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The $l_1$ Measure of Image Reconstruction Subject to Motion Corrupted $k$-space Data

Cishen Zhang∗, Zai Yang†, and Lihua Xie†

Abstract

For reconstruction of a nonnegative real valued object based on scanned data in the $k$-space, it is shown that the $l_1$ norm of the original motionless object is minimum among that of all objects with combined translational and rotational motions and is a sensitive measure of the object motions. These results provide an insight and theoretical fundamental for applying the $l_1$ optimization to the motion correction problem for image reconstruction in various practical applications.

Keywords: Image reconstruction, Motion correction, $k$-space data, The $l_1$ norm.

1. Introduction

There are various imaging applications where the physical sensing and detecting devices scan imaged objects and perform the data acquisition in the frequency domain. The acquired measurement data are commonly known as the $k$-space data and the follow up image reconstruction processing involves the inverse Fourier transform of the $k$-space data. A typical practical application based on the $k$-space data acquisition and reconstruction is magnetic resonance imaging (MRI) [1] and other practical and important applications include synthetic aperture radar (SAR) imaging [2], radio telescope imaging [3], fluorescence lifetime imaging (FLIM) [4], digital holographic tomography [5], optoacoustic imaging [6], etc.

If the imaged object is not stationary during the $k$-space data acquisition, it will result in motion corrupted $k$-space data and, consequently, motion artifacts in the reconstructed image. To reduce and remove the motion artifacts from the reconstructed image is known as the motion correction problem. This problem is associated with the multidimensional phase retrieval problem, see [7] and references therein, and has been an active research topic for a long time.

The motion correction problem is known to be ill-posed and, in general, has multiple solutions. The navigator based methods [8, 9] have been popular in dealing with motions of rigid objects by acquiring extra $k$-space lines and detecting more motion information. The iterative projection algorithms [10] and the convex optimization method [11] are also popularly applicable to the motion correction problem. Most recently, an $l_1$ optimization method is applied to motion correction in MRI [12] and the phase retrieval problem [13].

The optimization based motion correction method requires a practically feasible measure of the image reconstruction error caused by object motions, so to enable formulation and computation of the optimal image reconstruction solution. It is therefore important to derive a meaningful measure and understand how it represents the image reconstruction and optimization performance. In this paper, an analysis is carried out to show that, for reconstruction of a nonnegative real valued image based on $k$-space measurement data, the $l_1$ norm of the original motionless image is minimum among that of all images with motions and corrupted $k$-space data. It is further shown that, in most practical circumstances where the motions of the object are of irregular or random nature, the $l_1$ norm of the original motionless image is strictly less than that of images with motions and corrupted $k$-space data. These results demonstrate that the $l_1$ norm of the reconstructed image is a sensitive and meaningful measure and can provide theoretical fundamental and technical formulation for applying the $l_1$ optimization to the motion correction problem.

2. The $k$-space data of image with motions

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the sets of real, nonnegative real and complex numbers and $\mathcal{F}$ and $\mathcal{F}^{-1}$ the 2-dimensional (2D) discrete Fourier transform (DFT) and
inverse DFT operations, respectively. Consider a continuously distributed 2D rigid object \( s_c(x,y) \in \mathbb{R}_+ \) defined over a finite rectangular support \( \mathcal{S} \), where \((x,y)\) denotes the cartesian coordinates of the continuous 2D spatial plane. Since any motion of the rigid object in the 2D continuous spatial support \( \mathcal{S} \) can be represented as a superposition of a translation motion \((\tilde{x},\tilde{y})\) from the origin of the object \((x,y) = (0,0)\) and a rotation of angle \( \theta \) centred at \((x,y) = (0,0)\) and in the counter clockwise direction, the displaced object is written as

\[
\hat{s}_c(x,y) = s_c(\tilde{x},\tilde{y}),
\]

where

\[
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} - \begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix}.
\]

For digital imaging purpose, the continuous spatial support \( \mathcal{S} \) is uniformly discretized into an \( M \times N \) grid with cartesian coordinates \((m,n)\). Correspondingly, the discretized distribution of the 2D rigid object \( s_c(x,y) \) is written as

\[
s(m,n) = s_c(m\Delta_x,n\Delta_y),
\]

where \( \Delta_x \) and \( \Delta_y \) are the discretization sizes in the horizontal and vertical directions, respectively.

Suppose that the rigid object \( s(m,n) \) is scanned by some imaging device for acquisition of the \( k \)-space data. If a motion of the object, denoted by \( \hat{s}_k(m,n) \), occurs when the \( k \)-space data acquisition is at \( k = (u,v) \), the acquired \( k \)-space datum is

\[
\hat{S}(u,v) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{s}_k(m,n) e^{-j2\pi \left( \frac{mu}{M} + \frac{nv}{N} \right)}.
\]

It is noted that the motion of the discretized \( \hat{s}_k(m,n) \), at each \( k \), has taken into account any combined rotational and translational motions in the continuous spatial plane, in terms of the continuous motion parameters \((\tilde{x},\tilde{y})\) and \( \theta \), as specified for \( \hat{s}_c(x,y) \) in (1). If the acquired \( k \)-space data set \( \hat{S}(u,v) \) is used for image reconstruction, the resultant image, denoted by \( \hat{s}(m,n) \), is the IDFT of \( \hat{S}(u,v) \), i.e.

\[
\hat{s}(m,n) = \mathcal{F}^{-1}(\hat{S}(u,v)) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{S}(u,v) e^{j2\pi \left( \frac{mu}{M} + \frac{nv}{N} \right)}.
\]

It is further noted that (2) is a pointwise representation of \( \hat{S}(u,v) \), for each \( k = (u,v) \), in terms of the scanned object \( \hat{s}_k(m,n) \). Whereas, (3) is the IDFT of the \( k \)-space data set \( \hat{S}(u,v) \). The expressions (2) and (3) are therefore not a DFT pair and \( \hat{s}(m,n) \neq \hat{s}_k(m,n) \), in general, for any \( k \).

3. The \( l_1 \) norm of image with motions

The analysis of the \( l_1 \) norm of the reconstructed image \( \hat{s}(m,n) \) is to gain understanding of how the object motion affect the image reconstruction in terms of the \( l_1 \) measure so to provide guideline and method for motion artifact correction. For such an analysis and without loss of generality, it is assumed in the rest of this paper that the following assumptions are satisfied.

**A1:** \( s_c(x,y) \in \mathbb{R}_+ \) is non-constant and continuously distributed and, consequently, \( s(m,n), \hat{s}_k(m,n) \in \mathbb{R}_+, \forall k \), are non-constant.

**A2:** The motions of \( s_c(x,y) \) and, hence, \( s(m,n) \) and \( \hat{s}_k(m,n) \), \( \forall k \), are confined within the finite support \( \mathcal{S} \).

**A3:** The reference object image is set at \( k = 0 \), i.e. \( s(m,n) = \hat{s}_0(m,n) \).

**Theorem 1** The image \( s(m,n) \) of the motionless object possesses minimum \( l_1 \) norm among all images \( \hat{s}(m,n) \) reconstructed from the motion corrupted \( k \)-space data set, i.e.

\[
\|s\|_1 \leq \|\hat{s}\|_1 = \|\mathcal{F}^{-1}\hat{S}\|_1.
\]

**Proof:** It follows from (2), (3) and assumption A3 that

\[
\hat{S}(0,0) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{s}(m,n)
\]

\[
= \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{s}_0(m,n)
\]

\[
= \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(m,n).
\]

Since \( s(m,n) \in \mathbb{R}_+ \),

\[
\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(m,n) = \|s\|_1
\]

As a result

\[
\|\hat{s}\|_1 = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} |\hat{s}(m,n)| \geq \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{s}(m,n)
\]

\[
= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(m,n) = \|s\|_1.
\]

\( \square \)

Theorem 1 shows that the \( l_1 \) norm of the motion corrupted image \( \hat{s}(m,n) \) is lower bounded by its averaged value, which is the \( l_1 \) norm of the motionless image \( s(m,n) \). It implies that \( \|\hat{s}\|_1 > \|s\|_1 \) if some element of \( \hat{s}(m,n) \) has strictly negative or complex value. Further analysis of the conditions for \( \|\hat{s}\|_1 > \|s\|_1 \) is given below.
Lemma 1 Let $\hat{\mathcal{F}}$ be the set of reconstructed images $\hat{s}$ given by (3) as the IDFT of $\hat{S}(u,v)$ and $\hat{\mathcal{F}}_+^*$ be the set of $\hat{s}$ with all $\hat{s}(m,n) \in \mathbb{R}_+$. Then $\hat{\mathcal{F}}_+^*$ is a measure zero set of $\hat{\mathcal{F}}$.

Proof: Let $(\hat{s}_k, \hat{y}_k, \theta_k)$ be the combined translational and rotational parameters of $s(m,n)$, confined in $\mathcal{P}$, at each $k$-space point $k$ and $\mathbb{P}$ be the set of $(\hat{s}_k, \hat{y}_k, \theta_k)$ for all $k$. The displaced object at $k$ and $\hat{k} = (M - u, N - v)$, respectively, is

$$
\hat{s}_k(m,n) = s_c(n\Delta, \cos \theta_k + m\Delta, \sin \theta_k - \hat{x}_k, \\
- n\Delta, \sin \theta_k + m\Delta, \cos \theta_k - \hat{y}_k),
$$

$$
\hat{s}_k(m,n) = s_c(n\Delta, \cos \theta_k + m\Delta, \sin \theta_k - \hat{x}_k, \\
- n\Delta, \sin \theta_k + m\Delta, \cos \theta_k - \hat{y}_k).
$$

In view of the formulations of the above $\hat{s}_k(m,n)$ and $\hat{S}(u,v)$ in (2), it is clear that the set formed by the motion parameter pairs $(\hat{s}_k, \hat{y}_k, \theta_k)$ and $(\hat{s}_k, \hat{y}_k, \theta_k)$, such that $\hat{S}(u,v) = \hat{S}(M - u, N - v)$, is a measure zero set of $\mathbb{P}$. It follows, from that the non-constant $\hat{s}_k(m,n)$ and, consequently, $\hat{S}(u,v)$ are continuous functions in $(\hat{s}_k, \hat{y}_k, \theta_k)$, that the set of the pairs $\hat{S}(u,v)$ and $\hat{S}(M - u, N - v)$, such that $\hat{S}(u,v) = \hat{S}(M - u, N - v)$, is a measure zero set of the set of $\hat{S}(u,v)$ for all $k$. Finally, the result of the lemma follows from that $\hat{S}(u,v) = \hat{S}(M - u, N - v)$, for all $k$, is a necessary condition for any $\hat{s} \in \hat{\mathcal{F}}$. $\square$

Lemma 2 If $\hat{s} \not\in \hat{\mathcal{F}}_+^*$, then $||\hat{s}||_1 > ||s||_1$.

Proof: If $\hat{s} \not\in \hat{\mathcal{F}}_+$, at least one $s(m,n) < 0$ or $\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(m,n)$ has a strictly non-zero imaginary part. In either or both of the cases, $||\hat{s}||_1 > ||s||_1$. $\square$

The result of Lemmas 1 and 2 directly lead to the following result.

Theorem 2 Under assumptions A1-A3, the $l_1$ norm of the reconstructed image is strictly greater than that of the motionless image, i.e. $||\hat{s}||_1 > ||s||_1$, for almost all possible motions characterized by the translational and rotational motion parameter set $\mathbb{P}$. $\square$

Theorem 2, strengthens the result of Theorem 1 to show that the $l_1$ norm value of reconstructed images is, in general, increased when object motions occur, so is a practically meaningful and sensitive measure of object motions.

If $s(m,n)$ is sparse with a considerable number of zero elements, the oscillatory harmonic components of $\hat{s}(m,n)$ caused by motions spread uniformly over the whole spatial support which can very likely make the real part of some $\hat{s}(m,n)$ negative where $s(m,n)$ is zero. Thus the $l_1$ norm of sparse $s(m,n)$ can be sensitive to object motions.

4. Image reconstruction by $l_1$ optimization

Consider that the object only has translational motions, i.e. it satisfies

A4: The motions of $s_c(x,y)$ is translational only with $\theta = 0$.

Under assumption A4, the displaced object with the translational motion $\bar{r}_k = (\bar{x}_k, \bar{y}_k)$ at each $k$ is

$$
\hat{s}_k(m,n) = s(m - \bar{x}_k, n - \bar{y}_k),
$$

and its corresponding $k$-space datum is

$$
\hat{S}(u,v) = \Lambda_k(u,v)S(u,v),
$$

where $S(u,v)$ denotes the motion free $k$-space datum and

$$
\Lambda_k(u,v) = e^{-j2\pi (\bar{x}_k u + \bar{y}_k v)}.
$$

For motion correction and image reconstruction, an $l_1$ optimization based LASSO type problem can be formulated as follows.

$$
\min_{s,F} \left\{ ||s||_1 + \frac{1}{2} ||\Lambda_k \hat{F} \odot (\hat{S} - \hat{s})||_F^2 \right\}, \text{ subject to } s \succeq 0,
$$

(6)
where $\lambda > 0$ is a regulation parameter, $\odot$ denotes the Hadamard product, $\| \cdot \|_F$ denotes the Frobenius norm and the acquired $k$-space data $\hat{S}(u,v)$ may contain measurement noise. An equivalent formulation of the $l_1$ optimization problem is of the form

$$\min_{s, \hat{r}} \|s\|_1, \text{ subject to } \|\Lambda_r \odot \mathcal{F}s - \hat{S}\|_F \leq \varepsilon \text{ and } s \succeq 0,$$

where the parameter $\varepsilon$ controls the fidelity of the reconstruction solution to the measured data. In the noise free case and as $\lambda \to \infty$ and $\varepsilon \to 0$, the problems (6) and (7) both approach to

$$\min_{s, \hat{r}} \|s\|_1, \text{ subject to } \Lambda_r \odot \mathcal{F}s = \hat{S} \text{ and } s \succeq 0. \tag{8}$$

The following theorem is a straightforward consequence of Theorem 1.

**Theorem 3** The image $s(m,n)$ of the motionless object satisfying assumptions A1-A4 is an optimal solution of the optimization problem (8).

\[\square\]

## 5. Examples

### 5.1. MRI motion correction

This example demonstrates the effectiveness of $l_1$ optimization for MRI motion correction in the case where the image of interest is nonnegative real valued. It is associated with our recent work [12] where the image is assumed to be sparse under some appropriate basis but not necessarily nonnegative real valued. A 2D human brain image of size $256 \times 256$ shown in Fig. 1(a) is considered in the simulation which is obtained from BrainWeb (http://www.bic.mni.mcgill.ca/brainweb). The same motion is assumed to be shared among every phase-encoding line. Motion artifacts are introduced by applying continuously varying linear phase shifts to the motion free $k$-space data based on (5). Then complex white Gaussian noises are added to the measured $k$-space data such that the measurement SNR is 30dB. The sparse relaxed averaged alternating reflection (SRAAR) algorithm proposed in [12] is used for motion correction, which iteratively computes a joint solution for $s(m,n)$ and $\tilde{r}_k$. Fig. 1(c) shows that a small amount of motions (within 5 pixels along both the readout and phase-encode directions) can cause severe image artifacts. The motion corrected image using SRAAR is shown in Fig. 1(d) where few artifacts remain. The image is obtained within 97.8s and has a reconstruction SNR of 20.4dB. Finally, we note that the motion-free, motion-corrupted and corrected images, respectively, have an $l_1$ norm of $1.284 \times 10^4$, $1.389 \times 10^4$ and $1.291 \times 10^4$.

![Image 322x474 to 430x582](Image 322x607 to 430x715)

**Figure 2.** Results of motion correction for a satellite image with (a) motion free image, (b) translational motion, (c) motion corrupted image and (d) corrected image using the algorithm in [13] with SNR = 24.6dB. The $k$-space measurement SNR is 30dB.

### 5.2. Satellite image motion correction

This example is on reconstruction of a 2D nonnegative satellite image of size $256 \times 256$ subject to random translational motions. The motion along each direction with respect to the frequency index is modeled as a random walk with each step independently distributed. Since we do not attempt to exploit the relationship among the motions at different frequencies, the optimization problems (6) and (7), respectively, can be equivalently written as

$$\min_s \left\{ \|s\|_1 + \frac{\lambda}{2} \|\mathcal{F}s - \hat{S}\|_F^2 \right\}, \text{ subject to } s \succeq 0, \tag{9}$$

$$\min_s \|s\|_1, \text{ subject to } \|\mathcal{F}s - \hat{S}\|_F \leq \varepsilon \text{ and } s \succeq 0. \tag{10}$$

The problems (9) and (10) are to solve the image $s(m,n)$ from the magnitude of the $k$-space data set $\hat{S}(u,v)$ and are exactly the popularly known phase retrieval problem [7, 10, 11]. It is noted that the solution for the phase retrieval problem is ill-posed and subject to a global phase ambiguity. Augmented Lagrangian alternating direction method (ADM) based algorithms are recently de-
veloped in [13] to solve (9) and (10), which are fast and accurate.

In our simulations, a nonnegative satellite image shown in Fig. 2(a) is considered. Random motions shown in Fig. 2(b) and measurement noises with SNR = 30dB are introduced in the measured k-space data. As a result, the image is totally corrupted as shown in Fig. 2(c). The corrected image using the algorithm in [13] is shown in Fig. 2(d), where a global phase ambiguity exists. The image is obtained within 9.2s only and has an SNR of 24.6dB without accounting for the physical ambiguities. Again, we present the $l_1$ norm of the motion-free, motion-corrupted and corrected images which, respectively, are $1.029 \times 10^6$, $2.556 \times 10^6$ and $1.031 \times 10^6$.

6. Conclusion

For reconstruction of a nonnegative real valued object based on scanned data in the k-space, this paper has carried out analysis to show that the $l_1$ norm is a meaningful and sensitive measure of the reconstruction error caused by object motions. The results can thus provide a theoretical fundamental and a technical approach to the development of methods and solutions for the ill-posed motion correction problem. The effectiveness of the $l_1$ optimization based solutions for image motion correction and reconstruction has been demonstrated by simulation examples.

References