MOT systems, where we use the transitions labeled. It is important to have the repump beam as any atoms decaying from the $|F' = 5/2\rangle$ state into the $|F = 1/2\rangle$ state can be re-cycled into the $|F = 3/2\rangle$ state to access the cooling beam once again.
atoms trapped while they undergo cooling in the optical molasses a magnetic trap is used. This is the magneto- part in the name. Two magnetic coils in anti-Helmholtz configuration provide a spatially varying magnetic field with a zero-field point in the centre - increasing in field radially outwards in all three dimensions. This field lifts the degeneracies of the \( m_F = 0, \pm 1 \) states of the atom and provides a position-dependent resonance condition.

In our apparatus we use six individual MOT beams in a six-way cross formation, detuned by four natural linewidths to the \( |F = 3/2\rangle \rightarrow |F' = 1/2, 3/2, 5/2\rangle \) and \( |F = 1/2\rangle \rightarrow |F' = 1/2, 3/2\rangle \) transitions, with intensity twice the saturation intensity. Initially we load the MOT for 30-40s to capture on the order of \( 10^9 \) atoms. Maximizing the atoms trapped requires that we maximize the phase-space density of the trap (see 2.1). This is maximized by moving from the initial MOT to a compressed MOT. A compressed MOT is the same as the original MOT except we ramp up the magnetic field confinement from 20 G/cm to 50 G/cm, reduce the detuning of the transition to \( \Gamma/2 \) and reduce the intensity of the lasers to well below the saturation intensity \( I_s \). At this stage we have achieved a temperature of 280(15)\( \mu \)K, which is very cold but still twice the Doppler limit for \( ^6 \text{Li} \). The number of atoms trapped at this temperature is of magnitude \( 10^9 \). We can’t hold the trap here for long due to the presence of inelastic collisions causing heat and loss of atoms. It is at this point we move to a dipole trap by overlaying it over the MOT and ramping up the power. We will use this dipole trap to perform the next cooling stage, that of evaporative cooling.

**Evaporative Cooling**

To perform evaporative cooling we use a very high powered fibre laser in the 100W YLR-100 from IPG. With this we create a red-detuned single beam optical dipole trap very far from resonance at 1075 nm, focused down to a waist of around 38 \( \mu \)m. Axial confinement is provided by curvature of the Feshbach magnetic field (see 3.3.2 for details on the magnetic potential). The method of evaporative cooling is to gradually lower the depth of the optical potential, allowing atoms with the greatest kinetic energy to escape, cutting off the upper tail of the Boltzmann distribution of velocities present in the gas. This method is effective at cooling, but reduces the total number of atoms trapped. Since this evaporative cooling relies on elastic collisions to redistribute the kinetic energy and re-stack the high momentum tail of the Boltzmann distribution so they can escape the trap; (taking much kinetic energy with them, cooling the gas) we
load the trap with as many atoms initially trapped as we can manage, which currently is around $10^7$ atoms per spin state: here again we are manipulating atoms in the two lowest hyperfine states of $^6$Li, $|F = 1/2, m_F = \pm 1/2\rangle$. To perform the evaporation cooling optimally the gas should be allowed time to rethermalize as the top of the potential is reduced, necessitating a slow reduction of this potential, but if this process is too slow the gas can lose many atoms to three-body recombination scattering events, spontaneous emissions or collisions with background gas molecules. Balancing between these two competing factors requires careful calibration of this cooling method. While performing the evaporative cooling we sweep the Feshbach field slightly to the BEC side of the resonance, as a gas of bosonic molecules reaches higher densities - as well as being trapped twice as deeply as unbound atoms - leaving the unbound atoms to be evaporated before the pairs, providing us with a fifty-fifty mixture of the two lowest spins states, once evaporated. Once the gas is cooled using this method we have our final temperatures of 10-100nK absolute, which corresponds to around 0.1 $T/T_F$ and $\approx 3 \times 10^5$ atoms in each spin-state inside the 3D well of the evaporation laser dipole trap. We are then ready to transfer into the 2D trapping potential. This 2D trap potential will be detailed below, where we also go into the theory of dipole traps in general (of which the evaporation trap is an example).

3.3 2D Trap schematics

We use a hybrid system of two different trapping mechanisms to trap our gas of two-spin $^6$Li atoms in a single, 2D cloud - a blue-detuned laser dipole trap in a TEM$_{01}$ mode for transverse/tight confinement along the $z$-direction, and a magnetic bowl potential for azimuthal/weak directions. This method has been used by other groups on Bose gases[78], with other methods including red-detuned single-cloud traps[79, 80] as well as lattice traps (where there are many 2D clouds trapped side-by-side) can also be used[81, 82, 83]. Trade-offs between different methods are apparent: in our design we have a single cloud and may directly image the density profile of the cloud by absorption imaging (see 3.4), whereas with a lattice this is not so, but a lattice may be able to probe interference patterns between clouds once the potential is removed[83], as well as other benefits of holding many clouds at once, such as having more clouds per cooling cycle.

As we will see we achieve trapping potentials with confinement frequencies of on
order $\omega_{\text{tight}} = 2\pi \times 5 \to 6 \text{kHz}$ and the weak trapping potential of $\omega_{\text{weak}} = 2\pi \times 22 \to 26 \text{Hz}$ where the variance in the trapping potential for the blue-detuned laser is decided by the laser power and the trap geometry, and the variance in the magnetic trapping is due to the magnetic field we are performing the experiment at. Note that we vary the magnetic field for accessing different regimes of the pairing and not really to vary the trapping potential, but this is a consequence of using the same set of coils to perform two functions: the trapping potential in the ‘weak’ direction, with frequency $\omega_{\text{weak}} = 2\pi \times 22 \to 26 \text{Hz}$ for magnetic field $B \approx 660\text{G} \to 860\text{G}$; with the second function the varying of the scattering length $a$. The trapping frequencies here are the frequency at which an atom oscillates back and forth in the potential, and dictate the energy levels of the harmonic oscillator associated with the trapping potential, where those energy spacings for a harmonic oscillator are given to be integer multiples of $\hbar \omega$, i.e. $\hbar$ times the trapping frequency.

I will here take a moment to clarify terminology. When I refer to ‘tight’, ‘transverse’, (sometimes this will be assigned a cardinal direction $y$) - this refers to the thin dimension of the pancake geometry in 2D. So the ‘weak’, ‘azimuthal’, ‘radial’ and ‘$x$-$z$’ cardinal directions are referring to the plane of the 2D which forms the 2D surface, or the larger directions of the pancake geometry. This is because the $z$-direction is taken to be the direction of laser propagation in the following. See figure (3.3).

### 3.3.1 Transverse trapping potential: TEM$_{01}$ mode dipole

Dipole trapping of atoms in an electric field formed by a laser is done through the mechanism of optically induced dipole potentials[84]. In a naive picture, a laser field induces an electric dipole moment in an atom which can serve to lower or increase its energy, creating a preference to move toward, or away, from the region of greatest field. Whether it is compelled toward or away depends on the wavelength of the photons - if they are red-detuned to the electronic transition on which the atoms are being absorbed and emitted then they are compelled toward regions of higher intensity, and vice versa for blue-detuned wavelengths - in this case they are repelled from regions of greatest intensity. This effect is known as the ac-Stark shift. The polarizability is the quantity that determines the strength of the induced dipole in the electric field, and
is dependent on the dipole potential as the following:

$$U_{dip}(r) = -\frac{3\pi e^2}{2\omega_0^3} \left( \frac{\Gamma}{\omega_0 - \omega} + \frac{\Gamma}{\omega_0 + \omega} \right) I(r)$$  \hspace{1cm} (85)$$

with $c$ the speed of light, and the radial frequency of the photons given by $\omega$, with the resonant frequency of the electronic transition $\omega_0$ and its linewidth $\Gamma$. $I(r)$ is the intensity function of the optical field. We can also express the scattering rate of the transition, $\Gamma_{sc}$, as

$$\Gamma_{sc}(r) = \frac{3\pi e^2}{2\hbar \omega_0^3} \left( \frac{\omega}{\omega_0} \right)^3 \left( \frac{\Gamma}{\omega_0 - \omega} + \frac{\Gamma}{\omega_0 + \omega} \right)^2 I(r)$$  \hspace{1cm} (86)$$

Theoretically the formulation for the TEM$_{01}$ mode trap is given by the following. We start with the electric field equation[85], with $z$ here the direction of propagation of the laser:

$$E_{l,m}(x, y, z) = A_{l,m} \frac{W_0}{W_{x,y}(z)} G_l \left[ \frac{\sqrt{2}x}{W_{x,y}(z)} \right] G_m \left[ \frac{\sqrt{2}y}{W_{x,y}(z)} \right] \exp \left( -i (kz - \frac{x^2 + y^2}{2R(z)} + (l+m+1)\zeta(z)) \right)$$  \hspace{1cm} (87)$$

where the indexes $(l, m)$ refer to the order of the Hermite-Gaussian functions $G$ in the
radial $l$ and axial $m$ directions, $A_{l,m}$ is the amplitude, $R(z)$ is the spot size at position $z$ and $W_{x,y}(z)$ is the waist size at position $z$ along the laser propagation direction. With $\zeta(z)$ the well known Riemann-Zeta function. The waists in the various directions are given by the capital $W$ functions, and depend on the wavelength ($\lambda$) of the photons forming the field:

$$W_{x,y}(z) = W_0 \sqrt{1 + \left( \frac{z \lambda}{\pi W_0^2} \right)^2} \quad (88)$$

with the $W_0$ being the minimum waist and $\frac{\pi W_0^2}{\lambda}$ is the Rayleigh range. We recall that the intensity distribution then, is equal to the magnitude of the square of the electric field, which gives:

$$I_{l,m} = |A_{l,m}|^2 \left( \frac{W_0}{W_{x,y}(z)} \right)^2 G_l^2 \left[ \frac{\sqrt{2}x}{W_{x,y}(z)} \right] G_m^2 \left[ \frac{\sqrt{2}y}{W_{x,y}(z)} \right] \quad (89)$$

where here we can express the amplitude as the peak intensity $|A_{l,m}|^2 = I_0 = \frac{P}{\pi W_0^2}$, where $P$ is the power in watts (W). For our TEM$_{01}$ mode we have $l = 0, m = 1$, which, substituting in the appropriate Hermite-Gaussian functions gives us:

$$I(x, y, z) = \frac{2Py^2}{W_x(z)W_y^3(x)} \exp \left( - \frac{2x^2}{W_x^2(z)} - \frac{2y^2}{W_y^2(z)} \right) \quad (90)$$

We can now substitute this back in to the equation for the dipole potential, giving:

$$U_{dip}(x, y, z) = U_0 \frac{2y^2}{W_y^2(z)} \exp \left( - \frac{2x^2}{W_x^2(z)} - \frac{2y^2}{W_y^2(z)} \right) \quad (91)$$

$$U_0 = \frac{3\pi c^2}{2\omega_0^3} \left( \frac{\Gamma}{\omega_0 - \omega} + \frac{\Gamma}{\omega_0 + \omega} \right) \frac{P}{W_x(z)W_y(z)} \quad (92)$$

From the curvature of (91) we can gain the harmonic trapping frequencies in each direction. This is done by Taylor expanding to second order and gives, for the tightly trapped direction:

$$\omega_y = 2\pi \sqrt{\frac{U_0}{\pi^2 m W_{0y}^2}} \quad (93)$$

In the ‘weak’ trapping directions, there is no optical trapping potential due to the near-absence of light in the center of the mode. However, in these directions there is
Figure 3.4: Visualization of the intensity for the TEM$_{01}$ mode curve as given by equation (90). The scales here are in arbitrary units for visualization. The vertical axis is intensity.
a weak anti-trapping and the trapping frequencies are imaginary. This anti-trapping is due to the tight confinement along $y$ which has a high frequency $\omega_y$ leading to an energy offset of $\frac{\hbar \omega_y}{2}$ [84]. The effect of this anti-trapping is overcome by the magnetic trapping (see 3.3.2):

$$\omega_x = 2\pi \frac{j}{\sqrt{2\pi}} \frac{1}{W_{0x} \sqrt{W_{0y}}} \left( \frac{\hbar^2 U_0'}{m^3} \right)^{\frac{1}{4}}$$  \hspace{1cm} (94)

$$\omega_z = 2\pi \frac{j \lambda}{2\pi^2} \frac{1}{W_{0y}^{5/2}} \left( \frac{\hbar^2 U_0}{m^3} \right)^{\frac{1}{4}} \hspace{1cm} (95)$$

This results, for a perfect Hermite-Gaussian beam with tight waist of 10$\mu$m, weak waist of 1800$\mu$m in the 532nm beam of 7W power:

$$\omega_y = 2\pi \times 5.8 \text{kHz} \quad \omega_x = 2\pi \times 1.73 \text{iHz} \quad \omega_z = 2\pi \times 3.74 \text{iHz}$$  \hspace{1cm} (96)

with geometric mean anti-trapping in the radial direction of $\omega_r = \sqrt{\omega_x \omega_z}$. As we trap in the radial direction using a magnetic field (see next section) which has frequency $\omega_r \approx 2\pi \times 24.5$ at 832G, the contribution the optical anti-trapping makes is very small as the contributions are summed as $\omega_{r,\text{total}} = \sqrt{\omega_{r,\text{mag}}^2 + \omega_{r,\text{dip}}^2}$.

To create the TEM$_{01}$ mode in the laser we impart a $\pi$ phase shift in half the incident beam, by using a material of different refractive index to retard that half of the beam relative to the other. For more detail about creating the mode and fabricating an optic device capable of creating this mode, see Appendix A. Once we have the mode induced in the beam we characterize it by taking a series of images at various points upstream and downstream of the focal point.

When we stitch these together in an appropriate program we gain the information about the intensity profile in all 3 dimensions and consequently the waist sizes of the mode. An example of the kind of image we get from this characterisation is shown below.

Once we have calculated the expected trapping frequencies the next step in calibration is to deploy the 2D trap onto the cold atom gas and measure the actual oscillation frequency. I will briefly describe how we transfer the atoms from our 3D evaporative cooling laser to the 2D trap and then detail how we measure the tight trapping frequency.
Figure 3.5: Here we see images taken of the TEM\textsubscript{01} mode. These images are at the focal point of the mode, where a sweep of many images each 50\textmu m apart is taken, through the focus. We can use these images to calibrate the mode as shown below. Width of the valley (1/e\textsuperscript{2}) in the transverse direction $W_y \approx 10\textmu m$. The $z$-dimension is the direction of laser beam propagation. Camera used to take these images was BASLER acA2500 - 14um with attached confocal lens of numerical aperture $NA = 0.25$ placed at the end of a lens tube.
Figure 3.6: This is a scan-through of the intensity profile of the 2D trap. It is made up of a series of images taken through the focal point of the trap from the final lens, showing how the intensity distribution varies over this focal length and how the intensity of the ‘lobes’ - the two peaks - varies. We use images like this to calibrate the potential of the trap, as we can calculate the waist sizes and thus frequencies from this in our MATLAB program. The $1/e^2$ waist here is $W_y = 10\mu m$. The numbers shown on the axes correspond to camera pixels - the camera had pixel size $6.45\mu m$, and we use a magnification of $\times9.3$ to our confocal imaging lens, so this gave us a final conversion of $1\text{unit} = 0.65\mu m$ along the $x$-axis of this image. The $y$-axis is how far the stage is wound between successive images, which here is $2.5\mu m$ per unit. In the bottom image the wave-like imperfections are due to slight misalignment as the camera stage is wound by hand between slices.
At the end of our cooling phase the atoms are trapped in a red-detuned dipole trap which contains up to $3 \times 10^5$ atoms per spin state (with two spin states). To transfer to the 2D trap - where we trap approximately $80 \times 10^4$ atoms per spin state (due to geometry) - we hold the atoms in the shallowest potential of the evaporative cooling process, on the BEC side of the Feshbach resonance (to compress the cloud), while linearly ramping up the power of the 2D 532 nm blue-detuned potential over 150 ms to a very weak potential: approximately 4% of the final power. This very weak blue-detuned trap still holds the atoms without spilling them. We then ramp down the evaporation laser over 150 ms so that the atoms are now fully trapped by the weak TEM$_{01}$ mode and the magnetic bowl potential. To ensure that dipole modes are not induced (sloshing modes) or if induced they have had time to damp out, we ramp up the 2D trap slowly over 750 ms to its full power. This results in power of approximately 6W at the atoms, where we read 10W at the laser head; the loss being due to the optics in the system but mainly the acousto-optic modulator (AOM) and reflections from the surface of the glass cell (the optical setup for the 2D trap is detailed below in figure (3.8).

Once trapped in the 2D trap it is of importance to measure the tight trapping frequency. To do this we make use of our ability to image the atoms from the side of the trap using absorption imaging (see section (3.4)). By weakening (turning down the power) of the TEM$_{01}$ mode from maximum to around a third (6W to 2W) then ramping it back up over 20$\mu$s, while applying a strong magnetic field gradient to displace, sets up a dipole mode oscillation in the atoms in the tight trapping direction. We then allow this oscillation to continue for a time-step $\tau$ before taking an image from the side. This image is taken after a time-of-flight of 1.2ms, and we can determine the centre-of-mass momentum of the cloud. Fitting this centre of mass gives us a snapshot of the motion of the cloud. Doing this consecutively for many time-steps allows us to build up a picture of the oscillation over time as shown in figure (3.7). We are then able to fit an exponentially damped sinusoid to this curve and extract the tight trapping frequency.

So how do we produce the laser trapping potential? In figure (3.8) we can see a schematic layout of the optical setup that produces the trap. The laser production
Figure 3.7: Vertical displacement of the centre of mass in pixels vs time in ms. This measurement is of the center of mass measured from the side of the 2D trap at differing time steps. Each measured time step (blue) is the average of 3 images. The fit (red) is of an exponentially damped sine wave $ae^{-bx}\sin(cx + d) - ex + f$, with $a-f$ the fitting variables and $x$ the timestep. This exponential captures the reduction in amplitude over the range of the data, with the sine wave capturing the oscillation itself. The fit results in a tight trapping frequency of $\omega_z = 2\pi \times 5.65$kHz.
Figure 3.8: Schematic of the 2D trapping optical setup. Lengths are not to scale. A detailed description of the setup is provided in the text.
begins with our green laser, which is a continuous-wave (cw) 532nm Verdi G15 from Coherent. This laser is capable of up to 15W of cw power with low noise. We typically run the laser at 10-12W during operation to get the trapping frequencies we desire. Since this is quite a high power, all the optics need to be chosen to make sure they can handle the high power-density. To that end we attempt to use fused-silica lenses where possible, and air-gapped achromats.

Once the light is emitted from the laser it passes through a half-wavelength wave-plate which allows us to tune the linear polarization of the light, which we then pass through a beamsplitting cube if we want to pick off some of the light for another application or to turn down the output during calibration or alignment. From there we use a pair of mirrors to reflect the light up through a periscope from the optical table to a board mounted above the laser head. Another half-wave plate is placed before a 3:1 reducing telescope. These two elements are primarily to modify the beam to match the optimal specifications of the acousto-optic modulator (AOM), so that we achieve the greatest diffraction efficiency possible. Typical AOM efficiencies we achieve are between 80-85%. This AOM is driven by a combined digital and analogue signal modulated by a proportional-integral-derivative (PID) controller which is tuned to stabilize the laser intensity while the circuit is in operation. The AOM allows direct control over the intensity of the trap while in operation - we interface with the AOM through the PID instead of changing the power on the laser head when we require. This is done from our computer control program which allows output of both TTL digital signals and analogue -10 to 10V variable output signals. An example of when this is used is ramping up the power of the trap initially, we do this via a linear ramp which is fed to the PID circuit which controls the AOM in the form of a varying voltage signal. We trap using the +1 order output mode of the AOM, and dump the zeroth order. This is due to practical considerations: if there is a malfunction in the AOM system it is preferable for the system to be dumping the power.

After a few mirrors the beam reaches a 30mm achromatic lens which focuses into a 25\( \mu m \) pinhole. The purpose of focusing the beam tightly into a pinhole is to perform spatial filtering. With the pinhole efficiency of around 92-98% we are able to clip just enough of the edge of the beam to produce Fresnel rings and obtain a highly gaussian beam in the far-field. When calibrating the system we experimented with different pinholes - 50, 100 and 25 \( \mu m \) sized - and different lens sizes to produce the
optimal configuration in terms of the final trapping waist of the TEM$_{01}$ mode - this being dependent on the beam diameter at the final lens before the science cell, and the cylindrical lens. This is relevant as after the pinhole has done its spatial filtering the next optic (apart from an aperture to block out any higher order noise) is the 400mm lens which collimates the beam. At this point the beam is collimated and quite large - a visible spot of approx 20mm ($1/e^2$ diameter) can be seen. This large beam is then incident on the $\pi$-phase plate, which produces the interference pattern giving us the TEM$_{01}$ mode. This phase plate is an optical flat half coated with magnesium-fluoride (MgF$_2$) of height 600nm. This step produces the $\pi$-phase shift in half of the incident beam giving rise to the interference pattern of the TEM$_{01}$ mode. More information on this is given in the appendix as I spent some time exploring fabrication of a new $\pi$-phase plate.

Next the beam passes through a 1000mm cylindrical lens, which focuses the beam in the direction we want the final, 300mm achromat lens to collimate - the radial $x$-$y$ plane (with $z$ the direction of laser propagation). After passing through a narrow slit at the focus of the cylindrical lens (again, this slit is designed to clear up any higher order noise and restore a nice gaussian intensity profile to the beam) it arrives at the final focusing lens before the science cell. This lens is mounted on an $x$-$y$ micrometer stage which allows us fine control over the focal position of the laser beam. This is used to align the trap directly over the atoms which is necessary to ensure a clean transfer from the previous trap to this 2D trap. Lastly the beam passes out from the atoms and into a beam dump, where a glass picko (just a glass coverslip) sends approximately 1% of the total power into a photo-diode which converts the light signal into a voltage, allowing us to use a PID to control for intensity noise.

This setup of the TEM$_{01}$ dipole trap has been largely successful, although many lenses and a long beam path mean that any imperfections propagate through the path of the laser. To counter this we used the multiple stages of spatial filtering - namely the pinhole and the vertical slit. The alternative to the free-space propagation is to use a fibre to pipe the beam while keeping a nice gaussian profile, and this is certainly something we attempted initially but unfortunately the fibre we had couldn’t handle the power and was burnt. A new fibre setup is on the way to replace this method of trapping, and hope is that it will provide greater stability and robustness against mechanical movement over time, as well as a higher degree of spatial filtering.
Figure 3.9: The Feshbach coils in Helmholtz configuration from which the magnetic trapping potential is produced. Since the coils have radius 4cm and are placed 5cm apart, this results in the residual curvature.

3.3.2 Radial trapping potential: Magnetic field

Magnetic trapping is useful because when the fields originate from large coils they have very harmonic potentials[86]. We exploit the pre-existence of our Feshbach coils which provide us with a magnetic trapping potential in the radial direction with the form of a saddle potential. The hyperfine states we trap are high-field seeking, so they are attracted the central regions of higher magnetic field. The Feshbach coils we use in our setup are nearly in the ideal Helmholtz configuration, being ‘horizontally’ aligned in the plane of the earth (see figure (3.9));

Regulation Helmholtz coils are placed a distance $R$ from each other which is equal to the radius $R$ of the magnetic ring. In our setup they are slightly further apart from this ideal distance, which creates a ‘dip’ along the coil axis, resulting in a maximum in the radial plane. It is in this curvature that we trap our high-field seeking atoms.

The technical specifications are as follows. The coils are made up of hollow square copper wire of outer cross section $3.2 \times 3.2$mm with internal square hole $1.6 \times 1.6$mm through which we run chilled water to cool the coils while in operation. Each coil (of the two) are made up of 51 turns and together have total inductance of $433 \mu$H. This results in a characteristic switch off time of $\tau = \frac{L}{R} = \frac{433 \mu H}{(1/7) \Omega} = 3$ms.
\[
B_z(r, z) = \frac{IN_T \mu_0}{2\pi} \frac{1}{\sqrt{(a+r)^2 + (z-h)^2}} \left[ K \left( \frac{4ar}{(a+r)^2 + (z-h)^2} \right) 
+ \left( \frac{a^2 - r^2 - (z-h)^2}{(a-r)^2 + (z-h)^2} \right) E \left( \frac{4ar}{(a+r)^2 + (z-h)^2} \right) \right]
\]

\[
B_r(r, z) = (z-h) \frac{IN_T \mu_0}{2\pi r} \frac{1}{\sqrt{(a+r)^2 + (z-h)^2}} \left[ - K \left( \frac{4ar}{(a+r)^2 + (z-h)^2} \right) 
+ \left( \frac{a^2 + r^2 + (z-h)^2}{(a-r)^2 + (z-h)^2} \right) E \left( \frac{4ar}{(a+r)^2 + (z-h)^2} \right) \right]
\]

The equations above describe the magnetic field intensity across the vertical \((z)\) direction and along the radial plane. In them the \(K()\) and \(E()\) functions refer to the complete elliptic integrals of the first kind, \(I\) is the current in Amps, \(N_T\) the number of turns of the coil, \(a\) is the radial distance and \(h\) the height, as the coils may be offset from the axis in the \(z\)-direction. These equations are for coils of fixed size - the magnetic field is simulated to emulate from the centre of the coil structure, so \(r\) is the radius to this point. \(\mu_0\) is the permeability of free space. The result from computing our intensity distributions (the code is available for mathematica here [https://github.com/Zholistic/Lithium6/blob/master/magFieldSimulation.nb](https://github.com/Zholistic/Lithium6/blob/master/magFieldSimulation.nb)) is, for the 51 turns and with 111 Amps is displayed in figure (3.10).

The radial magnetic trapping frequency correspondingly varies depending on the which field we are performing the experiment at. Since the field is also tied to the Feshbach resonance and therefore the interaction strength between the particles of different spin state, we get a frequency range of around \(\omega_r = 2\pi \times 22 \rightarrow 26\text{Hz}\). We verify this by measuring the frequency at different fields. The measurement is done by perturbing the cloud of atoms and looking at the rate at which it sloshes - also known as measuring the dipole mode. The resulting frequencies vary with the magnetic field as:

\[
\omega_r = \alpha \times \sqrt{B}
\]

(97)

Where here \(\omega_r\) refers to the radial trapping frequency and the constant \(\alpha \approx 0.84(5)\) for our setup. More discussion on this calibration and detailed graphs calibrating the
Figure 3.10: Here we have a slice of the magnetic field intensity distribution, for 111 Amps modeling our in-house Feshbach fields in Helmholtz configuration. You can see that for the center slice (left), there is an inverted bowl potential for high-field seeking atoms, which peaks at around 832G for input current of 111 Amps. For the slice through of the height (right) we have a small dip in the magnetic potential as our coils are displaced slightly more than a regulation Helmholtz configuration.

Figure 3.11: A contour surface slice-through of the magnetic field for the 111 Amps as in the figure above. Here we see the magnetic field intensity variation at all positions around the coils (black dots - they are infinitely thin in this model), with the saddle that exists in between the coils. The axes here are in meters, with the field intensity colours labeled on the contours themselves.
radial frequency depend on the magnetic field can be found in the chapter on the collective oscillation result, as the quantum anomaly is sensitive to this value.
3.4 Imaging Techniques

To extract information about the density of the gas we image it using the method of absorption imaging. Other imaging techniques that can be used for cold-atom imaging include fluorescence imaging\cite{87} and phase-contrast imaging\cite{88}\cite{89}. Absorption imaging works by shining a resonant laser field onto the cloud of atoms, which absorb photons, allowing us to detect where this absorption takes place by comparing the image to a background image without absorption. This effectively images the shadow of the atoms, and is a form of destructive imaging, as once the atoms have absorbed photons, they heat and irreversibly change from their original state. We focus the shadow of the atom cloud onto a CCD camera and have the capability to image from both above the plane of the 2D atoms (the top) and to the side of the atoms.

Imaging from any direction this way by its nature integrates over that dimension, to gain the column density: \( n(x, y) = \int n(x, y, z)dz \). Since our cloud is 2D when we image from the top this allows us to directly infer the density of the atoms. We take three main images utilizing a kinetics mode available to the camera to image onto different sections of the CCD. The kinetics mode allows shifting of the information in each element of the CCD to another section of the CCD array, before saving the CCD information. This enables us to take these images as fast as possible: around 820\( \mu \)s apart, limited by the 3.2\( \mu \)s row shunt time. The three images are a beam image, an image of atoms in one spin state, and an image of atoms in a second spin state. A dark image is also taken, with the shutter closed, from which we can tell if there is stray light and also quantify the shot-noise of the camera setup. We tune our laser to the frequency of the spin state optical transitions from each using a frequency offset locking method which involves polarization locking to a saturation absorption spectroscopy cell. Offsets to the frequency are fed in via current control of the laser wavelength.

Absorption imaging invokes the Beer-Lambert law to relate light intensity to density via:

\[
\frac{dI}{dz} = -n\sigma(I)I
\]

where \( n \) is the density and \( I \) the imaging intensity, with \( \sigma(I) \) being the optical cross-
section taking saturation into account is equal to:

$$\sigma(I) = \frac{\sigma_0}{1 + (2(\omega - \omega_0)/\Gamma)^2 + I/I_{sat}}$$

(99)

with the resonant absorption cross-section of \(\sigma_0 = \frac{3}{8\pi} \lambda^2\) (with \(\lambda\) the wavelength of the light which is 671nm for our imaging setup); \(I_{sat} = 2.54\text{mW cm}^{-2}\) the saturation intensity, \(\Gamma = 5.9\text{MHz}\) the transition linewidth, \(\omega - \omega_0\) the laser detuning from the resonant transition. Putting the previous two equations together and integrating the first, we gain an expression for the column density as:

$$n(x, y) = \frac{1}{\sigma_0} \left[ \frac{I_i - I_f}{I_{sat}} + \left( \frac{2(\omega - \omega_0)}{\Gamma} \right)^2 + 1 \right] \ln \left( \frac{I_f}{I_i} \right)$$

(100)

Where the initial and final images \(I_f\) and \(I_i\) in our case are the images of atoms and the background image. Since we use a method of resonant absorption of light, our exposure intensity is measured versus the saturation intensity \(I_{sat}\). A typical image is taken for an exposure time of 10\(\mu\)s and at \(I \approx 0.25I_{sat}\). An issue can arise when we wish to measure accurately the atom number in the gas. If we expose for longer, we gain a better signal to noise ratio in the image, but we lose accuracy in the amount of atoms presented due to Doppler shifting of atoms that have already absorbed photons out of the original resonance. This problem is exacerbated with \(^6\text{Li}\), since this species is light and more susceptible to the effect. In fact the single photon recoil is equal to 147 kHz, so for imaging at \(I = 5I_{sat}\), ie taking images at five times the saturation intensity, which is typical of the experiment, we can calculate the number of photons scattered in 10\(\mu\)s to be:

$$N_{\text{photons}} = \frac{\Gamma \tau}{2} \frac{I/I_{sat}}{1 + I/I_{sat}}$$

(101)

which gives us a value of around 30 photons scattered per atom. So for our 10\(\mu\)s image we get a total red-shift of 4410 kHz. We need to compare this to the power broadening of the transition linewidth; which goes by a factor of \(\sqrt{1 + I/I_{sat}} = 1.12\). So 4410 kHz/(1.12 \times 5.9\text{MHz}) 2.4%. While this is a good amount less then 1, it still means the atoms will not scatter at the maximum rate possible during imaging and thus not contribute to the optical density measured. To overcome this we take an image at a very short exposure time of 1\(\mu\)s and high intensity \((I \approx 10 \times I_{sat})\), which scatters less photons during imaging, so we gain a much more accurate atom number
The optical density of the clouds is shown for each as the false-colour bar. Pixel length here is 2.84\(\mu \text{m}\) with the magnification of the top camera (\(\times4.58\)) already factored in. (but correspondingly worse signal to noise - see figure 3.12). This image allows us to evaluate a ratio between the atom number count and density in both the high and low intensity images, to accurately estimate the atom number present in the gas. The very low optical density of the cloud (due to its 2D nature) helps us here as all atoms see approximately the same intensity of imaging beam.

We also have our set up so we may image the cloud from the side, using a secondary imaging beam and camera. This is an Allied Vision Prosilica GC1290 where we have a magnification of 1.9\(\times\) onto the CCD. This camera does not have the kinetics mode feature, and so the imaging is done with a sequence of shots of the background and of only one of the imaging states. It is useful to have both side and top imaging setups, as both of them are used to calibrate the trap and to gain different perspectives on the atom cloud.

This brings an end to this section, where I have outlined the imaging technologies and procedures we use to gain absorption image information about our atom cloud. We also conclude the general experimental methods chapter, where technical aspects of the cooling and trapping process were laid out. Detail on certain experimental techniques associated with particular results can be found in the later chapters - which deal directly with those results. The next chapter focuses on the computational and analytic
Figure 3.13: Here we can see an absorption image from the side camera setup, where this shot is taken after a time-of-flight of 600$\mu$s: in-situ images of the 2D cloud from the side are so thin as to be beyond the diffraction limit. We use side images such as these to calibrate the tight trapping frequency.

methods required to extract information from the data gained by these experimental techniques.
4 Analytic Methods

Experimental physics has always been about obtaining data from a physical apparatus and then analysing and interpreting these to gain meaningful results. This chapter deals less with the techniques of the apparatus and more with the analytical methods for processing the data: firstly with the methods of temperature fitting (thermometry) and secondly with the utilization of principal component analysis (PCA) to analyse the dynamics of a time varying sequence of images. Both are computation methods that use advanced algorithms to analyse the data to gain further information, and, as they are both extremely useful and important in the field of cold atom research, I include them here.

4.1 Thermometry

Determining the temperature of an interacting gas so close to absolute zero (typically 10 - 100nK) is a non-trivial affair. It is not simply a matter of placing a thermometer - a quantity of a substance with a known behaviour at different temperatures - close to the gas, as it would be difficult to do so without heating or disturbing the cloud. Another important consideration is that simple analytic models relevant to ideal and weakly interacting gases cannot be applied to the conditions of our atom cloud, which are usually strongly interacting and at non-ideal temperatures.

The central procedure to the technique of temperature measurement is to use what we can measure - the absorption image of the gas - and fit the resulting density profile to a known distribution where temperature is a free parameter of the fit, thereby determining it without prior knowledge. In our method, we transform the density profile using the local density approximation (LDA) and knowledge of the equation of state (EOS) to a form which we can fit to the virial expansion about the fugacity in the high temperature (low density) wings of the cloud, and a self-consistent T matrix theory in the so-called ‘GG’ theory[90] as the cloud becomes more dense. The explanation of these terms and the method follows.
4.1.1 Density Profiles and Equation of State

For an ideal gas considered in section (2), it is possible to fit polylogarithmic functions such as

\[ n(x, y)_{2D} = -\frac{1}{\lambda_{dB}^2} \text{Li}_1 \left( -e^{(\mu - \frac{1}{2}m \omega_x^2 x^2 - \frac{1}{2}m \omega_y^2 y^2)/(k_B T)} \right) \]

(102)
to the line profile of the gas. These profiles show that as the temperature increases, the deviation from the zero-temperature density profiles is most pronounced in the wings of the cloud. Fits to these functions do not, however, take interactions into account, and are thus not correct for our experiments which lie close to a Feshbach resonance, and for 2D cloud which always contain a weak bound state. What, then, is the equation of state (EOS) of our system?

Determination of the equation of state for various regimes is of great interest, as the equation of state defines the relation between thermodynamic variables. In the past, formulation of the EOS for 3D Fermi gases[91][92][93] has been reported as well as for 2D Bose gases[94][95][96]. The equation of state for the 2D Fermi gas was studied experimentally by our group[97] and others[53] with theoretical treatments[98][90]. One benefit of formulating the gas in terms of this equation of state is that it allows us to express the density profile in terms of a universal function \( f(\mu, \beta E_B) \) which takes into account the energy scales due to the temperatures \( \beta = 1/(k_B T) \), binding energy of the bound state \( E_B = \frac{\hbar^2}{m a_{1D}^2} \) and the chemical potential \( \mu \). This general function \( f \) can be calculated or approximated in various limits - the virial expansion becomes exact at high \( T \), and the GG function (described in section 4.1.3) performs well up to higher interaction strengths - so if we can express our data in terms of this equation of state function we can fit to it. The first step toward doing this is recognizing that if we know the shape of our trapping potential well (and we do), we may re-express how the density varies spatially to how the harmonic potential varies, using the local density approximation we first encountered when we looked at the polytropic equation of state:

\[ \mu_0 = \mu[n(r)] + V_{ho}(r) \]

(103)
The next step in applying this process is to express the density with respect to potential - \( n(V) \). This then enables quite straightforward ways to generate the pressure \( P \) and compressibility \( \kappa \) of the gas, by taking the integral and the derivative of this density-
potential curve (where we follow in the footsteps of a treatment for the 2D Bose gas\cite{96} - more detail on our approach can be found in Kristian Fenech’s thesis\cite{99}). Note that the following equations for the pressure and compressibility have had the LDA applied so are in terms of $n(V)$, the density as a function of potential:

\[ P = \int -n(V)dV \]  
\[ \kappa = -\frac{1}{n^2} \frac{dn(V)}{dV} \]

Which we can normalize by dividing by their respective zero temperature values in an ideal gas (T=0):

\[ P_0 = \frac{nE_F}{2} \]
\[ \kappa_0 = \frac{1}{nE_F} \]

where here the Fermi energy is $E_F$ and the $A$ is area under the curve. Figure 4.1 shows a numerical modelling of the normalized pressure and compressibility for a simulated cloud at zero and 25nK temperatures, using the ideal gas equations (as in section 2).

### 4.1.2 Virial Expansion

There are various approaches to take interactions into account, one of which being the virial expansion in terms of the fugacity $z = e^{\beta \mu}$ (where $\beta \mu$ is a small parameter), which is valid in regions of relatively high temperatures\cite{100}. The fugacity is a correction to the pressure term (effective partial pressure) which allows the ideal gas law to still be valid with interactions. The virial expansion is a perturbative method which produces predictions for the thermodynamic variables, including the density as:

\[ n(r) = \frac{2}{\lambda^3} \ln(1 + z(r) + 2\Delta b_2 z(r)^2 + 3\Delta b_3 z(r)^3 + ... ) \]
\[ \text{where } \quad z(r) = e^{\beta(\rho_0 - |V(r)|)} \]

Here the $\Delta b_n$ are the $n$th coefficients of the expansion, which are dependent on the interaction strength and temperature as characterised in $\beta E_B$. The second order
Figure 4.1: Results of the equation of state for ideal density profiles at T=0 (blue) and T=25nK (red). The plots are numerical simulation, at discrete points (hence the scatter). The value $\tilde{p}$ is the reduced pressure and the $\tilde{\kappa}$ the reduced compressibility. This theory could be used to calculate the temperatures, by fitting to our measured experimental density profiles, however it does not take into account interactions, and so we consider more complex theories in the next section. These curves model functions present in equations (110) on.
coefficient takes on a value, which can be obtained with the Beth-Uhlenbeck formalism, of
\[ \Delta b_2 = e^{\beta E_B} - \int_{-\infty}^{\infty} \exp\left(\frac{e^{2t}}{2\pi}\right) \left[ \frac{2}{\pi^2 + 4(t-x)^2} \right] dt \] (108)
with \( x = \frac{1}{2} \ln(2\pi\beta E_B) \). The third order virial coefficient is much more involved to compute. We get the numerical value from a program supplied by the author of [100] for given values of \( \beta E_B \), which follow the methods introduced in that paper and requires solving a three-body calculation. The C-program with MATLAB wrapper (enabling it to be called in MATLAB via command line) can be found here https://github.com/Zholistic/Lithium6/blob/master/huicoeffs.cpp.

4.1.3 The GG theory

A more sophisticated theory of the 2D Fermi gas that accounts for interactions and correlations between particles of differing spin is the GG theory[101][90][102]. This is a self-consistent T-matrix theorem, where the G’s refer to interacting Green’s functions. It has been studied extensively for 3D[103], and is also known as the Luttinger-Ward theory. The result we use is calculated numerically for a particular \( \beta E_B \), and ends with a set of curves \( f_n \) vs \( \beta \mu \) as shown in figure (4.2). In the limit of high temperature and negative chemical potential \( \mu \), it approaches the virial results. The primary advantage we gain by using this theory, as we will see from the fits, is that it enables us to continue our fitting routine up to higher \( \beta \mu \approx 1 \), corresponding to lower temperature \( T/T_F \), where the third order virial results fails for \( \beta \mu > -1 \).

4.1.4 Fitting Routine

To fit density profiles of our Fermi gas to the virial expansion and GG calculations we follow an involved procedure. We begin with the experimental absorption images of the cloud, taken from the top-down perspective, as described in the imaging section. Using a number of images, of clouds taken under the same conditions, we average them \textit{azimuthally}, that is, we average them radially around the center-of-mass of the cloud, taking advantage of the circular symmetry of our cloud. Typically the more images taken under the same conditions, when azimuthally averaged, give better (smoother - less noisy) density vs potential curves, and so we took upwards of twenty shots to fit the
Figure 4.2: This graph shows the comparison between the GG and the 2nd and 3rd order virial function for the choice of $0.2\beta E_B$. You can see they converge for high T or negative chemical potential $\mu$. 
Figure 4.3: Surface showing the GG data that was generated for different interaction strengths ($\beta E_B$) across the various $\beta \mu$. From the surface we can choose a fidelity that we require by interpolating where needed. Note that we don’t use the part of the surface where there are no generated points from the GG theorem.
curves below. As we saw that the GG theorem follows the curve of the data to higher \( \beta \mu \) than the virial expansion, we are able to fit using this GG theorem to more points than the virial expansion. If we like, we can use this advantage to fit to averages of less absorption images than with the virial expansion fits - we can reduce the amount of images we need to take under the same conditions and still produce a robust fit. As an example we ideally would take 50 or more images of the cloud under the same conditions and average that number, when using virial expansion fitting routines, due to the noise in the wings of the cloud (the low density regions), where the virial expansion is fit. Since the GG function gains us more points to fit over, we may take 20 or so images and fit to those. So this method has improved our ability to fit in that sense.

Since the GG function is numerically generated only for certain values of \( \beta E_B \), we fit a surface to all the data as a function of both \( \beta E_B \) and \( \beta \mu \) as show in figure 4.3. On that surface we may interpolate over the range of \( \beta E_B \) to generate a surface to the fidelity we desire. One this is done we have a surface which spans the range of GG theory points supplied to us by Dr Brendan Mulkerin. The next step we take is to fit analytic functions along each value of \( \beta E_B \). The reason we fit an analytic function like this is not only for convenience in the fitting routine. The analytic function allows us to connect the data from the GG and virial calculations. We use the virial expansion in the very low \( \beta \mu \) regions of the curve where the GG theory has not been generated, but where this known theory (the virial) is valid (from around \( \beta \mu = -3.5 \) the virial takes over), and extends correctly all the way to \( \beta \mu \to -\infty \). These analytic fits on each curve of virial expansion plus GG theory take the following form:

\[
 f_{GG\text{comp}}(x) = \left[ c(1) + c(2)x + c(3)x^2 + c(4)x^3 + c(5)x^4 + c(6)x^5 \right] / \left[ 1 + c(7)e^{c(8)x + c(9)} \right] \tag{109}
\]

where the \( c(a) \) represent a set of free coefficients of the fit for a given \( \beta E_B \). One of these curves exists for every \( \beta E_B \) spline we have on the surface in figure 4.3. The form of this equation ensures the fit tails off the same way as the virial expansion function in the high temperature, negative \( \beta \mu \) limit. A further reason for creating this ‘composite’ function is that minimization routines (computational fitting algorithms) fit better to more points, and we were constrained by the number of points that were generated by the GG theory.
From the experimental data, we proceed with the density vs potential curve in standard international (S.I.) units (so after averaging of many absorption images, and application of a radial averaging on the symmetry of the cloud, which gives us a 1D profile, then application of the LDA) with which we transfer the density and the potential to the equation of state \( f_n \) vs \( \beta \mu \) via the transformations:

\[
n(V) \rightarrow n(V) \frac{\hbar^2}{m} \beta = f_n(\beta E_B, \beta \mu) \tag{110}
\]

\[
V \rightarrow (\mu_0 - V) \beta \tag{111}
\]

Where here we have used the equation of state identity\[97\] that

\[
n \lambda^2 = f_n(\beta E_B, \beta \mu) \tag{112}
\]

and recall that the de Broglie wavelength squared \( \lambda^2 = \frac{2\pi \hbar^2}{m} \beta \). The GG theory supplied to us was generated numerically and in dimensionless units.

From the transformation equations (110) and (111) we can now express our density vs potential curves for the gas on top of the GG and virial \( f_n \) vs \( \beta \mu \) curves, provided we choose a starting \( \beta E_B \). Let us look where the unknowns exist. Every contains an unknown temperature. \( \beta E_B \), the binding energy, can be calculated, for a given \( a_{2D} \), which is also known; as the behaviour of the 3D scattering length around the Feshbach resonances is well-documented\[11\][57]. It turns out only \( \mu_0 \), the chemical potential at the center of the cloud, is unknown apart from the temperature \( T \).

When calculating the scattering length \( a_{2D} \) we use the result for the quasi-2D case which takes into account the confinement by the tight, \( z \), direction to the scattering. Recall that we had the result then (32), valid in the case of small \( E_B \):

\[
a_{2D} = l_z \sqrt{\frac{\pi}{B}} \exp \left(-\sqrt{\frac{\pi}{2} \frac{l_z}{a_{3D}}} \right) \tag{113}
\]

with \( B \approx 0.905 \).

The key elements of the algorithm for performing thermometry are (code avail-
able at https://github.com/Zholistic/Lithium6/blob/master/fitGGComp.m) as follows:

- From the absorption images of the cloud, that give a 2D density distribution, calculate azimuthal averages to obtain density vs potential curves using the local density approximation (LDA).

- Begin creating a new data structure with the experimental density vs potential, and the $a_{2D}$ of the cloud, in SI units.

- Determine which experimental points are good to fit; re-zero the experimental points if necessary. That is, make sure that the background level (the area of the curve where there are no atoms) has an average of zero, as offsets can cause problems.

- Input an initial $\beta E_B$ (so, temperature, as we know $E_B$ from $a_{2D}$) at which to try fitting the curve.

- Fit the data curve to the $f_{GGcomp}$ representation of the GG function, up to $\beta \mu$ of around 1, minimizing the two free variables: $(T, \mu_0)$. This involves using the transforms (110) and (111) detailed above.

- Evaluate the fitted $T$ to see if it matches the $T$ that was the initial guess in the $\beta E_B$. Iterate this procedure using different $\beta E_B$ until the fitted $T$ is the same as the initial $T$. Automation of this process uses a bisection algorithm, where the returned $\beta E_B$ contains a temperature that should converge to the $T$ that was the initial guess.

We can now find temperatures for a range of magnetic fields (interaction strengths). This method is fairly robust in that the points fitted can go up to around $\beta \mu$ of 1, which enables us to fit further into the cloud than just usual the virial approach, and to fit more accurately than using any non-interacting theory. We used this method as the background surrounding the cloud in some images was less-than-optimal, so a method where we can fit up into the more dense regions of the cloud is much preferred to work around any imaging artifacts/excess background noise that may exist. We note however that the accuracy of these fits may suffer due to systematic errors in the GG calculation. Agreement at high T is ensured by connecting to the virial result, but
Figure 4.4: Here is an example of the temperature fitting result, where we have on the y-axis the equation of state have used data at 725G and an average over 20 absorption images which are then radially averaged to produce the density profile. The green circles are the data points chosen to fit to the composite function (in brown), where the 2nd and 3rd order virial functions (dashed lines) are shown as well. The GG function is also show - but is identical to the composite function over this range. You can see in the negative $\mu$ or high T limit the virial functions overlap the composite function. Note that the temperature from the fit $T_{\text{fit}}$ is very close to the temperature to the initial guess - $T_{\beta_E\mu}$ - showing the converge via iteration over this value. This example also illustrates data that does not have the best background signal/noise, and this method of the composite GG function allows us to fit well up to around $\beta\mu$ of 0.7.
<table>
<thead>
<tr>
<th>Field (G)</th>
<th>$\log(k_Fa_{2D})$</th>
<th>$n_{pk}(\text{Atoms/m}^2)$</th>
<th>T (nK)</th>
<th>$T/T_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>860</td>
<td>2.36</td>
<td>6.683 e+11</td>
<td>26.1 ± 3.3</td>
<td>0.15 ± 0.02</td>
</tr>
<tr>
<td>725</td>
<td>2.27</td>
<td>5.418 e+11</td>
<td>27.7 ± 2.8</td>
<td>0.20 ± 0.02</td>
</tr>
<tr>
<td>725</td>
<td>2.195</td>
<td>5.002 e+11</td>
<td>23.7 ± 4.7</td>
<td>0.195 ± 0.03</td>
</tr>
<tr>
<td>700</td>
<td>1.41</td>
<td>8.338 e+11</td>
<td>30 ± 3.4</td>
<td>0.14 ± 0.01</td>
</tr>
</tbody>
</table>

Figure 4.5: Table reporting the results of the temperature fits for various magnetic fields for the $|1\rangle - |3\rangle$ mixture, where $|1\rangle = |F = 1/2, m_F = +1/2\rangle$ and $|3\rangle = |F = -1/2, m_F = -1\rangle$.

the GG at low T is model dependent and not exact[90][102]. We quote a 15% error on our fitted temperatures which includes a number of factors, including the uncertainty in our fitting function (standard deviation), noise in the experimental data and the observed differences between theory and experiment. The results are given in table (4.5).
4.2 Principal Component Analysis (PCA)

Extracting maximum information from the absorption images we take is of keen importance. A method which allows us to identify modes of greatest variance, from a time-series of images, is that of principal component analysis (or PCA)\[104][105][106][107]. One of a number of statistical analytic methods known as multivariate signal analysis methods - or more generally factor analysis - this model-free approach to extracting information is unique in its useful application to cold-atom research. In this section I will explain this method and how we applied it to identifying the collective oscillations in our experiment, along with some of the postives and negatives we found.

The method of PCA is as follows. To utilize, a time-varying series of images is required. From this a ‘mean’ image is computed, by averaging all the images, then subtracted from each. For \( N \) images each containing \( P \) pixels, the dataset can then be contained in one \( N \times P \) matrix, \( B \), where the index is such that \( B_{i,j} \) refers to the \( j \)th pixel of image \( i \). Now we wish to take \( B \) and compute a covariance matrix:

\[
S = B^T B / (N - 1)
\]  \hspace{1cm} (114)

where \( B^T \) is the transpose of the matrix \( B \) and this matrix \( S \) has at most \( N \) real eigenvalues \( \lambda \). Associated with each eigenvalue is of course an eigenvector, which are the principal components when the eigenvectors are normalised to unity. These may be computed by iterative, computational means, and in fact MATLAB has a built in function to generate these. Once reconstructed as images the principal components appear as in figure (4.6). They provide an orthonormal basis which spans the subspace of the dataset. This allows reconstruction of each original image by a weighted summation of a number of these basis ‘vectors’. For something like identifying the frequency of a collective mode within our gas, we are able to project the eigenvector associated with the collective mode (provided it actually exists) onto each image, calculating its contribution at each time step, then fitting to ascertain the frequency of the mode. An example of this is shown in figure (4.7).

We were able to use this method in our experiments to identify when certain collective modes were present and if there were multiple modes being excited by our excitation schemes. If you look at the figures below you can see that the set of images
I have chosen have breathing, dipole (corresponding to a center-of-mass or sloshing oscillation) and quadropole modes all visible. In a case like this during our experimental run, this would be a problem, and so we would have changed our excitation scheme (and in fact we did) to not excite disparate modes such as these all at once in the gas. Secondly, this method allowed us to extract both the breathing mode frequency and dipole frequency at the same time, from a single set of images. Since when calculating the later results for the quantum anomaly and other behaviours we express these as a ratio between the breathing and dipole modes, this enabled a robust method to extract both frequencies from the same images. The third advantage the PCA method gave us in our analysis was to see if the dipole mode was well-expressed or if there was any kind of sloshing about the trap going on - in a way that could indicate the trapping potential was misaligned. A clear and well-weighted dipole mode (and indeed breathing mode) shows that the laser TEM$_{01}$ mode dipole potential was aligned well to the radial bowl trapping potential from the magnetic field. This, combined with the other methods of trap calibration, were useful in determining whether our experimental setup was functioning correctly. Finally, when considering analytic techniques to parse time-varying data, an advantage of this method over a main competitor in Fourier decomposition is the time-steps that the images are taken at need not be even, fractional, intervals for the analysis to yield results. In truth this method can be done alongside a Fourier decomposition with only gain, as using the Fourier spectrum as a basis and projecting onto that - instead of in this case creating the basis out of the modes of greatest variance - only adds to the wealth of analysis tools at hand.

I will now speak briefly on things not so great about this method. In the end, we did not use this for our final results present in the later sections of this thesis, as the process we used to excite the collective modes turned out to be quite weak in effect - this being due to an experimental concern to not perturb the gas more than required, as to reduce unwanted heating and excessive oscillations - so this meant that the signal to noise of the collective mode was very weak when parsed by the PCA algorithm. Usually a breathing mode could be made out, but fits to that mode as projected onto the mean image were quite noisy and unreliable. Still, the method of PCA analysis allowed us great insight when calibrating the trap and optimising our measurements of dynamics within the cold atom gas, and for that it can be considered successful.
Figure 4.6: The result of the PCA algorithm for a set of 35 images, taken at 725G for a $|1\rangle, |3\rangle$ mixture and $N \approx 45K$ atoms. The first image is the ‘mean’ image which is the average of all the input images. The next three reconstructed images show eigenmodes of greatest variance, in this case a breathing mode, dipole mode and weak quadropole mode. The rest of the images show very weak to arbitrary noise signals which seem not to indicate any real systematic cause. Noise sources that could be indentified include imaging beam diffraction fringes, camera position (mechanical) variation and stray light which varies from shot to shot.
Figure 4.7: Here we see the first two principal components from the previous figure, when projected back on to the mean image. The top image shows the breathing mode oscillation, with the bottom showing a dipole mode. The right hand graph gives a measure of the proportion of this component contained in the mean image vs time (ms), which allows us to fit and gain the frequency of these modes.
5 The Dimensional Crossover

The study of dimensional crossovers - when, physically, we are in 3, quasi-2 or 2 dimensions, is of keen interest both theoretically[108][109] and experimentally[23][20][36][19]. As we have seen in (2.2), the scattering physics of the gas depends critically on dimension, giving rise to a persistent bound state in 2D for arbitrarily small attractive interactions and changing the mechanism of superfluidity[110][111]. These physical outcomes, being dependent on the dimensionality of the system, lead naturally to questions about such dimensionality - is the boundary between being physically in one dimension discrete? Is there a crossover region, and if so, how does it behave? What parameters does dimensionality depend on more, or less, strongly? In our study of interacting Fermi gases we attempted to experimentally answer questions such as these, using the method of collective oscillations as a dynamical probe of the system. In this chapter I present details of the experimental method we followed, with the results and insights we gained into the dimensional crossover by studying the collective oscillations of the gas. This comes after a theoretical discussion of the crossover and of a previous experiment of ours that I was involved in[36], which looked at how the widths of the trap changed with both atom number and scattering length: a method by which we were able to find the parameters of our experiment within which we are kinematically 2D.

5.1 2D Criteria

To gain insight into what it means to be in 2D or not, it is fundamental that we discuss what it means, physically, to be in two dimensions. From an ideal perspective (neglecting interactions) a spin-balanced system of fermions occupies states in the phase space of a harmonic oscillator, say with potential:

\[ V(x, y, z) = \frac{1}{2} m (\omega_r^2 x^2 + \omega_r^2 y^2 + \omega_z^2 z^2) \]  

where the \( \omega_r \) is the trapping potential in the radial direction, and \( \omega_z \) in the transverse (tight) direction, with \( m \) the mass of the atoms and \( \omega_z \gg \omega_r \). For it to be two dimensional, we can specify that this occupation, nor any energies associated with the system, occupy levels higher than the ground state in the transverse direction. So the
fermions are free to fill up the states of a harmonic oscillator such that the energy level of the highest occupied state in the radial dimensions does not exceed the first excited state of the harmonic oscillator potential in the transverse direction. This gives the criterion that $\hbar \omega_z \gg \hbar \omega_r$, or $E_F, k_B T \ll \hbar \omega_z$ where $\omega_z$ is the transverse trapping frequency. The second condition here incorporates the situation at $T > 0$, where to be within the 2D criterion all the energies associated with the system must be lower than the first excited state in that transverse direction (see Figure 5.1).

Where the ideal case is considered, we have a finite number of fermions that can fill these radial harmonic oscillator states before the next lowest-lying energy state would be the excited state in the transverse direction. Thus we have a limit on the number of atoms $N$ that can exist in our gas and still be in 2D. This counting limit is known as the ideal 2D number or $N_{2D}$ and can be found if one knows the ratio between the tight (transverse) and weak (radial) trapping potentials, where we express that ratio is as $\lambda = \omega_z/\omega_r$[20]:

$$N_{2D} = \frac{\lambda}{2}(\lambda + 1)$$

(116)

For typical values of our trap, given a tight trapping frequency $\omega_z \approx 5.6$ kHz and a
radial trapping frequency $\omega_r \approx 25$ Hz this gives a ratio $\lambda \approx 220$. From this we calculate $N_{2D} \approx 24000$ atoms per spin state.

This ideal criterion gives a clear picture for an ideal gas, but what happens when we introduce interactions and finite temperatures into the dimensionality? This question is one we explored in the form of an experiment[36], the results of which are displayed in figure (5.2). The investigation we performed looked at what happens to the transverse width of the gas as the atom number is varied from below to above the ideal criterion number, $N_{2D}$, as well as what happens when we vary the interactions by tuning the Feshbach field from the resonance to the far-BCS side of the resonance. Tuning to the far-BCS side of the resonance has two consequences. First, the (3D) scattering length becomes small compared to the harmonic oscillator length in the transverse direction $a < l_t$, where $l_t = \sqrt{\hbar/(m\omega_z)}$. Secondly the ratio of the binding energy of the bound state relative to the Fermi energy becomes very small $E_b/E_F \approx 0.001$. Varying the atom number to above the ideal criterion $N_{2D}$ has the effect of populating higher harmonic oscillator states in the transverse direction ($\hbar\omega_z$), which should show up as we look at the transverse width of the cloud. At this point we recover the behaviour expected for an ideal gas.

5.2 Transverse width experiment

The experiment we performed previously[36] to look at these effects applied the following method. We first prepared the gas by evaporatively cooling the two lowest hyperfine states of $^6$Li, $|F = 1/2, m_F = \pm 1/2\rangle$, then transferred the atoms by slowly ramping up the 2D TEM$_{01}$ mode trap. Once in the 2D trap the atoms were of temperature around $T/T_F \approx 0.1$. To fine-tune the atom number we applied a magnetic field gradient along the $z$-axis of the trap (the vertical, transverse direction) which spilled atoms and allowed us to reach a desired atom number. Once we had a desired atom number we used absorption imaging to image the cloud from the side. Since the thickness of the atom cloud from the side is below the diffraction limit, we imaged the atoms by switching off the TEM$_{01}$ trap and applying a 10 $\mu$s imaging pulse after a time-of-flight (TOF) of 600 $\mu$s.
Figure 5.2: The results from our analysis[36]. On top the three plots display graphs of the transverse width vs atom number for three different interaction strengths. In those plots, (b) and (c) show an elbow point where the trend goes from no increase in transverse width to a linear increase with atom number - this being indicative of a change from below 2D to quasi-2D dimensional behaviour. The bottom surface plot is of atom number on the vertical axis (represented by $E_F/\hbar \omega_z$), with interaction strength the horizontal axis (represented by the harmonic oscillator length divided by the 3D scattering length $l_z/a$). The colour is then the transverse width. The elbow points of the top graphs are plotted as the large grey dots in the bottom surface, showing a trend toward the ideal criterion of $\frac{E_F}{\hbar \omega_z}$ as the interaction strength shifts away from the Feshbach resonance on the BCS side.
Results from this experiment can be seen in figure (5.2). When the atom number is less than $N_{2D}$, and the interactions due to the Feshbach resonance are far from resonance (small) on the BCS side, the transverse width stays constant as it is raised to the level of the critical atom number. After this point the width increases approximately linearly with atom number, indicating that the 2D region has been left. If we refer to this point as the elbow, we can notice another interesting effect occurs as we approach the Feshbach resonance. As this happens the elbow moves to an atom number which is less than the ideal criterion. We see from the surface that at the $|F = 1/2, m_F = \pm 1/2 \rangle$ resonance, where $l_z/a_{3D} \to 0$, the elbow is below our atom number measurement (expressed here as the ratio of the Fermi energy to the energy of the first excited state in the transverse direction $\frac{E_F}{\hbar \omega_x}$). It is in the region, at atom numbers below the elbow point, that we can be sure we are in the 2D regime when interactions are taken into account. A recent theoretical treatment of this result can be found[112], and further discussion of the experimental procedure and result[99].

To gain further insight into this crossover phenomenon, we would like to conduct an experiment that probes directly what happens in the trap, rather than after expansions. This is one reason we looked to collective oscillation experiments as a new probe of Fermi gas system.
5.3 Collective oscillation experiment

The transverse width result sets the stage for further analysis of this crossover region to be performed by a different method, that of collective oscillations. As we have seen in 2.3.1, the polytropic equation of state can be tied to the breathing mode oscillation frequency via the polytropic coefficient[37]. This coefficient is sensitive to the dimensionality of the gas, and therefore if there is a change in the thermodynamic equation of state, we should be able to see it reflected in the breathing mode frequency. In this section I will detail the experimental method by which we examine the breathing mode frequency of our Fermi gas.

5.3.1 Experimental method: Breathing mode

This section describes the experimental method which produced our breathing oscillation results, a diagrammatic summary of which is presented in figure (5.3) below. Detail will be focused on the steps which deviate from the general experimental description explained in chapter 3.

We begin with the compressed magneto-optical trap (MOT) containing a mixture of $|\uparrow\rangle - |\downarrow\rangle$ spin states of $^6$Li. The next step, as outlined in the cooling and trapping chapter, is to transfer these atoms to the evaporative cooling trap and to proceed with the evaporation cooling process. This evaporation process spills atoms and allows us to control our final atom number: at the start of the process the trap contains around $3 \times 10^5$ atoms and at the end (after transfer to the 2D trap) we have from 6000 to 80000 atoms per spin state. As the evaporative cooling process is starting we transfer the atoms which are in state $|\downarrow\rangle$ to state $|\uparrow\rangle$ via application of a radio-frequency pulse, which gives us at the end of the evaporative cooling stage an even mixture of states $|\uparrow\rangle - |\downarrow\rangle$ (where the spin quantum numbers are $|\uparrow\rangle = |F = 1/2, m_F = +1/2\rangle$ and $|\downarrow\rangle = |F = -1/2, m_F = -1\rangle$) which have been cooled to a temperature around 0.15 $T/T_F$.

After we have the evaporatively cooled cloud of atoms we transfer them to the 2D trapping potential. We do this by turning on the blue-detuned 532 nm trapping laser TEM$_{01}$ mode to a small fraction of the total power (approx 8%), ramping off the evaporation trap in 150ms. We hold the atoms in this weak blue-detuned potential,
trapping many of the atoms that were left in the evaporative trap, before ramping up the power on the TEM$_{01}$ mode laser to full intensity in around 750ms. This slow ramp allows time for any sloshing modes in the trap to damp out. We take care to align the centre of the two trapping potentials over one another, which results in the maximum atoms transferred and minimizes any movements the cloud must make. In our 2D trap we trap anywhere between $5 \times 10^3$ and $80 \times 10^3$ atoms per spin state, chosen by the final intensity of the evaporation laser. The upper limit here ($80 \times 10^3$) is limited by the mismatched geometries of the two trapping potentials, and - compared to the 2D ideal criterion number - this atom number is far outside the 2D and into the quasi-2D regime. At this point in the process we now have a 2D trap of Fermi atoms, which contains a 50-50 mixture of the $|1\rangle - |3\rangle$ hyperfine states, trapped by the blue-detuned TEM$_{01}$ mode in the transverse direction and the bowl of the magnetic Feshbach field in the radial direction.
We induce the breathing mode oscillations in the gas by modulating the Feshbach magnetic field, which is also responsible for the radial trapping of the atoms as well as the interaction strength between their hyperfine states. This modulation, then, sweeps across a range of interaction strengths. Other experiments which induce such excitations in cold gases have used alternative methods, such as extra applied laser beams [46], or varying the intensity and angle of the primary trapping beams [113]. We tested similar methods - including varying the intensity of the trapping laser beam - but found them to be less controllable (compared to the control we have over modulating our weak trapping potential), and they tended to not discriminate between the modes excited. For our purposes the breathing mode was of interest, and inducing other modes in the gas simultaneously may have impacted on the results of the singular breathing mode measurements (especially in the hydrodynamic regimes), as well as it being prudent to not heat the cloud more than necessary. Dynamics excited in the non-2D plane could have also presented a problem. We tested to see if our method of using the radial trapping frequency resulted in raised temperature, but could detect very little heating, if any (there was no discernable difference in the temperatures), which we examined by comparing the density profiles of a cloud prepared in the same way both prior and after inducing the breathing mode, looking at the median image of the oscillating cloud.

Returning to the experiment, our chosen method of modulating the amplitude of the radial confining magnetic field shows why we choose the $|1\rangle - |3\rangle$ mixture over the usual $|1\rangle - |2\rangle$ mixture we have used in other experiments - because we have a smaller width of the Feshbach resonance (for the $|1\rangle - |3\rangle$ versus $|1\rangle - |2\rangle$, see section 2.2.2), the same Feshbach field modulation allows a sweep across a correspondingly larger range of interaction strengths - increasing the effectiveness of the stimulation method. We calibrated this method to provide the breathing mode oscillation but without higher-order oscillations (such as quadropole). The closer the test field was to the Feshbach resonance the greater the interaction sweep, and the less applied voltage variation required to stimulate the oscillation above a discernible signal-to-noise threshold. This is because the change in interaction strength also seems to couple to the width of the induced oscillation. The frequency of the magnetic modulation was matched to the expected frequency of the oscillation, which we found gave the clearest response for the breathing mode excitation. If we offset the frequency of the modulation - say we
modulated the magnetic field at 44 Hz when we expected a breathing mode response of 46 Hz (we find the system to oscillate into its natural frequency and is insensitive to small changes in the driving frequency) - we would see the same frequency of the resultant breathing mode (46 Hz), but with a reduced breathing mode signal. This was an important test to also show that we weren’t just getting any arbitrary response back - the cloud had its own natural breathing mode frequency which would respond with a larger signal when we applied an excitation equal to that frequency. We used a signal generator connected to the Feshbach field circuit to apply the magnetic field modulation, which was a generated sine wave with a triangular envelope, with maximum voltage difference corresponding to a peak to peak of ~ 1% of the overall trapping field. Typical ranges of the voltage were from 100-800 mV, which we varied to gain a clear signal of the oscillation. The modulation was applied for 5 cycles ($t_{osc}$) of the breathing mode frequency, after which the system was allowed to oscillate by itself for a variable hold time ($t_{hold}$). The short envelope of modulation translates to a broad Fourier width. After this hold time we took an in situ absorption image of the cloud from above.

After imaging we gain results in the form displayed in figure (5.4). For a particular magnetic field (B) and atom number (N) we take three images per $t_{hold}$, from 0 to 51 ms hold times, and then again from 175 to 226 ms. For each hold time, we azimuthally average the cloud (radially averaging about the centre of the cloud, taking advantage of its circular symmetry), producing a density distribution to which we fit a gaussian curve. We then find the radius of the cloud from the gaussian fit by taking the $1/e^2$ length. This radius varies as the cloud ‘breathes’ in the breathing mode oscillation. Plotting the radius versus the hold times, we average the three images at that magnetic field, and fit an exponentially damped sine wave to the resulting data points (again, see figure 5.4) . This allows us to extract the breathing mode frequency - from the sine wave fit - at each magnetic field. One cycle of the experiment takes around a minute, so producing the breathing mode frequency for a particular magnetic field and atom number took around 5 to 6 hours in optimal conditions. Some issues we ran into were: magnetic fields further from the $|1\rangle - |3\rangle$ resonance of 690 G had to be driven harder, as their interactions were less strong, and they produced weaker oscillations. It was also difficult to produce experimental data above an acceptable signal-to-noise ratio when the atom number fell below around 6000.
Figure 5.4: Result from a breathing mode oscillation measurement at a magnetic field of 700 G, with a cloud of atom number $N \approx 10 \times 10^3$ per spin state. Top left is the absorption image of the cloud from above, where the pixel scale has conversion $1\text{px} = 2.84 \mu m$. The false colour scale is the column density measured in atoms/pixel area, where the raw data has had an optical density correction of 1.28 applied (see 3.4). Top right is the same cloud after radial averaging about the centre, with a gaussian fit $f_{\text{gauss,fit}}(x) = ae^{-x^2/b^2}$, where $(a, b)$ are fit variables, in green. We take the $1/e^2$ radius of the fit, which is equal to the fit variable $b$. The bottom is the breathing mode data for this particular choice of magnetic field and atom number, where each data point is the average of three hold times $t_{\text{hold}}$. The fit function is an exponentially damped sine wave $f_{\text{sine,fit}}(x) = ae^{-bx}\sin(cx + d) + e$ which gives the breathing mode frequency $\omega_B$. In this case $\omega_B = 2\pi \times 44.8(44.63, 44.98)$, where the error values in brackets are determined from the fitting function and represent the 95% confidence interval. The error on each data point is the standard deviation from summing over the three radius measurements taken at each $t_{\text{hold}}$. 

88
5.3.2 Experimental method: Dipole mode

As the results given by [37] are in terms of the proportion of the breathing mode frequency divided by the radial trapping frequency ($\omega_r$), it was of importance to measure this frequency accurately. Since our gas is trapped radially by the residual curvature of the Feshbach magnetic field, generally there will be a single trapping frequency for each field (B). Our approach was to measure this trapping frequency at many Feshbach magnetic fields and fit a function which we could then extract the radial trapping frequency at any field. This approach was successful for the dimensional crossover experiment we discuss in this chapter, but a different approach was needed for the quantum anomaly results, since they were even more sensitive to the value of this radial trapping frequency. That is discussed in the following chapter. Here the basic idea is the same as for measuring the frequency of the breathing mode, except instead of the breathing mode we induce a dipole mode oscillation in the gas - where the centre of mass of the atom cloud varies with time - and fit to this a function from which we can find the frequency $\omega_r$.

Experimentally we evaporatively cool a cloud of atoms as with the breathing mode, ending up with a cloud containing an even mixture of the $|1\rangle - |3\rangle$ spin states of $^6$Li. To ensure a good signal-to-noise in our images we prepare this cloud with around $15 \times 10^3$ atoms per spin state. To create the dipole mode we slowly ramp on a small magnetic field gradient (using the magnetic field coils for the MOT) for 100 ms, which has the effect of displacing the magnetic centre that the high-field seeking atoms were radially trapped in. Turning this gradient field off suddenly we then let the atoms in the cloud oscillate about their undisturbed 2D magnetic field trap centre, taking images at various hold times $t_{\text{hold}}$.

Now that we have the process for a particular image, we vary $t_{\text{hold}}$ to build up a picture of the oscillation over successive shots. We took two images per hold time, varying the hold time from 0 to 84 ms in 6 ms steps, and again from 200 to 260 ms in 6 ms steps. For particularly clean oscillations we would also take data out at 800 ms, to improve the accuracy of our fit functions. Figure 5.5 shows the results of the dipole
Figure 5.5: Variation of the centre of mass of the atom cloud versus time for 1D profiles X and Y, where the profiles were created by averaging the cloud over the X and Y directions. This data was for a magnetic field of 700 G and atom number of $N \approx 17000$. Each point is an average of two images at that hold time, where the error is the standard deviation of the average of the center pixel of those two images. For the top camera 1 px = 2.84 $\mu m$. The fit function used here is an exponentially damped sine wave, $f_{sine, fit}(x) = ae^{-bx} \sin(cx + d) + e$, with the letters (a-e) being the fit variables. We see that the top image for Y has frequency $\omega_y = 2\pi \times 21.59 (21.41, 21.76)$ where the error is the 95% confidence interval on the fit variable. For X we have $\omega_x = 2\pi \times 22.39 (22.29, 22.49)$. This gives us the radial trapping frequency for this magnetic field, which is the geometric mean of the X and Y, $\omega_r = \sqrt{\omega_x \omega_y}$, here $\omega_r = 21.99 (21.85, 22.12)$. 
oscillation. Because we are tracking the centre point of our cloud for each image. To do this we would average over both dimensions of the top-down image, resulting in two one-dimensional density distributions of our data. These are the X and Y in the figure. For each density distribution we fitted a gaussian function and found the position of the peak of this fit, in both the X and Y directions. We then had a coordinate of the centre of mass \((x, y)\). Plotting this versus the time step gave us a distribution which we could fit an exponentially damped sine wave to (similar to the breathing mode), from which we extracted the frequency of the dipole mode in both directions, \(\omega_x\) and \(\omega_y\). Taking the geometric mean of these two frequencies \((\omega_r^2 = \omega_x \omega_y)\) gives us the frequency of the dipole oscillation. This is the frequency we were after, as it is equal to our radial trapping frequency \(\omega_r\).

Once we were able to extract the radial frequency for a particular magnetic field, we built up many measurements at the same and different fields. Our results are displayed in figure 5.6. We found a small variation in our measurements of the radial frequency from day to day, and again from week to week. It appeared that variations in the alignment of the anti-trapping from the laser in the 2D trap could affect the measurement of the radial frequency \(\omega_r\). To overcome this we took many measurements, and then fitted a function \(f(B) = \alpha \sqrt{B} + \beta\) to this average. From this we could recover a radial frequency for any of the fields we required, but with a relatively large error bar. We found this to be fine for the crossover results, but as will be explained in the next chapter, extra steps were needed to ensure the accuracy of the radial frequency when probing the quantum anomaly.

This completes the explanation of the experimental methods associated with creating and exploring the breathing and dipole collective oscillations within our Fermi gas, for looking at the dimensional crossover. In the next section we discuss the results we obtained and compare them to the predictions of the theory.
Figure 5.6: Results for the geometric mean radial frequencies $\omega_r$ for various magnetic fields. The blue points are $\omega_r$ taken on different days, their scatter we attribute to different lab conditions which could affect the anti-trapping caused by slight misalignments of the 2D laser trap, affecting the measurement of the dipole oscillation. We took many points and then fit the function $f(B) = \alpha \sqrt{B} + \beta$ to the averages at each magnetic field (represented by the green crosses).
5.3.3 Results discussion

Performing the breathing mode oscillation procedure for varying atom numbers (both above and below the ideal 2D criterion $N_{2D}$) and for varying interaction strengths we were able to build up a picture of what happens in various regimes. Our results for these are presented in the graph below, figure (5.7), and compared with the predictions of the polytropic theory outlined in 2.3.1. These results were obtained at temperatures around $T/T_F \approx 0.15$. The results presented are for four different interaction strengths, quantified by $l_z/a_{3D}$, where $l_z = \sqrt{\hbar/m\omega_z}$ is the harmonic oscillator length in the transverse direction, and $a_{3D}$ the 3D scattering length. The Feshbach resonance occurs at $l_z/a_{3D} = 0$.

Our results show a well clustered region at around $2\omega_r$ for atom numbers far below the ideal criterion of $N_{2D}$, which would agree with the table below for results from Stringari et al [37] (table 5.8) which predict two times the radial trapping frequency for both the $T=0$ and high temperature 2D regime (notation: the $\omega$ of the table is the same as the radial trapping frequency $\omega_r$). That the data clusters a little above the $2\omega_r$ for very small atom numbers is a consequence (and evidence) of the quantum anomaly and will be gone into detail in the next chapter. Here we are interested what happens as we leave the 2D regime by increasing the atom number above the 2D criteria, and we see for interactions around the Feshbach resonance that this deviation comes faster for the strongly interaction gases than for the far-BCS regime (represented by the $l_z/a_{3D} = -2.12$ dataset). The datasets appear to converge at around the $\sqrt{3}\omega_r$ value for the ratio of the breathing to radial trapping frequencies, which appears to agree well with the prediction for the ‘Unitary Pancake’ regime. This is a quasi-2D regime where the local density approximation still holds in all three dimensions, allowing for the expression of the equation of state using the quasi-2D formalism [37]. Where they don’t converge at the $\sqrt{3}\omega_r$ value, they seem to trend toward the $\sqrt{10/3}\omega_r$ value, this being the predicted value for a ‘pancake’ thermodynamically 3D Bose gas. This is to be expected as the further we go on the negative side of the resonance the more bosonic the gas would tend to behave. It should be noted this is the first time (note: actually there was another experiment by the group of Selim Jochim et al at the same time as we performed these experiments, which has similar results and can be found [114]) - the $\sqrt{3}\omega_r$ value has been shown experimentally for a breathing mode in a
Figure 5.7: Results for the breathing mode oscillation across the dimensional boundary, quantified by the atom number relative to the 2D ideal criterion atom number, $N/N_{2D}^{id}$. The vertical axis is the breathing mode frequency proportional to the radial trapping frequency, $\omega_B/\omega_r$. The four different interactions we explored are the different colours, with one on the near-BEC side of the resonance ($l_z/a_{3D} = 0.53$ red), which shows a similar trend to that of the resonance ($l_z/a_{3D} = 0$ green) and the near-BCS side of the resonance ($l_z/a_{3D} = -0.48$ blue). The $\sqrt{10/3}\omega_r$ result is the breathing mode frequency we would expect for a ‘pancake’ quasi-2D Bose gas at $T = 0$. As we travel further from the resonance on the BCS side ($l_z/a_{3D} = -2.12$ purple) we see the cloud remain closer to the expected $2\omega_r$ of the 2D breathing mode behaviour for atom numbers $N/N_{2D}^{id} \leq 1$ relative to the 2D criterion, which still appears to converge to the $\sqrt{3}\omega_r$ of the quasi-2D regime as the 2D atom number criterion is exceeded.
Figure 5.8: Table of breathing mode frequencies and polytropic coefficients as in [37]

<table>
<thead>
<tr>
<th>Regime</th>
<th>Polytropic Coefficient</th>
<th>Breathing Mode Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unitary 3D</td>
<td>5/3</td>
<td>$2\omega_{ho}$</td>
</tr>
<tr>
<td>Unitary Pancake (LDA)</td>
<td>3/2</td>
<td>$\sqrt{3}\omega_{\perp}$</td>
</tr>
<tr>
<td>2D Mean Field (T=0)</td>
<td>2</td>
<td>$2\omega_{\perp}$</td>
</tr>
<tr>
<td>2D Mean Field (high T)</td>
<td>2</td>
<td>$2\omega_{\perp}$</td>
</tr>
</tbody>
</table>

Looking again at the results, we note the trend of the transition as we go from 2D to 3D. The strongly interacting data ($l_z/a_{3D} < -1$) cluster around the same, faster transition than that of the more weakly interacting result ($l_z/a_{3D} = -2.12$). The indicates the transition from 2D to 3D seems to be affected by the atom number more in the case of strong interactions. In fact for the weaker interactions we see they change from the 2D to 3D regime at a much higher N than the stronger interactions, suggesting that the interactions are again important in determining the dimensional crossover.

Finally we note that these results are in essence directly probing the thermodynamics of the system via the breathing mode, which is perhaps why we get results that agree so well with the polytropic coefficients associated with the equation of state. We conclude this chapter having shown collective oscillations to be a powerful probe of the gas which allows us to map the dimensional crossover as we had done in our previous experiments looking at the transverse widths of the cloud, but now we are interrogating the gas directly with in situ imaging, rather than using time-of-flight imaging where we have to account for the momentum distribution of the atoms to properly interrogate the images.
5.4 The Quantum Anomaly

We now return to the discussion of the quantum anomaly, with respect to our experimentation on the breathing mode in the 2D Fermi gas. As discussed in section 2.4, the quantum anomaly is the result of anomalous symmetry breaking, which occurs in our 2D Fermi system due to the necessity of renormalizing the interaction to introduce a missing length scale[40], which breaks the SO(2,1) symmetry otherwise present. This leads to a theoretical deviation from the $2\omega_0$ breathing mode frequency expected in a non-symmetry-broken system [43][44][62], where the $\omega_0$ refers to the dipole mode frequency and is equal to the radial breathing mode frequency in our trap $\omega_r$. In this section I will begin with a description of the experimental procedure we undertook to measure accurately the breathing mode and dipole mode oscillations in the Fermi gas, followed by a presentation of the results and discussion of where these results sit with respect to previous experimentation[46] and current theoretical understandings[43][44].

5.4.1 Experimental method

To investigate the quantum anomaly, we created and examined breathing mode oscillations at various magnetic fields as in the crossover experiment detailed in 5.3.1, but now focussing on low atom numbers (N) for the 2D regime[115]. We also probed the dipole oscillations of the gas in a similar manner to the crossover experiment 5.3.2 but with a few changes. Since we knew that our dipole mode frequency measurements - the measurements of the radial trapping frequency $\omega_r$ - could be prone to variation based on the lab conditions (humidity, temperature of the lab, slight week to week differences in the optics), we sought to remedy any issues that could arise due to this by taking both the breathing and dipole mode frequency measurements simultaneously. This meant for a particular magnetic field and atom number, we could use both measurements (of $\omega_B$ and $\omega_r$) and have confidence they were taken under the same conditions. The scheme we employed was to take a run of breathing mode values - one set of data points at hold times every 3ms from 0 to 51 ms and then from 175 to 226 ms - and then to immediately take a set of dipole values, then again a set of breathing, etc. We ended up with a scheme:

$$B \rightarrow D \rightarrow B \rightarrow D \rightarrow B$$

(117)
where the $B$ represents a set of breathing mode data and the $D$ a set of dipole mode data. We then analyzed them to extract their frequencies, effectively monitoring the radial mode frequency $\omega_r$ at the same time we measured the breathing mode frequencies $\omega_B$. This was an intensive measurement, with a set of data taken over a continuous 14 hour session.

5.4.2 Results discussion

Our results for detection of the quantum anomaly are displayed in figure 5.9 and discussed in[115]. The data was taken at a temperature of around $T/T_F = 0.15$ and atom number $N = 0.3N_{2D}^{ld}$, which is 30\% of the critical atom number to be in the 2D regime for an ideal gas. We see a distinct deviation from the $2\omega_r$ frequency that would be expected if the anomaly wasn’t present. For $-1 < \ln(k_Fa_{2D}) < +3$, across the Feshbach resonance of our mixture, we saw an approximate 2.5\% deviation. This is lower than the theoretical predictions of up to around 10\% deviation. This can be explained by the finite temperature of our gas but more importantly that even at $N = 0.3N_{2D}^{ld}$, due to interactions we are not strictly in the 2D regime, as shown by our ‘elbow’ experiment to determine the 2D regime (see figure 5.2). At weak interactions (as $\ln(k_Fa_{2D}) \rightarrow 6$) we see a return to the $2\omega_r$ frequency as expected. So both finite temperature effects and the trapping potential in the transverse region not being strong enough to take us to the deep 2D region may explain our deviation from the theoretical curves. Given this, we consider the signature of the quantum anomaly to be clear enough to show the phenomena to be a real effect.

We can also compare these results to proposed experiments in a 2D Bose gas by Olshanii and Perrin[116], where they expect a much lower value for the frequency shift, on the order of at most 1\%. This would be a consideration for Fermi gases on the BEC side of the Feshbach resonances, where due to pairing they can behave like a gas of bosonic dimers. Our data was however taken on the BCS side of the resonance.

A previous exploration of breathing modes was performed by Vogt[46] (data points of which are shown in the left panel in figure 5.10), an experiment which paved the way for and motivated our exploration. Their results however do not see significant shift from the $2\omega_r$ value for the breathing mode frequency (see the data points in figure 5.10). We believe the reason for this is due to the higher temperature of their Fermi gas, at around $T/T_F = 0.37$. This is similar to the reason we believe our results to not
Figure 5.9: The results of our breathing mode oscillation frequency divided by the radial trapping frequency $\omega_B/\omega_r$ plotted versus interaction strength $\log(k_F a_{2D})$. We see a deviation of around 2.5% across the strong interacting regime up to around $\log(k_F a_{2D}) = 3$, where we see a return to the 2D breathing mode frequency of twice the radial trapping frequency. This data was taken at temperature of around $T/T_F = 0.15$, and at an atom number $N = 0.3N_{2D}^{1D}$, well below the 2D ideal atom number criteria. The error on the points is due to the error in the frequencies for the breathing and radial modes, which stems from the uncertainty in the fitting functions.

lie exactly on the theoretical curves by Hoffmann[44] and Gao[43] (see figure 5.10). Our finite temperature of the gas (our $T/T_F$ lies in the range 0.14 to 0.22) and the strong interactions may take us outside a deeply 2D regime, giving an explanation for our experimental deviation.

A review of the results presented in this chapter are available in a paper published in Physical Review Letters[115].
Figure 5.10: Theory curves from the literature. In the left panel is a graph from the review article by Levinsen and Parish [49] which displays the theory curve from Hofmann [44] in blue. The vertical axis on the left is the frequency shift $\delta \omega$ from the breathing mode $\omega_B$, divided by the dipole mode frequency $\omega_0$. Both graphs plot versus the interaction strength in terms of $\ln(k_F a_{2D})$. The data points in the left panel are the results from the experiment of Vogt [46], which deviate from both the theory and our results for values of $\ln(k_F a_{2D}) < 3$. This is likely due to the temperature they were taken at, which was around $T/T_F = 0.37$, contrasted with our results at $T/T_F = 0.15$. The right hand graph is the theory from the work of Gao and Yu [43], where the vertical axis is the same as the left hand graph but for a factor of two. A combined graph is presented in figure 5.11.
Figure 5.11: Plot of our data points (in red) for the anomaly overlaid on the theory curves from the literature. They include the Hofmann results in blue[44], Gao & Yu as the purple points[43] and the results of Mulkerin et al in green[117]. This is the plot presented in our paper[115] and displays the deviation of the breathing mode frequency $\delta \omega$ from the dipole mode $\omega_r$. The scale invariant prediction of $2\omega_r$ is the dashed black line. Again the $x$-axis is the interaction strength quantified by $\ln(k_F^{\text{H}} a_{2D})$. 
6 Conclusion & Outlook

The use of collective oscillations as a probe to explore 2D Fermi gases is a powerful method, which helped us to illuminate the behaviour of the gas over the dimensional crossover from 2D to quasi-2D, at varying interaction strengths, and allowed us to probe the phenomenon of the quantum anomaly.

For the crossover from 2D to quasi-2D, we saw that predictions of the breathing mode from the polytropic equation of state supported our results in the limiting cases - that of the breathing mode frequency being twice the radial trapping frequency ($\omega_B = 2\omega_r$) for atom numbers well below the ideal atom number criterion for the 2D regime $N \ll N_{2D}^{Id}$. As we increased the atom number, reaching the point where $N = N_{2D}^{Id}$, we saw for strong interactions the breathing mode frequency greatly deviate from the 2D frequency, but for weak interactions the breathing mode frequency persisted at the expected 2D breathing mode frequency for these higher atoms numbers. For all interaction strengths we saw that as the atom number went higher - $N > N_{2D}^{Id}$ - that the quasi-2D prediction of $\omega_B = \sqrt{3}\omega_r$ was the point where the data was converging to, showing good evidence that this polytropic approach to the equation of state gives an accurate prediction of the breathing mode frequency of the gas, and that - vice-versa - the breathing mode frequency can be used as a determination of the dimensional regime of the gas. This is to our knowledge the first time the $\sqrt{3}\omega_r$ result has been shown experimentally for a 2D Fermi gas. Looking to the future, a further study in this vein could include a look at varying temperature as well as interaction strength and atom number, to build up a surface of the dimensional crossover relative to those variables.

The anomalous breaking of the SO(2,1) symmetry in a 2D Fermi gas, due to the renormalization procedure of the interaction Hamiltonian (which necessarily introduces an extra length scale), also known as the quantum anomaly, was shown to be a real phenomenon through the increase of the breathing mode frequency of our 2D Fermi gas from the expected value of $2\omega_r$ in the non-symmetry-breaking case. We showed a clear signature of this effect, but due to the finite temperature, we did not see the magnitude of the anomaly as high as expected in T=0 theoretical predictions. Future outlook in this area includes a possible confirmation of the quantum anomaly.
in high-energy physics, which would be an exciting confirmation of this anomalous symmetry breaking effect in two very distinct experimental disciplines, along with further confirmation of the effect at lower temperatures.

This ends the main section of the thesis, and I would thank you for reading! As well as again to thank the team I worked with to achieve these results - in particular Chris Vale, Paul Dyke, Sascha Hoinka and Peter Hannaford - the colleagues I worked with for the vast bulk of these results. More acknowledgments may be found at the beginning of this thesis.
A Nano-Fabrication of Phase Plate

The π-phase plate is an optical component critical to the formation of our 2D trap. It provides the interference pattern responsible for the creation of the TEM$_{01}$ mode potential, trapping the gas in the tight direction. Throughout the course of our experiments we were making use of the optical piece that had been in use for many years, and was showing wear. Any kind of wear such as tiny scratches or imperfections in this phase plate can cause irregularities in the trapping potential, which is not desirable for many reasons, including potential heating on transfer between traps, imaging imperfections and of course the further we stray from a harmonic trapping potential the further we are from many theoretical formulations of the gas - it introduces a systematic error.

The phase plate works by imparting a half-wavelength ($\pi$) phase shift in half of the incoming gaussian intensity distribution. When focused down to a waist, this results in an intensity distribution with two lobes like that presented in section 3.3. To create this effect we use a single glass optic made from high quality quartz glass (fused silica). The phase shift is due to the refractive index difference the top half of the beam sees as it passes through the glass optic and is given by the equation:

$$\frac{\lambda}{2} = x\Delta_n$$ (118)

<table>
<thead>
<tr>
<th>Refractive indexes n at 532nm [118]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{\text{air}}$</td>
</tr>
<tr>
<td>$n_{\text{MgF}_2}$</td>
</tr>
<tr>
<td>$n_{\text{FusedSilica}}$</td>
</tr>
</tbody>
</table>

where in the above equation $\Delta_n$ is the refractive index difference between the two materials ($n_2 - n_1$), and $x$ is the distance traveled by the light - which corresponds to the thickness we require the materials. The pre-existing optic uses a magnesium fluoride (MgF$_2$) coating on one half of the optic, with air on the other half, which from the equation gives a required thickness of $x_{\text{MgF}_2} = 702$nm for the MgF$_2$.

Magnesium fluoride was chosen as a compound because it is very transparent to
Figure A.1: Phase plate schematic. The top diagram shows the optic with magnesium fluoride deposited on the surface, which should be of thickness $x_{\text{MgF}_2} = 702\text{nm}$ to create the $\pi$-phase shift. The bottom is the method to etch away the fused silica quartz flat. For the phase shift this would be of thickness $x_{\text{air}} = 577.4\text{nm}$.

The $532\text{nm}$ light that we use, so the high power of our continuous wave laser (up to $12\text{W}$) would not produce much heat to distort the optic nor would absorb much light and so be a source of loss. The problem with MgF$_2$ is that it is a very soft dielectric compound, and so cleaning the optic with compounds such as acetone or isopropyl alcohol couldn’t be done. This, as well as it being very easy to scratch, lead us to consider other options. An idea that occurred was if we could just create an optic with the correct spacing out of the fused silica flat itself, we could clean this with the regular methods we used to clean other optics. A schematic of the two creations is available in figure (A.1).

Then what spacing would the gap need to be? Using the above equation and the refractive index of the fused silica, we get a spacing thickness of $x_{\text{air}} = 577.4\text{nm}$. A method that appeared promising to create such a step in the middle of a glass flat was that of plasma etching, which bombards the surface of a material with plasma, stripping away surface layers. The next section will detail this process and the attempts
to create a step of the correct size.

A.1 Plasma Etching

To produce the etched step we looked to plasma etching. Collaborating with the Nano Lab clean-room at Swinburne we gained access to a reactive-ion etching (RIE) machine. Specifically we would like to thank Saulius Juodkazis and Xijun Li of the Swinburne Centre for Micro-Photonics for this access and training. Using a SiO\textsubscript{2} (quartz) optical flat (Thorlabs WG41050) we tested the ability for the plasma etcher to erode the surface. Using a plasma formed from Fluoroform CH\textsubscript{3}F gas, at 5 Pa of pressure and \textit{inductively coupled plasma} (ICP) power of 200 W (RF power of 100 W) we gained an acceptable etch effect. Calibration of the exact rate was done by etching for various times and interpolating a linear function, and was found to be 1.13 nm/s. This meant that to etch to desired height of 577 nm we bombarded the sample for 512 seconds. These numbers were approximate however, and in practise we found accuracy to be within the ±10 nm range.

To create the line of the step we used a sacrificial mask, covering half the optical flat. We tested various different masks, including a razor blade, high temperature tape and a glass coverslip. The glass coverslip created the best result, as the razor blade heated up and marked the surface, and the tape didn’t have a straight enough edge. To hold the glass to the optical flat - to keep it from moving during the placement inside the plasma oven and during the etch process - vacuum grease was applied under the coverslip. This was thin enough, and only a small amount was needed, so the glass touched the optical flat where the edge was required. The grease was easily removed after by acetone bath.

Once we had etched the edge into the optical flat, we needed to measure the step height to determine if we had etched the correct amount or not. Due to the transparent nature of the optic and the small size of the step we used two methods to ensure we had a correct measurement of the height. The first method was using Bruker interferometer, which uses the interference pattern from a laser to produce fringes, which can then be counted by imaging software to produce a 3D measurement of the surface and return a value for the step height. As this method can be inaccurate for transparent
mediums, we complemented this method by also using atomic force microscopy (AFM) imaging to gain another value for the height. The AFM imaging uses a cantilever tip to trace the surface where the deflection of this tip is measured by laser. For the AFM use we would like to thank Hayden Webb. Results of this method are shown in figure (A.2). Finally we tested the height of the step by looking at the TEM mode created by the optic in our laser system.

What we found from this etching process was that while the step could be created to the correct height, the step itself seemed to be eaten away and not as sharp as the pre-existing optic we had created by evaporative deposition. This, as well as the difference in surface roughness for the etched half compared to the non-etched half gave a result, at least for our exploration, not as good. For these reasons we proceeded with the deposition method outlined in the next subsection.
A.2 Evaporative Coating of MgF₂

The second method, one which have used succesfully before, to create a step as shown at the top of figure (A.1), is to coat half of the optic with magnesium fluoride (MgF₂). The reason we use this element is that it is very transparent to the wavelength we are using (532 nm), as well as being a metallic substance which can be deposition coated using an e-beam evaporative coating machine.

To go about this we made use of the facilities at the Melbourne Centre for Nano-Fabrication (MCN) located next to Monash University in Clayton, Victoria. This is a large cleanroom filled with state-of-the-art nano-fabrication technologies. For this access and help we would like to thank Yang Lim at MCN. The deposition machine was a Intlvac Nanochrome II e-beam coater, which works by melting an amount of the coating material via application of an electron beam at very low pressure. The coating material is scattered throughout the interior and covers the face of the sample. In our case the coating material used was magnesium fluoride (MgF₂), which is a soft compound that evaporates easily. During operation this has a tendency to melt at a non-linear rate, as the granules of the coating material melt individually. Due to this a careful calibration of the deposition rate was done prior to coating the optical flat, as well as implementing a wide sweep across the coating material while in operation.

In preparing the sample we placed a razor blade stuck with high temperature vacuum tape across the centre of the optical flat (Thorlabs WG41050). This provided a sharp edge for the step. Previously when coating we had heated the sample up, to help the MgF₂ stick to the surface, but for my coating I found there to be no problem with the material sticking; although I made sure to do a quick plasma etch on the material before coating (this was an option available on the coating machine) and the MgF₂ had no problem sticking to the surface. The thickness required at 702 nm is large for this process, and the deposition took 3-4 hours including pump-down time. After deposition we tested the optic in the laser beam and found it to produce a good TEM mode, similar to the step we had created years ago, using a different machine but similar method.
Figure A.3: The Intlvac Nanochrome II thin film e-beam evaporative coater.
Bibliography


112


113


[72] J. M. W. Fuchs, Molecular Bose-Einstein condensates and p-wave Feshbach molecules of 6Li2. Swinburne University of Technology, Centre for Atom Optics and Ultrafast Spectroscopy, 2009. 36, 37


[99] *Kinematics and Thermodynamics of a Two-Dimensional Fermi Gas*. PhD thesis. 65, 83


118


119


