Declaration

This thesis is my own work, and no part of it has been submitted for a degree at this, or at any other, university.

Andrew Johnson
Abstract

This thesis concerns the phenomena labeled *Dark Energy*. Herein we present our attempt to illuminate this most intriguing, perplexing, and demanding problem. Our efforts focus on three separate directions which outline below.

In **Chapter 2** we present our analysis of the 6-degree Field Galaxy Survey velocity sample (6dFGSv). Using this peculiar velocity sample we infer scale-dependent measurements of the normalised growth rate of structure \( f\sigma_8(k, z = 0) \). We constrain the growth rate in a series of \( \Delta k \sim 0.03h\text{Mpc}^{-1} \) bins to \( \sim 35\% \) precision, including a measurement on scales \( > 300h^{-1}\text{Mpc} \), which represents one of the largest-scale growth rate measurements to date. Our measurements are consistent with a scale-independent growth rate as predicted by General Relativity. Moreover, the measured amplitude of the growth rate is consistent with that predicted by the *Planck* cosmology. Combining these measurements, we measure the (scale-independent) normalised growth rate at \( z = 0 \) to \( \sim 15\% \). We emphasize this measurement is independent of galaxy bias and in excellent agreement with the constraint from the measurements of redshift-space distortions from 6dFGS. Motivated by upcoming peculiar velocity surveys, in this analysis we focus on understanding and correcting for systematic errors. In particular, we focus on non-Gaussian errors, zero-point errors, and combining (correlated) velocity surveys.

In **Chapter 3** we search for deviations from General Relativity. To this end, we present measurements of both scale- and time-dependent deviations from the standard gravitational field equations. We allow separate deviations for relativistic and non-relativistic particles, by way of the parameters \( G_{\text{matter}}(k, z) \) and \( G_{\text{light}}(k, z) \). To define each of these functions we use two bins in both scale and time, with transition wavenumber 0.01 Mpc\(^{-1}\) and redshift 1. To constrain these parameters, we highlight the use of two dynamical probes: galaxy power spectrum multipoles and the direct peculiar velocity power spectrum. These probes are highly complementary in constraining \( G_{\text{matter}}(k, z) \) as they probe density fluctuations on different scales. We use the multipole measurements as derived from the WiggleZ and BOSS Data Release 11 CMASS galaxy redshift surveys, while the velocity power spectrum is measured from the velocity sub-sample of the 6-degree Field Galaxy Survey. To extend our constraints we include additional probes including baryon acoustic oscillations, Type Ia SNe, the cosmic microwave background (CMB), lensing of the CMB, and the temperature–galaxy cross-correlation. Using a Markov Chain Monte Carlo likelihood analysis, we find the inferred best-fit parameter values of \( G_{\text{matter}}(k, z) \) and \( G_{\text{light}}(k, z) \) to be consistent with the standard model at the 95% confidence level. Additionally, we
preform Bayesian model comparison to further understand whether the data prefers the introduction of the parameters \(G_{\text{matter}}, G_{\text{light}}\).

In Chapter 4, motivated by weak lensing experiments, we present a new approach to determining the redshift distribution of galaxies in a photometric survey, in the form of a \textit{quadratic estimator}. The estimator infers a redshift probability distribution using the angular auto- and cross-correlations between the photometric and overlapping spectroscopic sample. We derive this estimator by expanding on the analysis by McQuinn & White (2013), where we suggest a number of corrections to their original formulas. To test our estimator we use a series of N-body simulations to construct mock galaxy catalogues. Within each 60 deg\(^2\) mock we generate a photometric and spectroscopic sample using pre-defined redshift distributions. For the spectroscopic sample we use a uniform distribution from \(z = 0.1\) to \(z = 0.9\), while for the photometric sample we use a Gaussian probability distribution. We find that the reconstructed redshift distributions, for individual mocks, are consistent with the input redshift distribution at a level of 2\(\sigma\), once we have accounted for the bias evolution of the lens sample. Averaging over the mock results, thus reducing the statistical error, we find that above a redshift of 0.5 there are statistically significant deviations from the underlying distribution (i.e., greater than 2\(\sigma\)). We interpret these deviation as the modelling of the correlation function breaking down and introducing a systematic bias.
Preface

Work performed as part of this thesis has been submitted for publication in two papers:

1. **The 6dF Galaxy Velocity Survey: Cosmological constraints from the velocity power spectrum** (Chapter 2)
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2. **Searching for Modified Gravity: Scale and Redshift Dependent Constraints from Galaxy Peculiar Velocities** (Chapter 3)
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Chapter 1

Introduction

The intention of this introduction is to provide the reader with the material necessary to appreciate the work that follows. In doing this we hope to convey plainly the motivations for the starting points of the three subsequent chapters.

1.1 A Short History of the Universe

Pointing a sensitive radio telescope at the sky one would make the discovery that, independent of the direction, there exists a uniform background of microwave radiation, at a temperature of 2.7 K (Penzias & Wilson, 1965). Further inspection would reveal tiny temperature fluctuations in this background radiation, at a level of 1 part in 100,000 (Spergel et al., 2003). This light, which permeates throughout our Universe, is the afterglow of the Big Bang, and the temperature fluctuations reveal the first observable structure in the Universe. For early work see Peebles & Yu (1970).

Today the universe is cold and matter is sparsely distributed. During very early cosmic times it was the opposite, hot and dense. So hot, in fact, that stable atoms could not form, and matter consisted of free electrons and atomic nuclei (Weinberg, 1972). As the temperature dropped below 3000 K, the thermal equilibrium between photons and matter was broken, see figure 1.1. Photons began a free expansion. This is the light one can observe with radio telescopes. It is a cool remnant of a
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phase transition that occurred when the universe was a mere 380,000 years old.

After 13.8 billion years of cosmic evolution we find ourselves capable of both studying the structure of the universe surrounding us and understanding it. We have good reasons to suspect that the primordial perturbations in the microwave background originate from quantum fluctuations, stretched by a period of rapid expansion (Guth, 1981). We understand the physical processes in the early universe that facilitated the production of light elements (Alpher, 1948). And, for the most part, we understand the process by which the primordial temperature fluctuations grew, via gravitational instability, into the beautiful hierarchy of structures we observe today.

Our current picture was developed and refined using a number of distinct cosmological probes. Using optical telescopes we have created 3-dimensional maps of the local universe. These maps contained a wealth of information on how fast structure is growing and the expansion history of the universe. Using observations from space we have measured the acoustic oscillations embedded within the microwave background. The amplitude and frequency of these oscillations have told us about the energy content of the early universe, and about the primordial power spectrum generated during inflation.

There are, of course, many missing components to this picture. First and foremost we expect gravity to pull; General Relativity predicts that for a universe filled with matter and radiation, the expansion rate will decrease as a function of time. Yet we observe the opposite. The expansion rate is currently accelerating. This is a profound mystery, which, undeniably, presents us with an opportunity to learn something about gravitational or particle physics, or even both. Furthermore, following an abundance of observational evidence, we have been forced to introduce dark matter, an exotic unknown form of matter that currently comprises $\sim 25\%$ of the energy density of the universe.
1.2 Gravity and Space-time Geometry

Gravity is the dominant force on the scale of the Universe. Along with the total density, it dictates the expansion rate of the universe and the growth of large- and small-scale structure: Its influence extends from galaxy clusters to individual galaxies to stars, planets, and nebula. Suffice it to say an understanding of gravity is of importance in understanding the universe we find ourselves in. In this section we will describe our current theory of gravity, general relativity, and the application of this theory to the universe.

Our current theory of gravity is described in terms of the differential geometry of curved space-time. To describe this geometry we use a metric tensor $g_{\mu\nu}(x, t)$ (Carroll, 2003). The spatial and temporal variation of this tensor is induced by the configuration of matter and energy in the universe. This variation results in gravitational attraction.

Finding a suitable metric tensor on small scales presents a significant problem. Fortunately, on large scales, which we define as 150 Mpc, the universe has two very useful symmetry properties which simplify matters. The symmetry properties are

Figure 1.1: The evolution of the fractional energy density of the components of the universe. Radiation dominates the energy budget at early times, while dark energy only very recently become the dominant energy component. Figure taken from http://www.damtp.cam.ac.uk/user/db275/Cosmology.pdf
isotropy and homogeneity. Isotropy implies that at any given point the universe (or, more precisely, a manifold) looks the same in every direction. This is a symmetry under rotation. Homogeneity is a symmetry under translation, it implies that the metric tensor is the same in every location (Carroll, 2003). Note, these properties are defined across space and not time, as the Universe expands the metric changes.

Using observations of the cosmic microwave background radiation one can observe that the universe looks incredibly similar in all directions, once we account for our local motion (see, Spergel et al. (2003)). This observation implies isotropy. The Copernican principle states that we do not inhabit a unique part of the universe. Given isotropy and the Copernican principle one can infer the Universe is homogeneous. Alternatively, one can test homogeneity using observations of large-scale structure (e.g., Scrimgeour et al., 2012).

Enforcing these symmetries implies a Robertson-Walker line interval (Weinberg, 1972), which has the form

$$ds^2 = a^2(\tau) \left[ d\tau^2 - (d\chi^2 + S_k^2(\chi)d\Omega^2) \right].$$

(1.1)

Where $d\Omega^2 = d\theta^2 \sin^2 \theta d\psi^2$, and $a(\tau)$ is the scale-factor. While $\tau$ is the conformal time, defined as $d\tau = dt/a(t)$, where $dt$ is the physical time. Here we are using units where $c = 1$.

This line-interval is dependent on the (normalized) curvature $k$, specifically,

$$S_k(\chi) = \begin{cases} 
\sinh(\chi) & \text{for } k = 1, \\
\chi & \text{for } k = 0, \\
\sin(\chi) & \text{for } k = -1.
\end{cases}$$

(1.2)

For a three-dimensional spatial slice of this metric, $k = 1$ implies a constant positive curvature on this surface, $k = 0$ implies zero curvature, and $k = -1$ implies negative curvature.

The background dynamics of this space-time are described by a single degree of freedom, the scale factor $a(\tau)$. This is the only time-dependent component of the
metric. To determine this function we invoke the Einstein equation:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} . \] (1.3)

This formula relates the matter content of the universe, as described by the energy-momentum tensor \( T_{\mu\nu} \), to the geometry of the universe, as described by the Einstein tensor \( G_{\mu\nu} \). Our imposed symmetries (homogeneity and isotropy) force the energy-momentum tensor to be that of a perfect fluid, and hence for a comoving observer

\[
T^\mu_\nu = \begin{pmatrix}
\rho & 0 & 0 & 0 \\
0 & -P & 0 & 0 \\
0 & 0 & -P & 0 \\
0 & 0 & 0 & -P
\end{pmatrix} \]

(1.4)

Where \( \rho \) is the density, and \( P \) is the pressure. Here we have left the summation over the various species \( i \), implicit, viz., \( P = \sum_i P_i, \rho = \sum_i \rho_i \). Three main components are relevant: radiation, matter, and dark energy. Here matter refers to both baryonic and non-baryonic matter, and at early times radiation includes neutrinos.

The evolution of each component is then set by energy conservation, i.e., \( \nabla_\mu T_{\mu\nu} = 0 \). Therefore, invoking energy conservation, the components evolve as

\[
\frac{\dot{a}}{a} = \begin{cases}
  a^{-3} & \text{for matter} \\
  a^{-4} & \text{for radiation} \\
  0 & \text{for a cosmological constant (or vacuum energy)}
\end{cases}
\]

(1.5)

As one would expect, the density of vacuum energy is constant with time, and the density of matter dilutes proportionally to the volume of the universe.

Combining Eqns. (1.1,1.3,1.5) we find the the Friedmann equations:

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} ,
\]

(1.6)

\[
\left( \frac{\dot{a}}{a} \right) = -\frac{4\pi G}{3} (\rho + 3P) .
\]

(1.7)

Where \( H \) is the Hubble parameter, which describes the expansion rate of the universe,
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and a dot indicates a time derivative, \( \dot{a} = da/dt \). Eqn. (1.6) gives the connection between the global expansion rate \( (H) \) and the total energy density \( \rho \) and the curvature \( (k) \) of the universe. Hence by determining the local expansion rate we can measure how much material there is in the universe. Rewriting Eq. (1.7) in terms of the equation of state of each species \( w_i = P_i/\rho_i \) one finds \( \ddot{a}/a \propto \sum \rho_i(1+3w_i) \). Hence in order for the expansion rate to accelerate, that is, \( \ddot{a} > 0 \), an energy component must have an equation of state \( w < -1/3 \).

For clarity, the first Friedmann equation (1.6) is normally rewritten in terms of dimensionless density parameters \( \Omega_A \), where

\[
\Omega_A = \rho_{A,t=t_0}/\rho_{\text{crit},t=t_0}.
\]  

(1.8)

Where \( (A) \) can take the value ‘\( m \)’ for matter, ‘\( r \)’ for radiation, or ‘\( \Lambda \)’ for dark energy. The critical density \( \rho_{\text{crit}} \) is defined as the density corresponding to a \( k = 0 \) universe at the current time \( t_0 \), specifically, \( \rho_{\text{crit}} = 3H_0^2/8\pi G \sim 10^{-5}h^2 \) protons cm\(^{-3} \).

Neglecting curvature, Eq. (1.6) simplifies to

\[
\frac{H(a)^2}{H_0^2} = \Omega_t a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda.
\]  

(1.9)

Here we have used the normalization \( a_0 = 1 \), and the energy densities are all evaluated today at \( t = t_0 \).

To observe the universe in 3-dimensions we require information about the distances to galaxies. The simplest estimate comes from the redshift of a galaxy. We can related the redshift of a galaxy to the background expansion rate as follows. Following from the geodesic motion of massless particles, within a FRW background the energy of photons scales as

\[
E \propto 1/a.
\]  

(1.10)

This scaling explains the \( a^{-4} \) evolution in Eqn. (1.5). A factor of \( a^{-1} \) arises because of the loss of energy, and \( a^{-3} \) because of the dilution of photons as the Universe expands.

In quantum mechanics the wavelength of light is proportional to the momentum of the photon, \( \lambda = h/p \), where \( h \) is the Planck constant, and \( p \) is the momentum. As
a result, the expansion of the universe changes the energy of the photon and hence the wavelength of the light. At a time $t_0$, the wavelength of light emitted at $a_{t_1}$ with wavelength $\lambda_1$ becomes

$$\lambda_0 = \frac{a_0}{a_1} \lambda_1.$$  \hspace{1cm} (1.11)

As a function of redshift, the scale factor is $a = 1/(1 + z)$.

### 1.2.1 The Inhomogeneous Universe

The treatment above disregards the very apparent structure we see around us in the universe (see, fig. 1.2). This structure contains a wealth of information about gravitational physics, dark matter and dark energy. To describe the evolution of this structure we solve the Einstein equation when including very small fluctuations to the metric and energy-momentum tensor.

We will not introduce relativistic perturbation theory in any detail here. For a comprehensive account we refer the reader to Ma & Bertschinger (1995). This section is intended as background for Chapter 3 where we modify the gravitational
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field equations.

For very small perturbations about a Friedmann-Robertson-Walker background, one can decompose the metric tensor into a background and a perturbed component. Henceforth, we will indicate background quantities using a bar, e.g., \( \bar{x} \). At linear order \( g_{\mu\nu} \approx \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \). In the Newtonian gauge this perturbation is characterized by two degrees of freedom:

\[
g_{\mu\nu} = \begin{pmatrix} 1 + 2\Psi & 0 \\ 0 & -(1 - 2\Phi)\delta_{ij} \end{pmatrix}
\]  

(1.12)

Where \( \Phi \) and \( \Psi \) represent scalar fluctuations to the metric: \( \Phi \) a spatial fluctuation and \( \Psi \) a temporal fluctuations. Vector and tensor mode contributions to large-scale structure are negligible, hence they are ignored.

Furthermore, the Einstein and energy-momentum tensor can be decomposed into background and perturbed components. Using the perturbed metric, one can compute the components the Einstein tensor; and, ignoring anisotropic stress, the perturbed energy-momentum tensor can be written as (Ma & Bertschinger, 1995)

\[
T^0_0 = \bar{\rho} + \delta \rho, \\
T^i_0 = (\bar{\rho} + \bar{P})v^i, \\
T^0_j = (\bar{\rho} + \bar{P})v_j, \\
T^i_j = \bar{P} + \delta^i_j\delta P.
\]

(1.13) (1.14) (1.15) (1.16)

Where \( v^i \) is the peculiar velocity, \( \delta \rho \) is the density perturbation, and \( \delta P \) is the pressure perturbation. For simplicity we will restrict our discussion to late times, hence we will ignore all pressure contributions as they are negligible. We will also work in terms of the dimensionless density perturbation \( \delta \equiv \delta \rho/\bar{\rho} \).

To describe structure growth we require a system of equations relating the gravitational degrees of freedom (\( \Phi \) and \( \Psi \)) to the density and velocity fields of the cosmic fluid. Two equations are found from energy conservation, \( \nabla_\mu T_{\mu\nu} \): the continuity and Euler equation,

\[
\delta' + \nabla \cdot \mathbf{v} - 3\Psi = 0,
\]

(1.17)
\[ \mathbf{v} + H \mathbf{v} = -\nabla \Psi. \]  

(1.18)

Where \( a' \equiv da/d \tau \), and \( H = a'/a \), i.e., the Hubble parameter in conformal time. From these equations one can observe that the divergence of the velocity field drives variation in the density field; and that the velocity field \( \mathbf{v} \) is driven by gradients in the metric \( \nabla \Psi \).

Two more equations are found using the Einstein equation for the perturbed components. Firstly, \( \Phi - \Psi = 0 \), which implies the temporal and spatial potentials must be identical and also reduces the gravitational degree of freedom to one\(^1\). The second equation is the Poisson equation:

\[ \nabla^2 \Psi = 4\pi G a^2 \bar{\rho} \Delta \]  

(1.19)

Where \( \Delta \) is the fractional overdensity in the co-moving gauge, \( \bar{\rho} \Delta = \bar{\rho} \delta - 3H \bar{\rho} v \).

Thus we have a closed system of equations. Together these equations control the evolution of large-scale structure in the universe. And with fixed initial conditions they allow us to predict statistical properties of the local density and velocity fields. This sets the foundation for the use of large-scale structure in constraining properties of the universe.

### 1.3 Beyond the Standard Model

Within the last decade a standard model of cosmology has emerged: the ΛCDM model. The foundations of this model are general relativity and the standard model of particle physics. Within this model we are free to specify the initial conditions (viz., the amplitude and slope of the primordial power spectrum) and the current energy density of all constituent components. Once these input components are fixed the ΛCDM model makes definite predictions for a range of observables; consistently, these predictions have been shown to agree with observational data (e.g., Planck 2018).

\(^{1}\)This is an important prediction of general relativity. In Chapter 3 we will discuss extensions to GR that violate this condition, and the cosmological probes available to test it. Note, anisotropic stress also causes this relation to break down; however, this type of stress only occurs in the very early universe.
The CDM model has two key components. Structure is seeded by dark matter over-densities, and the energy density of the universe today is dominated by dark energy, which causes the current expansion rate to accelerate. Dark energy is a uniformly distributed energy component with a constant density, and a negative equation of state. Within the standard model this energy component is interpreted as vacuum energy. Current observations are consistent with this interpretation. In particular, by probing the expansion history and large-scale structure we can test the vacuum energy predictions for the sound speed, the equation of state, and the evolution of the density (see, de Putter, Huterer & Linder, 2010; Copeland, Sami & Tsujikawa, 2006).

Problems arise when one calculates the expected energy density contribution from vacuum energy. In doing so one discovers the cosmological constant problem: The predicted energy density contribution from vacuum energy is discrepant by $\sim 120$ orders of magnitude with observations (Weinberg, 1989). Motivated by this crisis in theoretical physics, many extensions to the standard model and alternative dark energy models have been developed. We divide these into four categories:

- **Scalar Field Models**: Motivated by inflation, a single slowly rolling scalar field has been suggested to explain dark energy (for a review see, Copeland, Sami & Tsujikawa, 2006). The equation of state of a scalar field is determined by the field’s kinetic and potential energy. For a slowly rolling field the potential energy dominates and one recovers $w \approx -1$. However, the potential and kinetic energy of the field vary with time, hence the equation of state will evolve. Examples include quintessence, and K-essence.

- **Modified Gravity**: To induce a period of accelerated expansion one must modify either the energy-momentum tensor or the space-time geometry. Scalar field theories change the former, while modified gravity theories change the latter. Examples include $f(R)$, Galileon gravity, and Horndeski models (see, Clifton et al., 2012). Within metric theories of gravity the Einstein-Hilbert action is only unique under certain conditions given by Lovelocks Theorem. Effectively, to modify the gravitational dynamics one must add a new field, increase the
number of spatial dimensions, or add higher order derivative terms to the action.

- **Anthropic Arguments**: String theory predicts a very large number of metastable vacua, each with a different amount of vacuum energy: this large number corresponds to the number of different ways one can compact nine dimensions to three. It is possible to populate all these separate vacua (the ‘landscape of string theory’ Susskind, 2003) via eternal inflation, which is a generic prediction from inflationary models. Within such an ensemble of universes we find ourselves inhabiting a universe with a particularly small vacuum energy, because had the value been different we would not exist as observers (Weinberg, 1989; Susskind, 2003; Bousso, 2012).

- **Backreaction**: In general relativity the formulation of non-linear structure can influence the background expansion. This effect is known as Backreaction (Clifton, 2013). For linear perturbation theory and Newtonian N-Body simulations this effect averages out. However, in non-perturbative general relativity this effect can become significant. It has been suggested that Backreaction could influence the expansion rate enough to mimic the influence of dark energy (Buchert & Räsänen, 2012).

In Chapter 2 and Chapter 3 we concentrate on the first two categories of models for the following reasons. Anthropic Arguments are theoretically compelling, however, new observable signatures remain either unobservable or ambiguous. Regarding backreaction, no consistent framework for modeling these effects has been developed, this makes observational signatures unclear (Buchert & Räsänen, 2012). Hence, for both cases theoretical rather than observational progress is required.

### 1.4 Evidence For Dark Energy

A period of intense scrutiny preceded the acceptance of Dark Energy. Of particular importance for this transition was the agreement between a number of diverse observational probes. In this section we describe these critical pieces of evidence for
Dark Energy, in particular, we focus on the evidence coming from two cosmological probes: Type Ia Supernova and the CMB.

The first significant piece of evidence for Dark Energy came from the analysis of Type Ia Supernova (Riess et al., 1998; Schmidt et al., 1998). Supernova function as standard candles, i.e., there is a mapping between some external properties of Supernova, a property we can observe, and their intrinsic brightness. The cosmic distance ladder allows us to calibrate this mapping by estimating distances to the host galaxies of local supernova. The final product of this analysis is a measurement of the distance moduli, \( \mu = 5 \log_{10}(d_L(z;\ldots))/10 \text{pc} \), for a sample of supernova. Fig 1.3 shows the standard model predictions for the redshift evolution of the distance modulus in addition to measurements from type Ia SN.

The redshift evolution of the distance modulus is dependent on the redshift evolution of the various energy density components, given \( d_L(z;\ldots) \propto 1/H(z) \). Thus, using the measured distance modulus, we can infer the various energy density component. The early results of both Riess et al. (1998) and Schmidt et al. (1998) suggested \( \Omega_\Lambda \approx 0.7 \). Effectively, at a redshift of around 0.5 the results suggested the SN were 20% dimmer than expected in a universe with \( \Omega_m = 0.2 \) and \( \Omega_\Lambda = 0 \).

The Measurement of the first acoustic peak in the CMB provided the second strong piece of evidence for Dark Energy. The position of the peak determines the angular size of the sound horizon at \( z = 1100 \). In addition, the physical size of the sound horizon can be estimated based on the sound speed of the photon-baryon fluid. The ratio of these known quantities constraints the angular diameter distance at \( z = 1100 \) and hence the total energy density. Very early results from BOOMERANG (Netterfield et al., 2002) suggested \( \Omega_{\text{tot}} \approx 1 \). This measurement provided a new degeneracy direction in the \( \Omega_m - \Omega_\Lambda \) parameter space, this resulted in significant improvements of \( \Omega_\Lambda \). The agreement between these separate probes and the degeneracy breaking is shown in Figure 1.4.
1.5 Probing the Growth of Structure

There are two basic types of cosmological probes: \textit{geometric probes} which measure the background expansion history of the universe, and \textit{dynamical probes} which measure the growth of inhomogeneous structure in the universe. In this section we describe two examples of the latter, \textit{peculiar velocities} and \textit{redshift space distortions}. Moreover, we outline our motivations for adopting these particular probes for the analysis presented in Chapter 2 and Chapter 3.

A measured deviation from the standard model predictions could provide a significant clue as to the origin of dark energy. There are two clear approaches to search for deviations. Either reduce the statistical uncertainty on the measurement in question, or test a new prediction of the standard model. Given focus has historically been placed on measuring the expansion history using \textit{geometric probes}, \textit{dynamical probes} offer a relatively new way to test the predictions from the standard model. For recent measurements of the growth rate see Beutler et al. (2014); Samushia et al. (2014); Blake et al. (2011a). Moreover, generically modified gravity models can mimic the $\Lambda$CDM expansion history, but at the cost of changing how structure...
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Figure 1.4: Constraints on the dark matter energy density and the dark energy density from SN, BAOs, and the CMB. Figure taken from Suzuki et al. (2012)
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grows (Linder, 2005; Linder & Cahn, 2007; Huterer & Linder, 2007). *Dynamical probes* are therefore necessary to avoid this degeneracy.

A key variable to describe the inhomogeneous universe is the growth rate of structure, which quantifies the rate at which structure is growing:

\[ f(z) \equiv \frac{d \ln(D(z))}{d \ln(a)} \]  (1.20)

Where \( D(z) \) is the linear growth function, which quantifies the change in amplitude of density perturbations as a function for time, viz., \( \delta(a) = D(a)\delta(a = 1) \). The growth rate is the variable we aim to measure with the cosmological probes introduced in the following two sections. The growth rate is a particularly interesting quantity to measure as it can effectively distinguish between different modified gravity models. We illustrate this point in Fig 1.5.

1.5.1 Peculiar Velocities

The physical velocity of an observer, in terms of physical coordinates \( x^{\text{phys}} \), can be written as

\[ v^i_{\text{phys}} = \frac{dx^i_{\text{phys}}}{dt} = v^i_{\text{pec}} + Hx^i_{\text{phys}}. \]  (1.21)

So the total velocity of an object can be decomposed into a *peculiar velocity* \( v^i_{\text{pec}} \) and the motion induced by the Hubble expansion \( Hx^i_{\text{phys}} \). *Peculiar velocities*, therefore, represent a departure from a uniform and isotropic Hubble expansion. The peculiar motions of galaxies are induced by local density perturbations, where, for example, a galaxy is gravitationally attracted towards a local supercluster. Therefore, by measuring the local velocity field we can learn about the inhomogeneous universe (Kaiser, 1988; Strauss & Willick, 1995).

Only the line-of-sight direction of a galaxy’s peculiar velocity (\( S \)) changes the observed redshift; therefore, this is the only component of the velocity field which...

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is observable. The line of sight velocity is given by

\[ S \equiv v^i_{\text{pec}} \cdot \hat{r} = cz - H_0 r \]  \hspace{1cm} (1.22)

Where \( z \) is the redshift measured in the co-moving (or CMB) frame. \( \hat{r} \) is a unit vector pointing towards the galaxy, and \( r \) is the total distance to the galaxy, \( r = |\mathbf{x}_{\text{phys}}| \).

The total perturbation to the Hubble expansion by peculiar motion is small and decreases with redshift. To observe this trend, note the average peculiar velocity is \( \sim 300 \text{ km/s} \); therefore, at a redshift of \( z = 0.06 \) (0.1) the ratio of peculiar motion to Hubble flow is \( \sim 0.015 \) (0.01).

Using Standard ruler or Standard candle techniques we are able to infer the physical (or, redshift independent) distance to nearby galaxies (\( r \)). And following Eqn. (1.22), by combining this distance estimate with a galaxy’s redshift we can infer the line-of-sight peculiar velocity. Typically, to determine redshift independent distance estimates galaxy scaling relations are used. Two key examples of scaling relations are the Fundamental Plane and the Tully-Fisher relation, for recent applications see (Colless et al., 2001; Springob et al., 2007; Magoulas et al., 2010). A common feature of these methods is that the error on the distance scales linearly with distance, because of this peculiar velocity surveys are restricted to the low-redshift universe, i.e., \( z < 0.1 \).

The relationship between the velocity and density field in configuration space is given by

\[ \mathbf{v}(\mathbf{r}) = \frac{H_0 a f(a)}{4\pi} \int d^3 r' \frac{\delta(\mathbf{r}') (\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3}. \] \hspace{1cm} (1.23)

This equation highlights two reasons we are motivated to use velocities as a cosmological probe, both of which will also be emphasized elsewhere. Firstly, peculiar velocities are sensitive to very large distance scales in the local universe. Additionally, peculiar velocities directly probe the underlying dark matter density field, and hence are not dependent on modeling galaxy bias.

In Chapter 2 we present an analysis of the 6-degree Field Galaxy peculiar velocity survey. This sample contains \( \sim 9000 \) galaxies each with a measured peculiar velocity

\(^2\)Note, this expression is a common approximation to the full relation, viz., \((1 + z) = (1 + v_{\text{pec}}/c)(1 + H_0 r/c)\).
velocity. By analyzing the distribution of peculiar velocities in this sample we derive constraints on the growth rate of structure.

1.5.2 Redshift-Space Distortions

In galaxy spectroscopic surveys one infers the distance to galaxies in the sample using their redshifts. This conversion is done using Hubble’s law, \( z \approx \frac{H_d}{c} + v_{pec} \cdot \hat{r} \). Note, the redshift of a galaxy includes a contribution from the line-of-sight peculiar velocity, so the distance estimates will be slightly shifted (or distorted) from the galaxy’s true distance. We label the measured distance to a galaxy its redshift-space distance \( s \) and the physical distance to a galaxy its real-space distance \( r \).

This distortion of galaxy positions changes how clustered galaxies appear (Kaiser, 1987; Hamilton, 1998). Therefore, by measuring the difference between galaxy clustering in real-space and redshift-space one can measure statistical properties of the peculiar velocity field. On large scales this signal is dominated by the infall of galaxies into galaxy clusters, this provides us with a measurement of the growth rate \( f(a) \). In Chapter (3) we use Redshift-Space Distortions to constrain modified gravity models. As an introduction to this content, below we outline the theory behind redshift-space distortions on linear scales, as developed by Kaiser (1987); Hamilton (1998).

The relation between the real-space and redshift-space distance is given by

\[
\begin{align*}
    s &= r + \frac{v(r) \cdot \hat{z}}{aH(z)} = r + U_z(r) \cdot \hat{z}.
\end{align*}
\]  

(1.24)

Where we have defined \( U_z(r) \equiv \frac{v(r)}{aH(z)} \). For this derivation we will keep the line of sight component fixed for all galaxies in the survey: this is known as the ‘plane-parallel’ approximation. The Jacobian for the real to redshift space mapping is

\[
\begin{align*}
    \frac{ds^3}{dr^3} &= \left(1 + \frac{U_z}{z}\right) \left(1 + \frac{dU_z(r)}{dz}\right) 
\end{align*}
\]  

(1.25)

To simply this expression we only consider scales much smaller than the mean distance to the pair, because of this we can neglect the term \( U_z/z \) (see, Pápai & Szapudi, 2008).
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Using this Jacobian we want to derive a relation between the real-space overdensity \( \delta_r \) and the redshift-space overdensity \( \delta_s \). Because the total mass will be conserved in this transformation we know \((1 + \delta_s)ds^3 = (1 + \delta_r)dr^3\). Combining this relation with Eqn. (1.25) one finds

\[
\delta_s = \delta_r \left[ 1 + \frac{dU_z(r)}{dz} \right]. \tag{1.26}
\]

We can further simplify this expression by assuming the velocity field is irrotational\(^3\). With this we can express the velocity field as a gradient of a scalar field, in particular, we can write \(dU_z/dz = d/dz \nabla^{-2} \theta\), where \(\theta\) is the velocity divergence \((\theta = \nabla \cdot U)\) and \(\nabla\) is the Laplacian operator. Now converting Eqn. (1.26) to Fourier space we find

\[
\delta_s(k) = \delta_r(k) + \mu^2 \theta(k) = \delta_r(k)(1 + f \mu^2) \tag{1.27}
\]

Where \(\mu\) is the cosine of the line-of-sight angle. The final expression above is derived using the linear continuity equation, which implies \(\theta = f \delta_r\). Finally, with the above relation and when assuming a linear bias, we can relate the real-space matter power spectrum \(P_m(k)\) to the redshift-space galaxy power spectrum \(P^*(k)\)

\[
P^*(k, \mu) = b^2 \left( 1 + \frac{f}{b} \mu^2 \right)^2 P_m(k). \tag{1.28}
\]

Therefore, redshift-space distortions enhance clustering along our line of sight, and leave clustering unchanged perpendicular to our line of sight. Given this angular dependence, measuring \(P^*(k)\) as a function of angle provides a convenient way to extract information on the growth rate. We expand on this process in Chapter 3.

1.5.3 Weak Gravitational Lensing

The images we observe of distant galaxies are subtly distorted. This distortion changes both the apparent shape and size of galaxies by of order \(\sim 1\%\). For a given patch of sky, we can describe the changed appearances of galaxies in terms of a unique mapping from the intrinsic images to the apparent images, we label

\(^3\)This is a good assumption, see Percival & White (2009)
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this mapping a distortion field. This field is induced by the gravitational effects of distribution of matter along the line of sight. And, as we will show, by measuring specific components of this field one can infer the underlying matter distribution.

The foreground matter distribution distorts the image of background galaxies by inducing metric fluctuations: fluctuations in space-time change the path of null geodesics and this results in an increase or decrease in the galaxy’s observed size and an anisotropic stretching of the image. The lensing signal is generated by the underlying matter distribution, therefore it is not dependent on galaxy bias. Thus, lensing avoids a significant modelling challenge present in many alternative cosmological probes.

The effects of weak lensing can be described in terms of a deflection angle $\alpha(\theta, D_C)$. This angle quantifies the net deflection of a light ray relative to a fiducial (unperturbed) light ray, it is a function of both the propagation direction $\theta$ and the co-moving distance $D_C$. The deflection angle can be calculated as (Bartelmann & Schneider, 2001)

$$\alpha(\theta, D_C) = \frac{2}{c^2} \int dD'_C \frac{D_A(D_C - D'_C)}{D_A(D_C)} \nabla_\perp \Phi[D_A(D_C)\theta, \chi'] .$$

Where $D_A$ is the angular diameter distance, $\Phi$ is the spatial metric perturbation—the sole gravitational degree of freedom within GR—and $c$ is the speed of light. Because we do not know the intrinsic shape of each galaxy we cannot measure absolute deflection angles, viz., $\alpha$. However, we can measure the gradient of the deflection angle over the sky, we label this observable scalar quantity the convergence:

$$\kappa(\theta, D_C) = \frac{1}{2} \nabla \cdot \alpha(\theta, D_C) .$$

Using Eqn. (1.29) and the Poisson equation, i.e.,

$$\nabla^2 \Phi = \frac{3H_0^2\Omega_m}{2a} \delta ,$$

In this section, for simplicity, we will focus on deriving statistics for the shear field in harmonic space, rather than real-space. Although, note most observational studies are done in real space, because it has advantages when dealing with data.
the convergence can be written in terms of an integral over the density perturbation field \( \delta \), specifically,

\[
\kappa(\theta, D_C)_{\text{eff}} = \frac{3H_0^2\Omega_m}{2c^2} \int dD_C' \frac{D_A(D_C - D_C')}{D_A(D_C)} \delta[D_A(D_C)\theta, \chi].
\]

This equation tells us the convergence field induced by sources at a constant redshift, corresponding to a distance \( D_C \). And yet, in cosmological applications we are interested in a source population that is distributed over a range of redshifts. To include this we average Eqn. (1.32) over the normalized source redshift distribution, \( P(D_C) \):

\[
\tilde{\kappa}(\theta)_{\text{eff}} = \int dD_C P(D_C)\kappa(\theta, D_C)_{\text{eff}}.
\]

To facilitate comparison with theory, which predicts statistical quantities, the two-point correlation function of the convergence field is taken. In Fourier space

\[
\langle \tilde{\kappa}(\ell)\tilde{\kappa}(\ell') \rangle = (2\pi)^2 \delta_D(\ell - \ell')P_\kappa(\ell),
\]

where \( \delta_D \) is the Dirac delta function. And \( \tilde{\kappa} \) is the Fourier transform of the convergence \( \kappa(\theta) \), it is a function of the 2D wave vector \( \ell \). Finally, \( P_\kappa(\ell) \) is the convergence power spectrum. Combining the above equations one finds

\[
P_\kappa(\ell) = \frac{9}{4}\Omega_m^2 \left( \frac{H_0}{c} \right)^4 \int dD_C \frac{[q(D_C)D_A(D_C)]^2}{a^2(D_C)} P_\delta \left( k = \frac{\ell}{D_A(D_C)}, D_C \right).
\]

We define \( q(D_C) \) as the lensing efficiency given by

\[
q(D_C) = \int dD_C' P(D_C') \frac{D_A(D_C' - D_C)}{D_C'D_C}.
\]

The convergence \( \kappa \) changes the size of galaxies. However, there is no standard galaxy size, so this measure is not used for weak lensing analysis. Rather, the ellipticity is used, because on average galaxies are spherical. The variation in ellipticity is described by the shear statistic \( \gamma \) (Bartelmann & Schneider, 2001). Fortunately, the power spectra of the convergence and shear fields are equal, i.e., \( P_\kappa = P_\gamma \). Thus, modelling the former (using Eqn. 1.35) we can predict the latter.
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By measuring the shear power spectrum we can extract cosmological information. Yet, to interpret these measurements we must be able to model them. Therefore, from Eqn. (1.35), we need to understand the matter density field (via. \( P_s(k) \)), the distance-redshift relation (via. \( q(D_C) \)), and the source redshift distribution (via. \( P(D_C) \)). To this end, in Chapter 4 we discuss a novel approach to determine the source galaxy redshift distribution.

1.6 Overview and Motivation

In this section we present an overview of the chapters that follow, in addition to a more detailed introduction for each chapter. This section is intended to build on the material already introduced. Note, some overlap exists between these sub-sections, which we retain to keep each self-contained.

1.6.1 Chapter 2

Recent interest in peculiar velocity (PV) surveys has been driven by the results of Watkins, Feldman & Hudson (2009), which suggest that the local ‘bulk flow’ (i.e. the dipole moment) of the PV field is inconsistent with the predictions of the standard ΛCDM model; other studies have revealed a bulk flow more consistent with the standard model (Ma & Scott, 2013). PV studies were a very active field of cosmology in the 1990s as reviewed by Strauss & Willick (1995) and Kaiser (1988). Separate to the measurement of the bulk flow of local galaxies, a number of previous studies have focused on extracting a measurement of the matter power spectrum in \( k \)-dependent bins (for example see, Freudling et al., 1999; Zaroubi et al., 2001; Silberman et al., 2001; Macaulay et al., 2012). This quantity is closely related to the velocity power spectrum. Other studies have focused on directly constraining standard cosmological parameters (Gordon, Land & Slosar, 2007; Abate & Erdoğan, 2009).

In Chapter 2 we present cosmological constraints from the peculiar velocity subsample of the 6-degree Field Galaxy survey, henceforth 6dFGSv (Springob et al., 2014). This survey represents a significant step forward in peculiar velocity science,
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in terms of both the volume probed and the total number of galaxies. Exploiting the
potential of this survey is a key point of motivation behind this work. The 6dFGSv
sample contains redshifts and velocity measurements for \( \sim 9000 \) galaxies, which are
distributed over most of the southern sky to a depth of \( z \leq 0.055 \): Prior to 6dFGSv,
the largest velocity survey was SFI++ (Springob et al., 2007) which consists of
\( \sim 3000 \) galaxies. The most recent PV survey to be completed is Cosmicflows-2
(Tully et al., 2013). By combining many PV surveys and presenting new distance
estimates, this sample contains \( \sim 8000 \) galaxies.

Furthermore, in this chapter we concentrate on improving upon the treatment of
systematics— in particular, zero-points, and the gaussianity of errors—and the mod-
eelling of the local velocity field. These methodological improvements will be highly
relevant for upcoming peculiar velocity surveys. In particular, we are motivated by
the potential applications to the TAIPAN survey (Colless, Beutler & Blake, 2013),
which will significantly expand on 6dFGSv—we will discuss this survey in detail in
the concluding Chapter.

The quantity we can directly measure from the 2-point statistics of PV surveys
is the velocity divergence power spectrum$^5$. The amplitude of the velocity divergence
power spectrum depends on the rate at which structure grows and can therefore be
used to test modified gravity models, which have been shown to cause prominent
distortions in this measure relative to the matter power spectrum (Jennings et al.,
2012). In addition, by measuring the velocity power spectrum we are able to place
constraints on cosmological parameters such as \( \sigma_8 \) and \( \Omega_m \) (the r.m.s of density fluctua-
tions, at linear order, in spheres of comoving radius \( 8h^{-1}\text{Mpc} \); and the fractional
matter density at \( z = 0 \) respectively). Such constraints provide an interesting consis-
tency check of the standard model, as the constraint on \( \sigma_8 \) measured from the
CMB requires extrapolation from the very high redshift universe.

The growth rate of structure \( f(k,a) \) describes the rate at which density pertur-
bations grow by gravitational amplification. It is generically a function of the
cosmic scale factor \( a \), the comoving wavenumber \( k \) and the growth factor \( D(k,a) \);
expressed as \( f(k,a) \equiv d \ln D(k,a)/d \ln a \). We define \( \delta(k,a) \equiv \rho(k,a)/\bar{\rho}(a) - 1 \), as

$^5$Note in this analysis we will constrain the ‘velocity power spectrum’ which we define as a
rescaling of the more conventional velocity divergence power spectrum (see Section 2.3).
the fractional matter over-density and $D(k,a) \equiv \delta(k,a)/\delta(k,a=1)$. The temporal dependence of the growth rate has been readily measured (up to $z \sim 0.9$) by galaxy surveys using redshift-space distortion measurements (Beutler et al., 2014; de la Torre et al., 2013), while the spatial dependence is currently only weakly constrained$^6$, particularly on large spatial scales (Bean & Tangmatitham, 2010; Daniel & Linder, 2013). The observations are in fact sensitive to the ‘normalized growth rate’ $f(k,z)\sigma_8(z)$, which we will write as $f\sigma_8(k,z) \equiv f(k,z)\sigma_8(z)$. Recent interest in the measurement of the growth rate has been driven by the lack of constraining power of geometric probes on modified gravity models, which can generically reproduce a given expansion history (given extra degrees of freedom). Therefore, by combining measurements of geometric and dynamical probes strong constraints can be placed on modified gravity models (Linder, 2005).

A characteristic prediction of GR is a scale-independent growth rate, while modified gravity models commonly induce a scale-dependence in the growth rate. For $f(R)$ theories of gravity this transition regime is determined by the Compton wavelength scale of the extra scalar degree of freedom (for recent reviews of modified gravity models see Clifton et al., 2012; Tsujikawa, 2010). Furthermore, clustering of the dark energy can introduce a scale-dependence in the growth rate (Parfrey, Hui & Sheth, 2011). Such properties arise in scalar field models of dark energy such as quintessence and k-essence (Caldwell, Dave & Steinhardt, 1998; Armendariz-Picon, Mukhanov & Steinhardt, 2000). The dark energy fluid is typically characterised by the effective sound speed $c_s$ and the transition regime between clustered and smooth dark energy is determined by the sound horizon (Hu & Scranton, 2004). The clustering of dark energy acts as a source for gravitational potential wells; therefore one finds the growth rate enhanced on scales above the sound horizon. In quintessence models $c_s^2 = 1$; therefore the sound horizon is equal to the particle horizon and the effect of this transition is not measurable. Nevertheless, in models with a smaller sound speed ($c_s^2 \ll 1$) such as k-essential models, this transition may have detectable effects$^7$.

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$^6$A scale dependent growth rate can be indirectly tested using the influence the growth rate has on the halo bias e.g. Parfrey, Hui & Sheth (2011).

$^7$The presence of dark energy clustering requires some deviation from $w = -1$ in the low redshift universe.
Motivated by these arguments we introduce a method to measure the scale-dependence of the growth rate of structure using PV surveys. Observations from PVs are unique in this respect as they allow constraints on the growth rate on scales inaccessible to RSD measurements. This sensitivity is a result of the relation between velocity and density modes \( v(k, z) \sim \delta(k, z)/k \) which one finds in Fourier space at linear order (Dodelson, 2003). The extra factor of \( 1/k \) gives additional weight to velocities for larger-scale modes relative to the density field. A further advantage arises because of the low redshift of peculiar velocity surveys, namely that the Alcock-Paczynski effect – transforming the true observables (angles and redshifts) to comoving distances – only generates a very weak model dependence.

A potential issue when modelling the velocity power spectrum is that it is known to depart from linear evolution at a larger scale than the density power spectrum (Scoccimarro, 2004; Jennings, Baugh & Pascoli, 2011). We pay particular attention to modelling the non-linear velocity field using two loop multi-point propagators (Bernardeau, Crocce & Scoccimarro, 2008). Additionally, we suppress non-linear contributions by smoothing the velocity field using a gridding procedure. Using numerical N-body simulations we validate that our constraints contain no significant bias from non-linear effects.

### 1.6.2 Chapter 3

The observation of an accelerating cosmic expansion rate has likely provided an essential clue for advancing our theories of gravitation and particle physics (Witten, 2001). Interpreting and understanding this feature of our Universe will require both observational and theoretical advancement. Observationally it is critical that we both scrutinise the standard vacuum energy interpretation and thoroughly search for unexpected features resulting from exotic physics. Such features may exist hidden within the clustering patterns of galaxies, the coherent distortion of distant light rays, and the local motion of galaxies; searching for these features is the goal we pursue in this Chapter.

Either outcome will facilitate progress: failure to detect unexpected features, confirming a truly constant vacuum energy, will give credence to anthropic argu-
ments formulated within String Theory (Susskind, 2003). New observational signatures should then be targeted (e.g., Bousso, Harlow & Senatore, 2015). Alternatively, an observed deviation from a cosmological constant would indicate a new dynamical dark energy component or a modification to Einstein’s field equations (Clifton et al., 2012; Copeland, Sami & Tsujikawa, 2006). Independent of observational progress, historical trends in science may offer an independent tool to predict the fruitfulness of each interpretation (Lahav & Massimi, 2014).

The possibility of new physics explaining the accelerating expansion has inspired an impressive range of alternative models. As such, a detected deviation from the standard model will not present a clear direction forwards, that is, interpreting such a deviation will be problematic. One potential solution, which we adopt, is to analyse observations within a phenomenological model that captures the dynamics of a large range of physical models (e.g., Bean & Tangmatitham, 2010; Daniel et al., 2010; Simpson et al., 2013). It should be noted that not all approaches that introduce modified gravity or dark energy invoke an artificial separation between the cosmological constant problem and the problem of an accelerating expansion (e.g., Copeland, Padilla & Saffin, 2012).

To characterise the usefulness of phenomenological models we consider their ability to describe known physical models: namely, their commensurability (Kuhn, 1970). This property can be understood as describing the degree to which measurements made in one model can be applied to others. The absence of this property implies that a measurement should only be interpreted in terms of the adopted model: a consistency test. Whereas given this property one can constrain a range of models simultaneously, alleviating the problem of having to re-analyse each model separately.

Specifically, the model we adopt allows extensions to the standard $\Lambda$CDM model by introducing general time- and scale-dependent modifications ($G_{\text{light}}$ and $G_{\text{matter}}$) to General Relativity (Daniel et al., 2010): these parameters vary the relationship between the metric and density perturbations (i.e., they act as effective gravitational coupling). In this case, the equivalence between the spatial and temporal metric perturbations is not imposed. The commensurability of our model to others can then be shown by proving that $G_{\text{light}}$ and $G_{\text{matter}}$ capture all the new
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physics in specific modified gravity scenarios.

For example, de Felice, Kase & Tsujikawa (2011) show that by introducing parameters equivalent to $G_{\text{light}}$ and $G_{\text{matter}}$ one can provide an effective description of the entire Horndeski class of models. Importantly, the Horndeski class of models contains the majority of the viable Dark Energy (DE) and modified gravity (MG) models (Silvestri, Pogosian & Buniy, 2013; Deffayet et al., 2011). An often disregarded caveat is that the mappings between these gravitational parameters and MG and DE theories are only derived at linear order. Therefore, until proved otherwise, the ability of the phenomenological models to describe physical models is lost when using observations influenced by non-linear physics. To avoid this reduction in applicability we will focus on observations in the linear regime. We note this point has been emphasized elsewhere by, for example, Linder & Cahn (2007) and Samushia et al. (2014).

In pursuit of deviations from the standard model we use a range of cosmological observations. In particular, two dynamical probes will be emphasised: the galaxy multipole power spectrum and velocity power spectrum (for example, Beutler et al., 2014; Johnson et al., 2014). Hitherto, in the context of phenomenological models with scale-dependence, neither probe has been analysed self-consistently. In addition we utilize the following cosmological probes: baryon acoustic oscillations, Type Ia SNe, the cosmic microwave background (CMB), lensing of the CMB, and temperature-galaxy cross-correlation (this correlation is caused by the Integrated Sachs–Wolf effect).

We adopt this combination of probes, direct peculiar velocities (PVs) and redshift-space distortions (RSDs), to maximise our sensitivity to a range of length scales. This range is extended as the sensitivity of both measurements is relatively localised at different length scales: redshift-space distortions at small scales, and peculiar velocity measurement at large scales (Dodelson, 2003). The benefit is an increased sensitivity to scale-dependent modifications. The properties of, and physical motivations for, scale-dependent modifications to GR are discussed by Silvestri, Pogosian & Buniy (2013), and Baker et al. (2014).

Our motivation for the work in Chapter 3 is two-fold: Observationally, we aim to improve upon current measurements of deviations to GR. To this end, as discussed
above, we use a larger range of probes than previous analysis. Theoretically, we want to re-emphasize issues that arise applying phenomenological models to non-linear scales. We then discuss how to limit analysis to linear scales, circumventing this problem.

1.6.3 Chapter 4

The gravitational potential wells generated by large-scale structure induce coherent distortions in the shapes of background galaxies. By studying these distortions—known as weak gravitational lensing—we can probe both the underlying matter distribution and the geometry of the universe. As such, weak lensing is a unique probe of the physics of the evolution of the Universe, in particular, gravitational physics and Dark energy. For recent reviews of weak lensing we refer the reader to Kilbinger (2015) and Weinberg et al. (2013).

Galaxy weak lensing studies require deep wide-field optical surveys: galaxy shear measurements are noisy, hence for an accurate measurement one needs to average over a large number of galaxies; moreover, to avoid cosmic variance dominating the error budget a large area is needed. An early and successful example is the CFHT Legacy Survey. CFHTLS completed observations in 2009 and using the MegaCam instrument mapped 170 deg$^2$ to $\sim 25$ mag (for the results see, Heymans et al., 2013, 2012; Simpson et al., 2013; Erben et al., 2013). Ongoing imaging surveys, expanding on CFHTLS, include the following: The Kilo-degree Survey\(^8\) (KIDS), which aims to map out 1500 deg$^2$ with $ugri$ bands; the HyperSuprime cam Survey\(^9\) (HSC survey), which is expected to map 2,000 deg$^2$ down to $i \sim 26$; and, the Dark Energy Survey\(^10\) (DES), $\sim 5000$ deg$^2$ to $\sim 24$th magnitude in the $grizY$ bands.

While there is much promise for weak lensing many challenges remain. The most significant of these challenges can be divided into four categories, these involve both the extraction and interpretation of the lensing signal.

First, **baryonic and non-linear physics**: The weak lensing signal is generated by density perturbations on very small, and hence very non-linear, scales ($\sim 10$\(^8\) m).
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kpc). On such scales perturbation theory approaches break down, and baryons can significantly influence the density field. **Intrinsic alignments:** The gravitational lensing signal is extracted from the coherent alignment of nearby galaxies. However, tidal forces create an intrinsic signal by aligning nearby galaxies. To isolate the cosmological signal a model for the intrinsic signal is needed. **Shape measurement:** To measure the alignment of galaxies we need to measure their shapes. This is complicated by many factors distorting the shape, the most significant being the point-spread function. Finally, **redshift information:** the amplitude of the lensing signal is dependent on the redshift distribution of the lensed galaxies. And yet, we only have photometric information for each galaxy (full spectroscopic follow up is infeasible).

Chapter 4 concerns the final problem: the inference of the source redshift distribution. To consider a few recent approaches, the CFHTLenS survey (Hildebrandt et al., 2012) used the template based code BPZ (Benítez, 2000) to infer redshift probability distributions, and KiDS (de Jong et al., 2015) adopt a similar template based approach. Alternatively, instead of concentrating on a single algorithm, DES compared the accuracy of a number of different methods (see, Sánchez et al., 2014; Bonnett et al., 2015). Based on the available training data, empirical approaches—in particular, Neural Networks and Random Forests—performed the best, with an estimated scatter of $\sigma_z \sim 0.08$

Both template and empirical machine leaning methods fall into the category of **direct calibration.** This approach proceeds as follows. Spectra are taken of a subsample of the full photometric sample. And using the galaxies with both spectral and photometric information, a mapping is derived from color-magnitude space to redshift space, i.e., to a **photometric redshift** estimate.

One significant limitation of **direct calibration** is that the distributions in color-magnitude space of the spectroscopic sample and full sample need to match (i.e., the completeness of the spec-z sample needs to be very high). Failing this requirement, systematic biases may be introduced into the photometric redshift estimates (see, Newman et al., 2013a; Schmidt et al., 2014; Cunha et al., 2014; Sadeh, Abdalla & Lahav, 2015). To achieve a high level of completeness, spectroscopic redshifts for very faint galaxies will be required. This is problematic because spec-z’s are very
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challenging to obtain for faint galaxies\textsuperscript{11}. Note that, as shown in Sánchez et al. (2014) and Bonnett et al. (2015), by weighting the spectroscopic sample one can reduce the required completeness.

As an alternative to \textit{direct calibration}, calibration via cross-correlation was suggested by Newman (2008). In this approach one cross-correlates the photometric lensing sample with an overlapping spectroscopic sample. The amplitude of the correlation signal is then used to reconstruct the redshift distribution of the photometric sample. Importantly, for this method there are no requirements on the overlapping spectroscopic sample, except that it overlaps with the lensing sample. Using cross-correlations therefore avoids the above problem. Motivated by this many cross-correlation based algorithms have been suggested in the literature (e.g., Matthews & Newman, 2010; Ménard et al., 2013; Schulz, 2010; McQuinn & White, 2013).

In Chapter 4 we build on this work. In particular, we develop improvements to the \textit{optimal quadratic estimation} method suggested by McQuinn & White (2013). And we test our new algorithm using a series of mock galaxy catalogs. We do this with the intention to apply this method to future data sets. Specifically, we intend to use our algorithm to infer the distribution of galaxies, in tomographic bins, in the Kilo-degree lensing survey (KiDS) (de Jong et al., 2015; Kuijken et al., 2015) using the 2-degree Field Lensing Survey to trace the surrounding large-scale structure (Blake et al., 2016).

\textsuperscript{11}We significantly expand on this point in Chapter 4.
Chapter 2

Cosmological constraints from the velocity power spectrum

*Johnson et al.*

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Abstract

We present scale-dependent measurements of the normalised growth rate of structure \( f\sigma_S(k, z = 0) \) using only the peculiar motions of galaxies. We use data from the 6-degree Field Galaxy Survey velocity sample (6dFGSv) together with a newly-compiled sample of low-redshift \((z < 0.07)\) type Ia supernovae. We constrain the growth rate in a series of \( \Delta k \sim 0.03h\text{Mpc}^{-1} \) bins to \( \sim 35\% \) precision, including a measurement on scales \( > 300h^{-1}\text{Mpc} \), which represents one of the largest-scale growth rate measurement to date. We find no evidence for a scale dependence in the growth rate, or any statistically significant variation from the growth rate as predicted by the *Planck* cosmology. Bringing all the scales together, we determine the normalised growth rate at \( z = 0 \) to \( \sim 15\% \) in a manner independent of galaxy bias and in excellent agreement with the constraint from the measurements of redshift-space distortions from 6dFGS. We pay particular attention to systematic errors. We point out that the intrinsic scatter present in Fundamental-Plane and Tully-Fisher relations is only Gaussian in logarithmic distance units; wrongly assuming it is Gaussian in linear (velocity) units can bias cosmological constraints. We also analytically marginalise over zero-point errors in distance indicators, validate the accuracy of all our constraints using numerical simulations, and demonstrate how to combine
different (correlated) velocity surveys using a matrix ‘hyper-parameter’ analysis. Current and forthcoming peculiar velocity surveys will allow us to understand in detail the growth of structure in the low-redshift universe, providing strong constraints on the nature of dark energy.

### 2.1 Outline

A flat universe evolved according to the laws of General Relativity (GR), including a cosmological constant $\Lambda$ and structure seeded by nearly scale-invariant Gaussian fluctuations, currently provides an excellent fit to a range of observations: cosmic microwave background data (CMB) (Planck Collaboration et al., 2013), baryon acoustic oscillations (BAO) (Anderson et al., 2014b; Blake et al., 2011b), supernova observations (Betoule et al., 2014), and redshift-space distortion (RSD) measurements (Samushia et al., 2014). While the introduction of a cosmological constant term allows observational concordance by inducing a late-time period of accelerated expansion, its physical origin is currently unknown. The inability to explain the origin of this energy density component strongly suggests that our current understanding of gravitation and particle physics, the foundations of the standard model of cosmology, may be significantly incomplete. Various mechanisms extending the standard model have been suggested to explain this acceleration period such as modifying the Einstein-Hilbert action by e.g. considering a generalised function of the Ricci scalar (Sotiriou & Faraoni, 2010), introducing additional matter components such as quintessence models, and investigating the influence structure has on the large-scale evolution of the universe (Clifton, 2013; Wiltshire, 2013).

Inhomogeneous structures in the late-time universe source gravitational potential wells that induce ‘peculiar velocities’ (PVs) of galaxies, i.e., the velocity of a galaxy relative to the Hubble rest frame. The quantity we measure is the line-of-sight PV, as this component produces Doppler distortions in the observed redshift. Determination of the line-of-sight motion of galaxies requires a redshift-independent distance estimate. Such estimates can be performed using empirical relationships between galaxy properties such as the ‘Fundamental Plane’ or ‘Tully-Fisher’ relation, or one can use ‘standard candles’ such as type Ia supernovae (Colless et al.,
A key benefit of directly analysing PV surveys is that their interpretation is independent of the relation between galaxies and the underlying matter distribution, known as ‘galaxy bias’ (Cole & Kaiser, 1989). The standard assumptions for galaxy bias are that it is local, linear, and deterministic (Fry & Gaztanaga, 1993); such assumptions may break down on small scales and introduce systematic errors in the measurement of cosmological parameters (e.g. Cresswell & Percival, 2009). Similar issues may arise when inferring the matter velocity field from the galaxy velocity field: the galaxy velocity field may not move coherently with the matter distribution, generating a ‘velocity bias’. However such an effect is negligible given current statistical errors (Desjacques et al., 2010).

For our study we use the recently compiled 6dFGSv data set (Springob et al., 2014; Magoulas et al., 2012) along with low-redshift supernovae observations. The 6dFGSv data set represents a significant step forward in peculiar velocity surveys; it is the largest PV sample constructed to date, and it covers nearly the entire southern sky. We improve on the treatment of systematics and the theoretical modelling of the local velocity field, and explore a number of different methods to extract cosmological constraints. We note that the 6dFGSv data set will also allow constraints on the possible self-interaction of dark matter (Linder, 2013), local non-Gaussianity (Ma, Taylor & Scott, 2013), and the Hubble flow variance (Wiltshire et al., 2013).

The structure of this paper is as follows. In Section 2.2 we introduce the PV surveys we analyse; Section 2.3 describes the theory behind the analysis and introduces a number of improvements to the modelling and treatment of systematics effects. We validate our methods using numerical simulations in Section 2.4; the final cosmological constraints are presented in Section 2.5. We give our conclusion in Section 2.6.
2.2 Data & Simulated Catalogues

2.2.1 6dFGS Peculiar Velocity Catalogue

The 6dF Galaxy Survey is a combined redshift and peculiar velocity survey that covers the whole southern sky with the exception of the region within 10 degrees of the Galactic Plane. The survey was performed using the Six-Degree Field (6dF) multi-fibre instrument on the UK Schmidt Telescope from 2001 to 2006. Targets were selected from the K band photometry of the 2MASS Extended Source Catalog (Jarrett et al., 2000). For full details see Jones et al. (2004, 2006, 2009). To create the velocity sub-sample from the full 6dF galaxy sample the following selection requirements were imposed: reliable redshifts (i.e. redshift quality Q = 3 – 5), redshifts less than \( cz < 16120 \, \text{km s}^{-1} \) in the CMB frame, galaxies with early-type spectra, sufficiently high signal-to-noise ratio (\( S/N > 5A^{-1} \)), and velocity dispersions greater than the instrumental resolution limit (\( \sigma_0 \geq 112 \, \text{km s}^{-1} \)).

This sample represents the largest and most uniformly distributed PV survey to date (Fig. 2.1 top panel). The final number of galaxies with measured PVs is 8896 and the average fractional distance error is \( \sigma_d = 26\% \). The redshift distribution for 6dFGSv is given in Fig 2.2. The PVs for 6dFGSv are derived using the Fundamental Plane relation (for details of the calibration of this relation see Magoulas et al., 2010, 2012). The complete 6dFGSv Fundamental Plane catalogue is presented in (Magoulas et al., 2012). Using the fitted Fundamental Plane relation, the final velocity catalogue is constructed in Springob et al. (2014). For each galaxy in the catalogue we determine a probability distribution for the quantity \( \log_{10}(D_z/D_H) \); where \( D_z \) and \( D_H \) are respectively the ‘observed’ comoving distance inferred from the observed redshift and the true comoving distance.

To briefly expand, for each galaxy in the 6dFGSv catalogues the following properties were measured: the effective radius, the central velocity dispersion, and the effective surface brightness, respectively, \( R_e, \sigma_0, \) and \( I_e \). The Fundamental plane is the plane early-type galaxies inhabit in the three-dimensional parameter space defined by \( r = \log(R_e), s = \log(\sigma_0), \) and \( i = \log(I_e) \). For the 6df galaxies, the galaxy velocity dispersion was measured using the width of the spectral lines (specifically, the FWHM) using template
galaxy spectra as a reference point. Furthermore, the velocity dispersion and surface brightness were calculated from the PSF corrected 2-MASS photometry. We refer the reader to Magoulas et al. (2012) for full details.

Intrinsic scatter causes galaxies to deviate from the Fundamental Plane relation. To account for this scatter, the sample of galaxies is fit to a 3-D Gaussian probability distribution, which is defined in terms of a covariance matrix $\Sigma$, which describes the intrinsic scatter of the galaxies from the FP, and an observational error matrix $E$. In total the Gaussian distribution is defined by eight parameters: three describe the central values of the plane, three describe the variance, and two describing the tilt and slope of the plane. For a normalized vector of the observables (i.e., $x = \{r - \bar{r}, s - \bar{s}, i - \bar{i}\}$) the distribution is defined by

$$P(x) = \frac{\exp[-1/2x^t(\Sigma + E)x]}{(2\pi)^{3/2}|\Sigma + E|^{1/2}f}.$$  \hspace{1cm} (2.1)

The free parameters are fixed by finding the maximum likelihood of Eq. (2.1), for the 6dFv galaxies this is done using a gradient based optimization algorithm (Magoulas et al., 2012). The parameter $f$ is a galaxy dependent normalization term, it takes into account the various selection cuts that were placed on the 6dFGSv sample. With the Gaussian distribution fixed, the velocity of local galaxies is derived by computing the difference between the observed angular size of the galaxy and prediction from the FP. For the 6df galaxies this is done in a probabilistic sense because Eq. (2.1) gives the probability of a given effective radius.

### 2.2.2 Low-$z$ SNe catalogue

To extend the velocity sample into the northern hemisphere and cross-check the results for systematic errors, we construct a new homogeneous set of low-redshift Type Ia supernovae. The sample contains SNe with redshifts $z < 0.07$ and the distribution on the sky is given in Fig. 2.1 (lower panel), and the redshift distribution is given in Fig 2.2. The sample contains the following: 40 SNe from the Lick Observatory Supernova Search (LOSS) sample (Ganeshalingam, Li & Filippenko, 2013), analysed using the SALT2 light curve fitter; 128 SNe from Tonry et al.
2.2. DATA & SIMULATED CATALOGUES

(2003); 135 SNe from the ‘Constitution’ set compiled by Hicken et al. (2009), where we choose to use the sample reduced using the multi-color light curve shape method (MLCS) with their mean extinction law described by $R_v = 3.1$; 58 SNe in the Union sample from Kowalski et al. (2008)\textsuperscript{1}; 33 SNe from Kessler et al. (2009), where we use the sample derived using MLCS2k2 with $R_v = 2.18$; and finally 26 SNe are included from the Carnegie Supernova Project (CSP) (Folatelli et al., 2010). Significant overlap exists between the samples, so for SNe with multiple distance modulus estimates we calculate the median value. This approach appears the most conservative given the lack of consensus between light curve reduction methods and the correct value of $R_v$; nevertheless, we find there are no significant systematic offsets between the different reduction methods once we correct for zero-point offsets. The final catalogue consists of 303 SNe with an average fractional distance error, $\sigma_d \sim 5\%$.

We update the redshifts in these samples with the host galaxy redshifts in the CMB frame given in the NASA Extragalactic Database (NED), excluding SNe with unknown host galaxy redshifts; this is necessary as the quoted error in the redshift given for SNe data sets is similar to the typical effect that PVs have on the observed redshift. A number of these data sets include an error component $\sigma_v \sim 300$ km s$^{-1}$ accounting for peculiar motion. Where applicable, we removed in quadrature this error component of $(5/\ln(10))\sigma_v/cz$ from the distance modulus errors. This component is removed so that we can treat the samples uniformly, and in our analysis we treat the velocity dispersion as a free parameter. The estimated intrinsic scatter in absolute magnitude $\sigma_{SNe}$ is included in the error budget in all the samples. We define $\delta m \equiv \mu_{\text{obs}}(z) - \mu_{\text{Fid}}(z)$, where $\mu_{\text{Fid}}$ is the distance modulus calculated in a homogeneous FRW universe at redshift $z$ assuming the fiducial cosmology: $\Omega_b = 0.0489, \Omega_m = 0.3175, n_s = 0.9624, w = -1.0, H_{\text{Fid}} = 67$ km s$^{-1}$Mpc$^{-1}$ (motivated by Planck Collaboration et al., 2013).

For a consistent determination of the line of sight PV, $S$, and the quantity $\delta m$, the value of $H_0$ used to derive the prediction for the fiducial cosmology $\mu_{\text{Fid}}(z)$ needs to be the same as the value assumed during the light curve fitting procedure (where $\mu_{\text{obs}}(z)$ is derived). The authors of different SNe samples have assumed different

\textsuperscript{1}The new union2.1 data set adds no additional low-z SNe.
values of $H_0$ when deriving the distance moduli. Therefore before calculating $\delta m$ and the PV we correct this using $\Delta \mu_i = 5 \ln(H_{0,i}/H_{\text{Fid}})$, where $H_{0,i}$ is the assumed $H_0$ value in the $i^{\text{th}}$ sample and $H_{\text{Fid}}$ is the expansion rate at which we choose to normalise the sample. The assumed value of $H_{\text{Fid}}$ here is simply used because it is a convenient normalization. As $\delta m$ is a ratio of distances it is independent of the assumed value of $H_0$ (the values used to derive both distance moduli simply need to be equivalent).

For the rest of the paper we set $H_0 = 100h$ km s$^{-1}$Mpc$^{-1}$. The line of sight PV is calculated as

$$S = \frac{\ln(10)}{5} \left( 1 - \frac{(1 + z)^2}{H(z) d_L(z)} \right)^{-1} \delta m. \quad (2.2)$$

where $d_L(z)$ is the luminosity distance and $H(z)$ the Hubble expansion rate calculated in the fiducial model at the observed redshift $z$ (the derivation of this equation should be clear from Eq. (2.18)).

2.2.3 Mock Catalogues

We construct two sets of mock catalogues (I) and (II) using the GiggleZ N-body simulation (Poole et al., 2015). The simulation was run inside a periodic box of $1h^{-1}\text{Gpc}$ with $2160^3$ particles of mass $7.5 \times 10^9 h^{-1}M_\odot$. The simulation used the GADGET2 code (Springel, 2005), and haloes and sub-haloes were identified using the Subfind algorithm (Springel et al., 2001). The simulation is run assuming a fiducial cosmology that is specified in Section 2.4. Using the GiggleZ simulations 10 non-overlapping realisations of PV surveys were constructed for both Mock set (I) and (II), with the following properties:

- (I) From each central ‘observer’ a random sample of $\sim 3500$ Dark Matter haloes were selected within $100h^{-1}\text{Mpc}$ from the full sample available in the simulation (i.e., full sky coverage). An uncertainty in the apparent magnitude of $\sigma_{\delta m} \sim 0.1$ was applied to each galaxy. This corresponds to a distance error of $\sigma_d \sim 5\%$ (viz., the approximate distance uncertainty for SNe).

\footnote{In the order that the SNe samples have been introduced the assumed velocity dispersion values are $\sigma_v = [300, 500, 400, 300, 300, 300, 300, 300, 300, 300] \text{km s}^{-1}$ and the assumed values of the Hubble constant are $H_0 = [70, 65, 65, 70, 65, 65, 72] \text{km s}^{-1}\text{Mpc}^{-1}$.}
Figure 2.1: Mollweide projection of the 6dFGSv sample (upper) and the low-$z$ SNe sample (lower) given in right ascension (RA) and declination (Dec) coordinates. We grid the RA and Dec coordinates onto a $25 \times 25$ grid for the upper plot and a $20 \times 20$ grid for the lower plot. The colour of each cell indicates the number of galaxies with measured PVs in that cell; as given by the colour bars on the right.
CHAPTER 2. COSMOLOGICAL CONSTRAINTS FROM THE VELOCITY POWER SPECTRUM

Figure 2.2: The redshift distribution for both the 6dFGSv and low-z SNe PV catalogues. Here we have scaled up the number count for the SNe sample in each redshift bin by a factor of 10 in order to allow the two distributions to be overlotted.

- (II) From each central observer $\sim 8000$ Dark Matter haloes within $150 h^{-1}$Mpc were selected from one hemisphere of the sky. An error in the apparent magnitude fluctuation was introduced by interpolating from the observed trend for the 6dFGSv galaxies of $\sigma_{\delta m}$ with redshift. Fitting a simple linear relationship to the 6dFGSv data we find $\sigma_{\delta m} = 0.51 + 2.985z$. The final range of introduced observational uncertainties is $\sigma_{\delta m} \sim [0.5, 0.75]$.

We subsample these haloes randomly from the chosen observer volumes. We limit the size of each hypothetical survey to reduce large scale correlations between the individual realisations, although we expect that the catalogues may still contain residual correlations through being drawn from the same simulation. This situation is more severe for Mock set (II). In general the purpose of mock set (I) is to test the validity of our algorithms, various systematic effects and potential bias from non-linear effects, since the geometry (sky coverage) of the PV survey is not important, at first order, to answer these questions. Mock (II) is used as an approximate realisation of the 6dFGSv survey.

In the mock simulations we apply a perturbation to the PVs that is similar to the scatter induced by observational error. The process proceeds as follows. We place an observer in the simulation box and extract from the simulation the line-of-
sight velocity $S$ and true comoving distance $D_H$ of each surrounding galaxy. These quantities allow us to determine the observed redshift $z_{\text{obs}}$, from $z_{\text{obs}} = (1 + z_H)(1 + S/c) - 1$, and hence the observed redshift-space distance $D_z$. We now calculate the magnitude fluctuation $\delta m = 5 \log_{10} (D_z/D_H)$ and apply an observational Gaussian error, using the standard deviations specified above. We do not attempt to include additional effects such as survey selection functions, which are not required for the analysis described here.

2.3 Theory & new Methodology

Here we discuss a number of issues, including some improvements, in the framework for analysing PV surveys. We pay particular attention to:

- The covariance matrix of the data (Section 2.3.1)
- The effects of non-Gaussian observational errors and the requirement, in order to have Gaussian observational errors, to use an underlying variable that is linearly related to the logarithmic distance ratio (Section 2.3.2)
- The information we can extract from measurements of the local velocity field using 2-point statistics (Section 2.3.3)
- Modelling the velocity power spectrum, including non-linear effects in redshift space (Section 2.3.4)
- Data compression using gridding methods (Section 2.3.5)
- Marginalization of the unknown zero-point (Section 2.3.6)
- Combining different correlated data sets using hyper-parameters (Section 2.3.7)

The goal of this analysis is quantifying and modelling the degree to which PVs fluctuate from one part of the universe relative to other spatially-separated parts. The magnitude of this fluctuation in the PV field is generated by tidal gravitational fields which are in turn generated by the degree of departure from a homogeneous FRW metric and the relationship between density gradients and gravitational fields.
We introduce a method for extracting scale-dependent constraints on the normalised growth rate of structure $f\sigma_8(z, k)$. We emphasise the unique ability of PV measurements to probe the growth rate of structure on scales that are not currently accessible to redshift-space distortion (RSD) measurements; and the complementarity that exists between velocity surveys and RSD measurements in constraining modified gravity theories. Fig. 2.3 (adapted from Lombriser et al. (2012)) shows the various length scales probed by different methods to constrain gravity.

These methods can also be applied to larger upcoming PV surveys, such as the all-sky HI survey (WALLABY), the Taipan Fundamental Plane survey, and the SDSS Fundamental Plane sample (Colless, Beutler & Blake, 2013; Saulder et al., 2013) for which it will become even more crucial to extract unbiased results with accurate error estimates. Furthermore the improvements considered here will be significant for other approaches for extracting information from velocity surveys, for example by using the cross-correlation between density and velocity fields.

### 2.3.1 Velocity covariance matrix

We start with the assumption that the velocity field is well described by a Gaussian random field, with zero mean. Therefore, considering a hypothetical survey of $N$ galaxies each with a measured PV $S(x, t) = v(x, t) \cdot \hat{r}$, one can write down the likelihood for observing this particular field configuration as

$$L = \frac{1}{|2\pi C^{(v)}|^{1/2}} \exp \left( -\frac{1}{2} \sum_{m,n} S_m(x, t) C_{mn}^{(v)} S_n(x, t) \right),$$

(2.3)

where $v(x, t)$ is the total velocity of the object evaluated at the spatial position $x$ and time $t$, and $\hat{r}$ is a unit vector in the direction of the galaxy. The desired (unknown) variable in this equation, which depends on the cosmological model, is the PV covariance matrix. By definition $C_{mn}^{(v)} \equiv \langle S_m(x_m) S_n(x_n) \rangle$. The validity of the assumptions described above will be discussed in later sections. The above approximation to the likelihood yields the probability of the velocity field configuration (the data $d$) given the covariance (as determined by the cosmological model $m$); this quantity is typically denoted $L \equiv P(d|m)$. The quantity we are interested in extracting is the
2.3. THEORY & NEW METHODOLOGY

Figure 2.3: Scales probed by different methods to constrain gravity. The cosmological probes shown in red lines probe gravity by its effect on the propagation of light i.e., weak and strong lensing (such measurements probe the sum of the spatial and temporal gravitational potential). Probes that use dynamical measurements are given as blue lines (these trace the temporal part of the gravitational potential). PVs probe the largest scales of any current probe. Figure adapted from Lombriser et al. (2012).
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probability of the model given our observations of the velocity field, viz. \( P(m|d) \). Bayes’ theorem relates these two quantities as \( P(m|d) = P(d|m)P(m)/P(d) \). \( P(d) \) can be absorbed into a normalization factor and we assume a uniform prior (i.e., \( P(m) = 1 \)), implying \( P(m|d) \propto \mathcal{L} \).

The physical interpretation of the components of the covariance matrix are as follows: The diagonal elements can be viewed as representing cosmic variance (later we add a further diagonal contribution from observational uncertainties and non-linear contributions). As the model cosmology is changed, altering the degree of clustering in the low-redshift universe, the magnitude of cosmic variance changes. The covariance between individual PVs (i.e., the off-diagonal elements) results from those velocities being generated by the same underlying density field. Large wavelength Fourier density modes will have very similar phases for close pairs of galaxies, thus a similar gravitational force will be exerted on these galaxies and therefore their PVs will be correlated.

Hitherto, the covariance matrix \( C^{(v)}_{mn} \) has been calculated in terms of the matter power spectrum, \( P(k) \). We suggest that a more natural approach is to express the covariance matrix in terms of the velocity divergence power spectrum. We define the velocity divergence as \( \theta(x, t) \equiv \nabla \cdot \mathbf{v}(x, t) \), therefore \( \mathbf{v}(k) = -i\theta(k) \frac{\mathbf{k}}{k} \), so the velocity covariance matrix is given by

\[
C^{(v)}_{mn}(x_m, x_n) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x_m} \int \frac{d^3k'}{(2\pi)^3} e^{-ik' \cdot x_n} \frac{\langle \mathbf{x}_m \cdot \mathbf{k} \rangle \langle \mathbf{x}_n \cdot \mathbf{k}' \rangle}{k'^2k^2} \langle \theta(k) \theta^*(k') \rangle \tag{2.4}
\]

The simplification results from \( \langle \theta(k) \theta^*(k') \rangle \equiv (2\pi)^3 \delta^3(k - k')P_{\theta\theta}(k) \), where \( P_{\theta\theta}(k) \) is the power spectrum of \( \theta(x, t) \), evaluated here at a redshift of zero. The advantage of this derivation is that one is not required to assume the linear continuity equation. The angular part of the integral in Eq. (2.4) defines the survey window function, explicitly

\[
W(k, \alpha_{ij}, r_i, r_j) \equiv \int \frac{d\Omega_k}{4\pi} e^{i\mathbf{k} \cdot \mathbf{x}_{i,j}} \langle \mathbf{x}_i \cdot \mathbf{k} \rangle \langle \mathbf{x}_j \cdot \mathbf{k} \rangle \tag{2.5}
\]

The analytic form for Eq. (2.5) is given in the Appendix of Ma, Gordon & Feldman

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Figure 2.4: The window function for five pairs of galaxies in the 6dFGSv galaxy peculiar velocity catalogue. Large scale density fluctuations generate correlations between the PVs of pairs of galaxies, and the window function quantifies the wavelengths of density fluctuations that contribute to a given correlation. Specifically, the parameters input to the above window functions are as follows: for $W^{(a)}$ to $W^{(e)}$ we input $[r_i, r_j, \alpha]^{(a)} = [86.6, 133.7, 0.393]$, $[r_i, r_j, \alpha]^{(b)} = [76.8, 127.6, 1.313]$, $[r_i, r_j, \alpha]^{(c)} = [59.16, 142.5, 0.356]$,$[r_i, r_j, \alpha]^{(d)} = [51.9, 91.1, 0.315]$, and $[r_i, r_j, \alpha]^{(e)} = [99.49, 158.4, 0.463]$. The distances are all given in units of $[h^{-1}\text{Mpc}]$ and angles in radians.
as

\[ W(k, \alpha_{ij}, r_i, r_j) = 1/3 \left[ j_0(kA_{ij}) - 2 j_2(kA_{ij}) \right] \hat{r}_i \cdot \hat{r}_j \]

\[ + \frac{1}{A_{ij}^2} j_2(kA_{ij}) r_i r_j \sin^2(\alpha_{ij}) \]  

(2.6)

where \( \alpha_{ij} = \cos^{-1}(\hat{r}_i \cdot \hat{r}_j) \), \( A_{ij} \equiv |r_i - r_j| \) and \( r_i \) is the position vector of the \( i^{th} \) galaxy. The window function \( W_{ij}(k) \equiv W(k, \alpha_{ij}, r_i, r_j) \) is plotted in Fig (2.4) for a number of galaxy pairs in the 6dFGSv catalogue. For convenience we change the normalisation of the velocity divergence power spectrum and define the ‘velocity power spectrum’ as \( P_{vv}(k) \equiv P_{\phi\theta}(k)/k^2 \). Therefore we have

\[ C_{mn}^{(v)} = \int \frac{dk}{2\pi^2} k^2 P_{vv}(k, a = 1) W(k, \alpha_{mn}, r_m, r_n). \]  

(2.7)

### 2.3.2 The origin of non-Gaussian observational errors

Observations of the Cosmic Microwave Background have shown to a very high degree of accuracy that the initial density fluctuations in the universe are Gaussian in nature, which implies that the initial velocity fluctuations are also well-described by a Gaussian random field. Linear evolution of the velocity field preserves this Gaussianity, as it acts as a simple linear rescaling. This simplifying property of large scale density and velocity fields is often taken advantage of by approximations to the likelihood such as Eq. (2.3), which require that the PV field, \( S_i \), be accurately described by a multivariate Gaussian distribution. Although this is true with regards to cosmic variance, a crucial issue is that the observational uncertainty in PV surveys is often highly non-Gaussian in velocity units. In this section we describe the origin of this non-Gaussian error component, with particular reference to a Fundamental Plane survey; we note our conclusions are equally valid for Tully-Fisher data sets. Furthermore, we propose a solution to this problem and test its validity using numerical simulations in Section 2.4.

The Fundamental Plane relation is defined as \( R_e = \sigma_0^b \langle I_e \rangle \), where \( R_e \) is the effective radius, \( \sigma_0 \) the velocity dispersion and \( \langle I_e \rangle \) the mean surface brightness. In terms of logarithmic quantities it is defined as \( r = as + bi + c \) (\( r \equiv \log_{10}(R_e) \)
and \( i \equiv \log_{10}((L_e)) \) where \( a \) and \( b \) describe the plane slope and \( c \) defines the zero-
point. The Fundamental Plane relation therefore is a simple linear relation when the relevant variables are described in logarithmic units. Within this parameter
space (or, ‘Fundamental Plane space’) a 3D elliptical Gaussian distribution provides a
elegant empirical fit to the observed scatter of the FP variables\(^3\). Changing
the distance measure \( \log_{10}(R_e) \) to a quantity not given in logarithmic units (i.e.,
simply \( R_e \)) one would find that the scatter of the new variables can no longer be
described by a simple Gaussian distribution. This argument can be extended to the
Tully-Fisher relation, as it has intrinsic scatter that appears to be modelled well by
a Gaussian in absolute magnitude units.

As discussed in Springob et al. (2014) the fundamental quantity derived from
the Fundamental Plane relation is the probability of a given ratio between the ob-
served effective radius (observed size) \( R_z \) and the inferred physical radius (physical
size) \( R_H \) of the specific galaxy viz., \( P(\log_{10}(R_z/R_H)) \). In order to find the resulting
probability distributions for peculiar velocities, \( P(v_p) \), in standard units [km s\(^{-1}\)]
from the measured quantity \( P(\log_{10}(R_z/R_H)) \) we need to calculate the Jacobian re-
lating these two quantities. Firstly we can convert the logarithmic ratio of radii to
a logarithmic ratio of comoving distances. Defining \( x = \log_{10}(D_z/D_H) \), one has

\[
P(x) \equiv P(\log_{10}(D_z/D_H)) = J(D_H, z_H)P(\log_{10}(R_z/R_H)).
\]  

(2.8)

The Jacobian term needed to transform the probability distribution from a size ratio
to a distance ratio is approximated by (Springob et al. 2014)

\[
J(D_H, z_H) \approx \left(1 + \frac{99.939D_H + 0.01636D_H^2}{3 \times 10^5(1 + z_H)}\right)
\]  

(2.9)

where \( z_H \) is the Hubble redshift. Any dependence on the assumed cosmology here
will be insignificant given the low redshifts of the observations. The probability
distribution \( P(x) \) is measured for each galaxy of the 6dFGSv survey using Eq. (2.8);

\(^3\)This scatter is generated by the PVs of the galaxies and the intrinsic scatter of the FP relation.
Fig. 4 in Magoulas et al. (2012) shows the scatter of the FP parameters, where one can see the
data is well described by a 3D elliptical Gaussian.
importantly this distribution is very accurately described by a Gaussian distribution. Fig. 2.5 gives some examples for individual galaxies in the 6dFGSv sample.

We can now determine if the transformation from this distribution into the probability distribution for the PV (i.e., \( P(x) \rightarrow P(v_p) \)) preserves the Gaussian nature of the distribution or if it introduce non-Gaussianity. The transformation between these two probability distributions can be accurately approximated by

\[
P(v) = P(x) \frac{dx}{dv} \approx P(x) \frac{(1 + z)^2}{D_H \ln(10) c(1 + z)} \frac{dD_H}{dz_H},
\]

where \( dD_H/dz_H = c/(99.939 + 0.01636D_H)^4 \). Applying this non-linear transformation Eq. (2.10) to the \( P(x) \) distributions given in the 6dFGSv sample we find the resulting velocity probability distributions, \( P(v_p) \), become significantly skewed (as shown in Fig. 2.5) and hence are poorly described by a Gaussian distribution. In Section 2.4 we use numerical N-body simulations to quantify the impact of this non-Gaussianity on cosmological parameter fits, concluding that a measurable bias is introduced. To avoid this problem one is required to adopt a variable for the analysis that is linearly related to the logarithm of the ratio of comoving distances.

### Changing variables

The velocity variable we use is the apparent magnitude fluctuation, defined by \( \delta m(z) = [m(z) - \bar{m}(z)] \), where both quantities are being evaluated at the same redshift (the observed redshift), see Hui & Greene (2006); Davis et al. (2011). So the fluctuation is being evaluated with respect to the expected apparent magnitude in redshift-space. The over-bar here refers to the variable being evaluated within a homogeneous universe, i.e. in a universe with no density gradients and as a result no PVs. Recalling that the apparent magnitude is defined as

\[
m = M + 5 \log_{10}(d_L(z)) + 25
\]  

\(^4 \)This result can be derived from the approximation between comoving distance and redshift given in Hogg (1999), and is valid to < 1% within the range of redshift we are interested in.
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Figure 2.5: Probability distributions for \(x = \log_{10}(D_z/D_H)\) and \(v_p\) for four 6dFGS velocity sample galaxies. We note that the distribution of \(x\) is well-described by a Gaussian, whereas the distribution of \(v_p\) contains significant skewness.

where \(M\) is the absolute magnitude and \(d_L(z)\) is the luminosity distance in Mpc, we find \(\delta m(z) = 5x(z)\). We must now determine the covariance of magnitude fluctuations \(C_{ij}^m \equiv \langle \delta m_i(z_i)\delta m_j(z_j) \rangle\). The full treatment of this problem, which is effectively the derivation of the luminosity distance in a perturbed FRW universe, includes a number of additional physical effects besides peculiar motion that act to alter the luminosity distance, namely: gravitational lensing, the integrated Sachs-Wolfe effect, and gravitational redshift (Bonvin, Durrer & Gasparini, 2006; Pyne & Birkinshaw, 2004). For the relevant redshift range all these additional effects are currently insignificant. Here we focus on an intuitive derivation that captures all the relevant physics.

We first define the fractional perturbation in luminosity distance about a homogeneous universe as \(\delta_{d_L}(z) \equiv [d_L(z) - \bar{d}_L(z)]/\bar{d}_L(z)\) and note from Eq. (3.9) that \(\delta m = (5/\ln 10)\delta_{d_L}\). Therefore the problem is reduced to finding \(C_{ij}^L \equiv \langle \delta_{d_L}(z_i)\delta_{d_L}(z_j) \rangle\). The relationship between the observed flux \(F\) and the intrinsic luminosity \(L\) is given by

\[
F(z) = \frac{L}{4\pi(1+z)^4} \frac{\delta \Omega_9}{\delta A_e},
\]  

(2.12)
where $\delta A_e$ is the proper area of the galaxy (emitter) and $\delta \Omega_0$ is the observed solid angle. The angular diameter distance and the luminosity distance are defined as

$$d_A = \sqrt{\frac{\delta A_e}{\delta \Omega_0}}, \quad d_L = d_A(1 + z)^2,$$

both of which are valid in homogeneous and inhomogeneous universes\(^5\) (J. E. Peebles, 1993). In a homogeneous universe we have

$$\tilde{d}_A(\tilde{z}) = \chi_e/(1 + \tilde{z})$$

$$\chi_e \equiv \chi(\tilde{z}) = c \int_0^{\tilde{z}} \frac{dz'}{H(z')}$$

$$\tilde{d}_L(\tilde{z}) = \tilde{d}_A(\tilde{z})(1 + \tilde{z})^2$$

where $\chi$ is the comoving distance and $H$ is Hubble’s constant. Introducing a PV component into this homogeneous system, i.e. perturbing the system, has two effects (at first order):

- The redshift of the object is perturbed (via the Doppler effect). For small velocities (i.e., $v \ll c$), as is applicable to local motions of galaxies, the relation between the redshift in the homogeneous universe $\tilde{z}$ and the inhomogeneous universe $z$ is given by

$$1 + z = (1 + \tilde{z})(1 + \vec{v}_e \cdot \hat{n} - \vec{v}_0 \cdot \hat{n}),$$

where $\vec{v}_e$ is the emitting galaxy’s velocity, $\vec{v}_0$ is the observer’s velocity relative to the CMB, and $\hat{n}$ is a unit vector in the direction of the emitter from the absorber;

- The angular diameter distance is changed as a result of relativistic beaming. This occurs as the angle of the galaxy is shifted by $\delta \Omega_0 \to \delta \Omega_0(1 - 2\vec{v}_0 \cdot \hat{n})$. The result is

$$d_A(z) = \tilde{d}_A(\tilde{z})(1 + v_0 \cdot \hat{n}).$$

\(^5\)For completeness we note that the term inhomogeneous universe is used somewhat liberally in this section, the term should be taken to refer to a weakly perturbed Friedmann-Lemaître-Robertson-Walker geometry. In the context of general inhomogeneous universes the nature of the luminosity distance relation is unknown in most cases, and other physical contributions may become significant.
Using Eq. (2.13), Eq. (2.15) and Eq. (2.16) the luminosity distance in the perturbed universe is given by

\[
d_L(z) = \delta_L(\bar{z})(1 + 2v_e \cdot \hat{n} - v_0 \cdot \hat{n}).
\] (2.17)

Taylor expanding \( \delta_L(z) \) about \( \bar{z} \) gives (Hui & Greene, 2006)

\[
\delta_{\delta_L}(z) = \frac{\delta d_L}{d_L} = \dot{\delta} \cdot \left( \bar{v}_e - \frac{(1 + z)^2}{H(z)d_L} [\bar{v}_e - \bar{v}_0] \right)
\] (2.18)

where we work in units with \( c = 1 \). This relation is accurate to first order in perturbation theory, ignoring other contributions. Our Galaxy’s motion is very accurately known from observations of the CMB therefore we can transform the observed PV to the CMB rest frame and correct for the effect of \( v_0 \)\(^6\). Given \( \delta m = (5/\ln 10) \delta_{\delta_L} \) and using Eq. (2.7) one finds

\[
C_{ij}^m = \left( \frac{5}{\ln 10} \right)^2 \left( 1 - \frac{(1 + z_i)^2}{H(z_i)d_L(z_i)} \right) \left( 1 - \frac{(1 + z_j)^2}{H(z_j)d_L(z_j)} \right)
\int \frac{dk}{2\pi^2} k^2 P_{vv}(k, a = 1) W(k, \alpha_{ij}, r_i, r_j).
\] (2.19)

In Section 2.3.5 we update the formula for the covariance matrix to account for a smoothing of the velocity field we implement; the updated formula is given in Eq. (2.31).

**Including the intrinsic error**

To complete the covariance matrix of magnitude fluctuations we must add the observational part of the errors, uncorrelated between objects. This has two different components: the error in the measured apparent magnitude fluctuation \( \sigma_{\text{obs}} \) and a stochastic noise contribution \( \sigma_v \), which is physically related to non-linear contributions to the velocity (Silberman et al., 2001). The total magnitude scatter per

\(^6\)We assume that the correlation between ‘our’ motion and nearby galaxies is insignificant (i.e., \( \langle v_e v_0 \rangle = 0 \)). This is justified given we are working in the CMB frame. Any residual correlations when working in this reference frame are introduced by the effects of relativistic beaming which is a function of our local motion.
object is given by

\[ \sigma_i^2 = \sigma_{\text{obs}}^2 + \left( \frac{5}{\ln 10} \right)^2 \left( 1 - \frac{(1 + z_i)^2}{H(z_i) d_L(z_i)} \right)^2 \sigma_v^2, \quad (2.20) \]

The updated posterior distribution is therefore given by

\[ P(\Sigma|\delta \mathbf{m}) = |2\pi \Sigma|^{-1/2} \exp \left( -\frac{1}{2} \delta \mathbf{m}^T \Sigma^{-1} \delta \mathbf{m} \right), \quad (2.21) \]

where

\[ \Sigma_{ij} \equiv C_{ij}^m + \sigma_i^2 \delta_{ij}, \quad (2.22) \]

where \( \delta \mathbf{m} \) is a vector of the observed apparent magnitude fluctuation. For the SNe sample \( \sigma_{\text{obs}} \) represents both the light-curve fitting error and the intrinsic dispersion, as derived by the original SNe analysis. We do not need to vary \( \sigma_{\text{obs}} \) as a free parameter because its effect is degenerate with the contribution from the velocity dispersion, which we allow to vary.

### 2.3.3 Methods to extract information from the local velocity field

The aim of this section is to outline the parametrisations of the velocity covariance matrix (Eq. 2.19) we consider, and hence the type of cosmological models we constrain.

#### Traditional parametrisations

We first discuss two different methods already present in the literature. Both compare data to model by calculating a model-dependent covariance matrix, but they differ in the power spectrum model used to generate that covariance matrix. In the first method power spectra are generated for a range of cosmological models (as described below), while in the second method the power spectra are generated in a single fiducial cosmological model, and then perturbed in a series of Fourier bins. The first method is more easily compared directly to physical models, while
the second allows detection of generic scale-dependent effects.

Within the standard cosmological model the velocity power spectrum $P_{vv}(k)$ can be calculated as a function of the cosmological parameters $(\sigma_8, \Omega_m, \Omega_b, n_s, w, H_0)$. The parameters not previously described are defined as follows: $\Omega_b$ is the baryon density divided by the critical density; $n_s$ describes the slope of the primordial power spectrum; $w$ is the dark energy equation of state; and $H_0$ is the current expansion rate. Current velocity data sets do not contain enough statistical power to constrain all these parameters, therefore we focus on the two most relevant parameters: $\sigma_8$ which describes the overall normalization and $\Omega_m$ which controls the scale-dependence of power. Therefore we fix $(\Omega_b = 0.0489, n_s = 0.9624, w = -1.0, H_0 = 67\text{km s}^{-1}\text{Mpc}^{-1})$ to the best-fitting Planck values (see, Planck Collaboration et al., 2013). Now we can parametrise the velocity power spectrum as $P_{vv}(k) = P_{vv}(k, \Omega_m, \sigma_8)$, and from Eq. (2.19) and Eq. (2.20) we can predict the covariance matrix as a function of these cosmological parameters, $\Sigma = \Sigma(\Omega_m, \sigma_8)$, such that

$$P(\Omega_m, \sigma_s|\delta m) = |2\pi \Sigma(\Omega_m, \sigma_8)|^{-1/2} \exp \left(-\frac{1}{2} \delta m^T \Sigma^{-1}(\Omega_m, \sigma_s) \delta m \right).$$

Note that the quantity $|2\pi \Sigma(\Omega_m, \sigma_8)|$ depends on the cosmological parameters, as a result we do not expect the posterior distributions to be exactly Gaussian. Similar parameterisations were explored by Zaroubi et al. (2001); Zehavi & Dekel (2000); Jaffe & Kaiser (1995).

The second method involves specifying a fiducial velocity power spectrum $P_{vv}^{\text{Fid}}(k)$ which we choose using the current best-fitting Planck constraints, explicitly $(\Omega_m = 0.3175, \sigma_8 = 0.8344, \Omega_b = 0.0489, n_s = 0.9624, w = -1.0, H_0 = 67\text{km s}^{-1}\text{Mpc}^{-1})$. The power spectrum is now separated into bins in Fourier space and a free parameter $A_i$ is introduced and allowed to scale the ‘power’ within the given $k$ range of a bin. One can hence constrain the amplitude of the velocity power spectrum in $k$-dependent bins. This parameterisation is similar in nature to that explored in Macaulay et al. (2012) and Silberman et al. (2001), although the specifics of the implementation are somewhat different. This approach is more model-independent.
than the first parametrisation because it allows more freedom in the shape of the velocity power spectrum. Considering a case with \( N \) different bins, we define the centre of the \( i \)th bin as \( k_i^\text{cen} \) and the bin width as \( \Delta_i \equiv (k_i^\text{max} - k_i^\text{min}) \). We define

\[
\Pi(k, \Delta_i, k_i^\text{cen}) \equiv \mathcal{H}(k - (k_i^\text{cen} - \Delta_i/2)) - \mathcal{H}(k - (k_i^\text{cen} + \Delta_i/2)),
\]

where \( \mathcal{H}(x) \) is a Heaviside step function, so \( \Pi(k, k_i^\text{cen}, \Delta_i) \) is equal to one if \( k \) is in the \( i \)th bin and zero otherwise. Including the free parameters \( A_i \) which scale the amplitude of the velocity power spectrum within each bin, the \textit{scaled} velocity power spectrum is given by

\[
P_{vv}^{\text{Scaled}}(k) = A_1 P_{vv}^{\text{Fid}}(k) \Pi(k, \Delta_1, k_1^\text{cen}) + A_2 P_{vv}^{\text{Fid}}(k) \Pi(k, \Delta_2, k_2^\text{cen}) + \ldots + A_N P_{vv}^{\text{Fid}}(k) \Pi(k, \Delta_N, k_N^\text{cen}).
\]

(2.25)

The free parameters \( A_i \) do not have any \( k \)-dependence, and as a result one finds

\[
\int \frac{dk}{2\pi^2} k^2 P_{vv}^{\text{Scaled}}(k) W(k, \alpha_{12}, r_1, r_2) =
\sum_{i=1}^{N} A_i \int_{k_i^\text{cen} - \Delta_i/2}^{k_i^\text{cen} + \Delta_i/2} \frac{dk}{2\pi^2} k^2 P_{vv}^{\text{Fid}}(k) W(k, \alpha_{12}, r_1, r_2)
\]

so the magnitude covariance matrix for the \textit{scaled} velocity power spectrum is given by

\[
C_{ij}^m (A_1, A_2 \ldots A_N) =
\left( \frac{5}{\ln 10} \right)^2 \left( 1 - \frac{(1 + z_i)^2}{H(z_i)d_L(z_i)} \right) \left( 1 - \frac{(1 + z_j)^2}{H(z_j)d_L(z_j)} \right)
\sum_{i=1}^{N} A_i \int_{k_i^\text{cen} - \Delta_i/2}^{k_i^\text{cen} + \Delta_i/2} \frac{dk}{2\pi^2} k^2 P_{vv}^{\text{Fid}}(k) W(k, \alpha_{i,j}, r_i, r_j).
\]

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From Eq. (3.12) and Eq. (3.11) we then have

\[
P(A_1, A_2, \ldots, A_N | \delta \mathbf{m}) = |2\pi \Sigma(A_1, A_2, \ldots, A_N)|^{-1/2} \exp \left( -\frac{1}{2} \delta \mathbf{m}^T \Sigma^{-1}(A_1, A_2, \ldots, A_N) \delta \mathbf{m} \right).
\]

The best-fitting values \( A_i \) can be used to check the consistency with the fiducial model \( (A_i = 1) \) or to obtain the effective measured power \( P_i \) in each bin:

\[
P_i = A_i \int_{k_i^{\text{cen}} - \Delta_i/2}^{k_i^{\text{cen}} + \Delta_i/2} dk \frac{P_{\text{ss}}(k)}{\Delta_i}.
\]

The \( P_i \) values can now be compared with the predictions of the velocity power spectrum from different cosmological models.

**Scale-dependent growth rate**

We can also relate the measured \( A_i \) values to the growth rate of structure at each scale, as follows.

Here we will assume linear perturbation theory to be valid for both the density and the velocity fields; the justification for this assumption will be given in Section 2.3.5. In this regime the linear continuity equation is valid i.e., \( \theta(k) = -fH \delta(k) \).

These assumptions are required to place constraints on the growth rate, but not required for the previous parametrisations. A shift in \( f(z)\sigma_8(z) \) from the fiducial value to a new value, viz., \( f\sigma_8(z)^{\text{Fid}} \rightarrow f\sigma_8(z) \), has an effect on the velocity divergence power spectrum that can be calculated as \( P_{\theta\theta}(k) \rightarrow A_1 P_{\theta\theta}(k) \), where \( A_1 = \left( f\sigma_8(z)/f\sigma_8(z)^{\text{Fid}} \right)^2 \). One can then write down a ‘scaled’ velocity divergence power spectrum as

\[
P_{\theta\theta}^{\text{Scaled}}(k) \equiv \left( f\sigma_8(z,k_1^{\text{cen}})/f\sigma_8(z)^{\text{Fid}} \right)^2 P_{\theta\theta}^{\text{Fid}}(k) \Pi(k, \Delta_1, k_1^{\text{cen}}) + \left( f\sigma_8(z,k_2^{\text{cen}})/f\sigma_8(z)^{\text{Fid}} \right)^2 P_{\theta\theta}^{\text{Fid}}(k) \Pi(k, \Delta_2, k_1^{\text{cen}}) + \cdots + \left( f\sigma_8(z,k_N^{\text{cen}})/f\sigma_8(z)^{\text{Fid}} \right)^2 P_{\theta\theta}^{\text{Fid}}(k) \Pi(k, \Delta_N, k_N^{\text{cen}}),
\]

(2.28)
where again $P_{\text{scaled}}(k) \equiv P_{\text{scaled}}^*(k)/k^2$, and there are $N$ different bins that span the entire $k$ range. The growth rate is considered to be constant over the wavenumber range of a given bin. The above relation Eq. (2.28) results from the approximation $P_{\theta\theta}(k, z) \propto \sigma_s f(k, z)^2$.

The velocity power spectrum is calculated (at $z = 0$) by assuming the standard $\Lambda$CDM expansion history and that the growth of perturbations is governed by GR. We note that modifying the expansion history and/or deviations from GR at higher redshifts will affect the current growth rate. Therefore in order to consistently examine the possibility of a scale-dependence of the growth rate of structure (i.e., moving beyond a consistency test) such effects would need to be taken into account. Such an approach is beyond the scope of this paper and left for future work; here we simply consider if the observed growth rate as a function of scale is consistent with that expected within the framework of the standard model.

2.3.4 Modelling of the velocity power spectrum

In this section we will outline the model we use for the velocity power spectrum in terms of the cosmological parameters.

We calculate the real-space velocity power spectrum using the code velMPTbreeze (an extension of MPTbreeze in Crocce, Scoccimarro & Bernardeau (2012)), which computes the velocity power spectrum using two loop multi-point propagators (Bernardeau, Crocce & Scoccimarro, 2008) in a similar way to renormalized perturbation theory (RPT) (Crocce & Scoccimarro, 2006). velMPTbreeze uses an effective description of multi-point propagators introduced in Crocce, Scoccimarro & Bernardeau (2012) which significantly reduces computation time relative to other RPT implementations. The results from velMPTbreeze were extensively tested against $N$-body simulations (Crocce and Scoccimarro, in prep).

2.3.5 Reducing non-linear systematics and computation time

The velocity field is directly driven by the tidal gravitational field $\nabla \Phi$, where $\Phi$ is the gravitational potential, which causes it to depart from the linear regime at larger
scales than the density field (Scoccimarro, 2004). While the off-diagonal elements of the covariance matrix Eq. (2.31) are dominated by large-scale modes, as a result of the survey geometry\(^7\), this is not the case for the diagonal (cosmic variance) elements where the small scale power contributes to the intrinsic scatter. Hence non-linear effects are important to consider and minimize.

In order to suppress non-linear contributions and hence reduce potential systematic biases we adopt a simple smoothing (gridding) procedure. Gridding the velocity field significantly reduces the computation time by reducing the size of the covariance matrix; this will be essential for next-generation data sets given the computational demands of the likelihood calculation (which requires a matrix inversion for each likelihood evaluation).

The binning method we implement was developed and tested in Abate et al. (2008). The grid geometry used is a cube of length \(L\), where the average apparent magnitude fluctuation \(\delta m\) and error \(\sigma_{\delta m}\) are evaluated at the centre of the \(i^{th}\) grid cell \(\vec{x}_i\):

\[
\delta m_i(\vec{x}_i) = \frac{1}{N_i} \sum_j \delta m^\text{gal}_j(\vec{x}_j) \Theta_{ij},
\]

\[
\sigma_{\delta m,i} = \frac{1}{N_i^{3/2}} \sum_j \sigma^\text{gal}_{\delta m,j} \Theta_{ij}, \tag{2.29}
\]

where \(N_i\) is the number of galaxies located within the \(i^{th}\) cell, \(\delta m^\text{gal}_j\) is the inferred fluctuation in apparent magnitude for a specific galaxy and \(\sigma^\text{gal}_{\delta m,j}\) is the error component as defined in Eq. (2.20). The optimal choice for the gridding length scale is evaluated using numerical simulations and is discussed in Section 2.4. Both the observational error from the distance indicators and the error introduced by the non-linear velocity dispersion \(\sigma_v\) are being averaged. The sum over \(j\) is taken over the entire sample, where \(\Theta_{ij}\) equals one when the galaxy is within the grid cell and

---

\(^7\)This can be seen when plotting the window function \(W(k) \equiv \left( \sum_{j=1}^{N} \sum_{i=1}^{N} W(k, \alpha_{ij}, r_i, r_j) \right) / N^2\) of the survey (where \(W(k, \alpha_{ij}, r_i, r_j)\) is defined in Eq. (2.6)) and \(N\) is the number of galaxies in the survey. This window function only influences off-diagonal elements of the covariance matrix. One finds that the amplitude of \(W(k)\) significantly reduces as small-scales are approached, therefore less weight is attached to the power spectrum on small scales.

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zero otherwise. The process of smoothing the velocity field effectively damps the velocity power spectrum, this acts to suppress non-linear contributions. The function describing this damping is given by the Fourier transform of the kernel $\Theta_{ij}$, introduced in Eq. (2.29). Letting $\Gamma(k) \equiv F[\Theta_{ij}]$ from above we have

$$\Gamma(k) = \left\langle \text{sinc}\left(k_x \frac{L}{2}\right) \text{sinc}\left(k_y \frac{L}{2}\right) \text{sinc}\left(k_z \frac{L}{2}\right) \right\rangle_{\vec{k} \in k},$$

(2.30)

where $\left\langle F(\vec{k}) \right\rangle_{\vec{k} \in k}$ is the expectation value of $F(\vec{k})$ in the phase space $\vec{k} \in k$ i.e., $\left\langle F(\vec{k}) \right\rangle_{\vec{k} \in k} = 1/4\pi \int d\Omega F(\vec{k})$. Examples of $\Gamma(k)^2$, for a range of different smoothing scales, are given in Fig. 2.6. This allows one to calculate the velocity power spectrum between separate grid points; therefore once the velocity field has been smoothed we alter the theoretical prediction of the velocity power spectrum by

$$P_{vv}(k) \rightarrow P_{\text{Grid}}(k) = P_{vv}(k)\Gamma^2(k).$$

Now the covariance of $\delta m$ between grid centres, $\tilde{C}_{ij}$, is given by

$$\tilde{C}_{ij} = \left(\frac{5}{\ln 10}\right)^2 \left(1 - \frac{(1 + z_i)^2}{H(z_i)d_L(z_i)}\right) \left(1 - \frac{(1 + z_j)^2}{H(z_j)d_L(z_j)}\right) \int \frac{dk}{2\pi^2} k^2 P_{vv}(k, a = 1) W(k, \alpha_{ij}, r_i, r_j) \Gamma^2(k).$$

(2.31)

Using numerical $N$-body simulations Abate et al. (2008) explore the dependence of the recovered best-fitting parameters ($\sigma_8$ and $\Omega_m$) on the smoothing length. Specifically they find that (relative to the statistical error) a smoothing scale greater than $10h^{-1}\text{Mpc}$ results in an unbiased estimation of the cosmological parameters of interest.

In order to derive Eq. (2.31) one must presuppose the PVs inside each cell are well-described as a continuous field. However the velocities inside a grid cell represent discrete samples from the PV field; therefore as the number density inside each cell becomes small this approximation becomes worse. In Abate et al. (2008) a solution to this ‘sampling problem’ was proposed and tested using $N$-body simulations. To mitigate the effects of this approximation one interpolates between the case of a discrete sample and that of the continuous field limit. The weight attached to each is determined using the number of galaxies within each cell $N_i$. The diagonal
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Figure 2.6: Examples of the smoothing kernel $\Gamma(k) \equiv \mathcal{F}[\Theta_{ij}]$ for different values of the smoothing length $L$, given in units of $h^{-1}$ Mpc. We plot the square of the kernel as this is the term that modulates the velocity power spectrum, i.e., the term occurring in eq (2.31).

Elements of the covariance matrix are now updated as

$$\tilde{C}_{ii} \rightarrow \tilde{C}_{ii} + (\tilde{C}_{ii} - C_{ii}^m)/N_i,$$

(2.32)

where $C_{ii}^m$ is defined in Eq. (2.19). For this correction the continuous field approximation is assumed for the off-diagonal elements\(^8\).

2.3.6 Effect of the unknown zero-point

The zero-point in a PV analysis is a reference magnitude, or size in the case of Fundamental Plane surveys, for which the velocity is known to be zero. From this reference point one is able to infer the velocities of objects; without such a reference point only the relative velocities could be determined. An incorrectly calibrated zero-point introduces a monopole component to measured PVs. To give an example,

\(^8\)This approach is valid given the off-diagonal elements of the covariance matrix are significantly damped at small scales, and hence the smoothing of the velocity field has only a small effect on these elements.
for supernovae the zero-point is determined by the absolute magnitude $M$ and the
Hubble parameter $H_0$.

When deriving PV measurements the zero-point is typically fixed at its maxi-
mum likelihood value found during the calibration phase of the analysis; this allows
the velocities of all the objects in the sample to be determined. However, this
zero-point may contain error. In this section we introduce a method to analytically
propagate the uncertainty in the zero-point into the final cosmological result.

We first consider the case of analysing a single velocity survey. We define $a$ as
an offset in the magnitude fluctuation; such that $\delta m \rightarrow \delta m + a$. This indirectly
represents a perturbation to the velocity zero-point. Given we have some prior
knowledge of the distribution of this variable we give it a Gaussian prior i.e.,

$$P(y | \sigma_y) = \frac{1}{(2\pi)^{1/2} \sigma_y} \exp[-y^2/2\sigma_y^2]. \quad (2.33)$$

We define $x$ as an $N$ dimensional vector where each element is set to one (i.e.,
$(x)_i = 1$, for $i = 1..N$). Here $N$ is the dimension of $\delta m$. The parameter $a$ alters the
theoretical prediction for the mean velocity, $<\delta m^p> = 0$, to $<\delta m^p> = y x$. Now we
can analytically marginalize over the unknown zero-point (Bridle et al., 2002)

$$P(\Sigma | \delta m) = \int dy \, P(\Sigma | \delta m, y) P(y | \sigma_y)$$

$$= [2\pi \Sigma]^{-1/2} (1 + x^T \Sigma^{-1} x \sigma_y^2)^{-1/2} \exp \left[ \frac{1}{2} \delta m^T \Sigma^{-1} \delta m \right], \quad (2.34)$$

where

$$\Sigma^{-1}_M = \Sigma^{-1} - \frac{\Sigma^{-1} x x^T \Sigma^{-1}}{x^T \Sigma^{-1} x} \sigma_y^2. \quad (2.35)$$

We may wish to combine a number of different PV surveys with potentially different
zero-point offsets. In this case it is necessary to consider how one can marginalise
over the independent zero-points simultaneously. We consider the example of two
different PV surveys but note that this approach can be readily generalised to a
larger number of surveys (Bridle et al., 2002).

Firstly we decompose the data vector into apparent magnitude fluctuations from
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the first and second surveys,

$$\vec{\delta m} = \begin{pmatrix} \delta m_1^{(1)} \\ \delta m_2^{(2)} \end{pmatrix},$$  \hspace{1cm} (2.36)

where the first survey has \( n_1 \) data points and the second has \( n_2 \), therefore the combined vector has length \( N = n_1 + n_2 \). The data from the two surveys needs to be smoothed onto two different grids, this is a simple modification to the binning algorithm:

$$\vec{\delta m} = \begin{pmatrix} \frac{1}{N_{1,i}} \sum_{j \leq n_1} \delta m_j^{\text{gal}}(\vec{x}_j) \Theta_{ij} \\ \frac{1}{N_{2,i}} \sum_{n_1 < j \leq n_2} \delta m_j^{\text{gal}}(\vec{x}_j) \Theta_{ij} \end{pmatrix},$$  \hspace{1cm} (2.37)

where \( N_{1,i} \) and \( N_{2,i} \) are the number of galaxies inside the \( i \)th cell from the first and second survey respectively.

We now introduce two free parameters \((y, b)\) which will allow the zero-point to vary for each survey, again both parameters are given Gaussian priors (i.e., are distributed according to Eq. (2.33)). To account for a changing zero-point we alter the theoretical prediction for the mean value of the apparent magnitude fluctuations \( \langle \delta m^{\text{p}} \rangle \). This quantity is normally set to zero as PVs are assumed to be distributed according to a multivariate Gaussian with a mean of zero, now we have \( \langle \delta m^{\text{p}} \rangle = y x^{(1)} + b x^{(2)} \) where \( x_i^{(1)} = 1 \) if \( i \leq n_1 \) and \( x_i^{(1)} = 0 \) otherwise and \( x_i^{(2)} = 1 \) if \( i \geq n_1 \) and \( x_i^{(2)} = 0 \) otherwise. The updated likelihood is then

$$P(\Sigma|\vec{m}, y, b) =$$

$$\frac{1}{2 \pi \Sigma}^{1/2} \exp \left( -\frac{1}{2} (\vec{m} + \langle \delta m^{\text{p}} \rangle)^T \Sigma^{-1} (\vec{m} + \langle \delta m^{\text{p}} \rangle) \right).$$

We desire a posterior distribution independent of the zero-point corrections therefore we analytically marginalise over these parameters

$$P(\Sigma|\delta \vec{m}) = \int dy \int db \ P(\Sigma|\vec{m}, y, b) P(y|\sigma_y) P(b|\sigma_b)$$

$$= \frac{1}{2 \pi \Sigma}^{1/2} \left( 1 + x^{(1)^T} \Sigma^{-1} x^{(1)} \sigma_y^2 \right)^{-1/2}$$

$$\left( 1 + x^{(2)^T} \Sigma^{-1} x^{(2)} \sigma_b^2 \right)^{-1/2} \exp \left[ -\frac{1}{2} \delta \vec{m}^T \Sigma_{\delta \vec{m}}^{-1} \delta \vec{m} \right],$$  \hspace{1cm} (2.38)
where
\[
\Sigma_M^{-1} = \Sigma^{-1} - \frac{\Sigma^{-1}x^{(1)}x^{(1)^T}\Sigma^{-1}}{x^{(1)^T}\Sigma^{-1}x^{(1)}} - \frac{\Sigma^{-1}x^{(2)}x^{(2)^T}\Sigma^{-1}}{x^{(2)^T}\Sigma^{-1}x^{(2)}} + \frac{\sigma_y^2}{\sigma_b^2}.
\] (2.39)

Here we need to consider the variation to the determinant as the covariance matrix is being varied at each likelihood evaluation. For all zero-points here we choose a Gaussian prior with a standard deviation of $\sigma_y = \sigma_b = 0.2$. We find the choice of width of the prior has an insignificant effect on the final results.

### 2.3.7 Combining multiple (correlated) velocity surveys

Given the limited number count and sky coverage of objects in velocity surveys it is common for different surveys to be combined in a joint analysis. In this situation individual datasets may contain unrecognised systematic errors, requiring them to be re-weighted in the likelihood analysis.

The first method we consider to do this is a recent upgrade to the hyper-parameter analysis. The original hyper-parameter method was developed to remove the inherent subjectivity associated with selecting which data sets to combine in an analysis and which to exclude (see Lahav et al., 2000; Hobson, Bridle & Lahav, 2002). This process is achieved by including all the available data sets but allowing free hyper-parameters to vary the relative ‘weight’ attached to each data set, the hyper-parameters are then determined in a Bayesian way. Consider two hypothetical surveys with chi squared of $\chi^2_A$ and $\chi^2_B$. The combined constraints are typically found by minimising the quantity
\[
\chi^2_{\text{com}} = \chi^2_A + \chi^2_B.
\] (2.40)

This gives both data sets equal weight. Introducing the hyper-parameters one has
\[
\chi^2_{\text{com}} = \alpha\chi^2_A + \beta\chi^2_B.
\] (2.41)

The hyper-parameters can be interpreted as scaling the errors for each data set, i.e., $\sigma_i \rightarrow \sigma_i\alpha^{-1/2}$, or equivalently the covariance matrix of each data set $C_i \rightarrow \alpha^{-1}C_i$. The final values of the hyper-parameters, more accurately their probability distri-
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Distributions $P(\alpha)$ and $P(\beta)$, give an objective way to determine if there are systematic effects present in the data (e.g., a value $\alpha > 1$ can be interpreted as reducing the errors or correspondingly increasing the relative weight of the data set).

The problem with the traditional hyper-parameter analysis for PV surveys is that it assumes that the individual data sets are not correlated (this assumption is required to write down equation Eq. (2.40) and Eq. (2.41)). If the surveys cover overlapping volumes or are influenced by the same large-scale modes this is not the case. Recently the hyper-parameter formalism has been extended to a hyper-parameter matrix method which includes the cross correlations between surveys (Ma & Berndsen, 2014). Here the hyper-parameters scale both the covariance between objects in a given data set and the covariance between the data sets:

$$C^{D_iD_j} \rightarrow (\alpha_i\alpha_j)^{-1/2} C^{D_iD_j}$$

(2.42)

$D_i$ represents the $i^{th}$ data set, so $C^{D_iD_j}$ gives the covariance between the $i^{th}$ and $j^{th}$ data sets. For simplicity here we outline the case of two different data sets. In this case there are two hyper-parameters $(\alpha_1, \alpha_2)$ which we treat as free parameters. The hyper-parameter matrix is defined as:

$$H = \begin{pmatrix} \alpha_1^{-1} & (\alpha_1\alpha_2)^{-1/2} \\ (\alpha_1\alpha_2)^{-1/2} & \alpha_2^{-1} \end{pmatrix}.$$  

(2.43)

The final likelihood function is

$$P(\delta m | \vec{\theta}, \vec{\alpha}) = \prod_{i=1}^{2} \left( \frac{\alpha_i}{2\pi} \right)^{n_i/2} \frac{1}{\sqrt{|C|}} \exp \left( -\frac{1}{2} \delta m^T \left( \hat{H} \circ C^{-1} \right) \delta m \right).$$

Here $\circ$ is an ‘element-wise’ product (or, Hadamard product) defined as $(\hat{H} \circ C^{-1})_{ij} = \hat{H}_{ij} \times (C^{-1})_{ij}$, and $\vec{\theta}$ represents the parameters of interest. $\hat{H}$ is the Hadamard inverse of the ‘hyper-parameter’ matrix (i.e. $\hat{H}_{ij} = P_{ij}^{-1}$), and $n_1$ and $n_2$ are the number of data points in the first and second surveys respectively.

As described in Section 2.3.2 a free parameter $\sigma_v$ is typically introduced to ac-
count for non-linear random motion. One issue with the likelihood function defined above is that $\sigma_v$ and the hyper-parameters are quite degenerate. Therefore for our hyper-parameter analysis we fix $\sigma_v$ at the values found when analysing the surveys independently.

2.4 Testing with simulations

We require simulations of PV catalogues for several aspects of this analysis. Firstly, to determine if non-linear effects from the growth rate of structure or redshift-space distortions cause systematic errors. Secondly, to determine the approximate survey geometry and distance errors for which the non-Gaussian observational scatter of PVs becomes important. Finally to determine the effect (on the final constraints) of marginalising over the zero-point uncertainty. Note the construction of the mock catalogues used in this section is outlined in Section 2.2.

All the cosmological parameters not allowed to vary freely here are set to those input into the simulation (i.e., $\Omega_\Lambda = 0.727, \Omega_m = 0.273, \Omega_k = 0, H_0 = 100h$ km s$^{-1}$Mpc$^{-1}$, $\sigma_8 = 0.812, n_s = 0.960$). For the velocity power spectrum fits we use a smoothing scale (defined in Section 2.3.5) of $10h^{-1}$Mpc, while for the analysis of $\Omega_m$ and $\sigma_8$ we adopt a length of $20h^{-1}$Mpc. We use a larger grid size for the analysis of $\Omega_m$ and $\sigma_8$ because the evaluation of the likelihood (i.e., Eq. 2.23) is more computationally demanding relative to the evaluation of of the likelihood given in Eq. (2.26), the larger grid size reduces the computational requirements. We first shift the haloes within the simulation to their redshift-space position, using $\mathbf{x}^s = \mathbf{x}^r + \mathbf{v}(\mathbf{x}, t) \cdot \hat{r}/H_0$. Now we transform the PVs within the simulation to apparent magnitude fluctuations, $\delta m$.

At small scales the predictions from RPT become less accurate and are known to break down (experience exponential damping relative to the expectations from $N$-body simulations) at $k \sim 0.15h$Mpc$^{-1}$ for the velocity power spectrum evaluated assuming the fiducial cosmology of the simulation at a redshift of zero. We therefore

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9This is the case because for each $\Omega_m$ and $\sigma_8$ posterior evaluation we are required to re-calculate the entire covariance matrix (Eq. 2.31). This is not the case for the other parametrisations considered here.
2.4. TESTING WITH SIMULATIONS

truncate the velocity power spectrum fits at this scale. We note that this scale varies for different cosmological parameters, therefore for the \((\Omega_m, \sigma_8)\) fits we test a range of values, \(k_{\text{max}}\), for truncating the integral when calculating the covariance matrix, to decide the optimal choice for the data.

Now using 8 different observers from mock set (I) we test the ability of each parametrisation to recover the input cosmology, under the conditions outlined above. Recall for mock set (I) the input distance error is \(\sigma_d \sim 5\%\), the approximate distance error for SNe. The derived constraints on \((\Omega_m, \sigma_8)\) for various values of \(k_{\text{max}}\) are given in Fig. 2.8; the black square symbols here give the input cosmology of the simulation. The velocity power spectrum measurements are given in Fig. 2.9 and the constraints for a scale-dependent growth rate, \(f\sigma_8(z = 0, k)\), are given in Fig. 2.10. The thick blue lines in Fig. 2.9 give the predictions for the average power within the defined bin ranges for the fiducial cosmology, this is calculated using Eq. (2.27) with \(A_i = 1\). In addition to giving the results for a single mock realisation we also average the results found for 8 different mock realisations in order to provide a more accurate systematic test. Again some care needs to be taken when interpreting the combined constraints given that on the largest scales the mock realisations are significantly correlated. This is most pronounced for the largest-scale bin in Fig. 2.9 and Fig. 2.10, for which we interpret the consistently ‘high’ measurement power as being produced by correlations. Also note the mock simulations considered here have significantly greater statistical power than current PV surveys, so we are performing a sensitive systematic check. We find that at the investigated error levels we are able to accurately recover the input cosmology of the simulation for all parametrisations considered. We conclude therefore that the bias from non-linear structure is currently insignificant, the linear relation between the PV and \(\delta m\) is valid and non-linear RSD effects do not bias our final constraints.

We performed additional checks by fitting for the velocity power spectrum using both non-linear perturbation theory and linear theory. This test, albeit simplistic, gives us a way to estimate the contribution that non-linear scales have on our final results. The test show very little variability when changing between the two models of the velocity power spectrum. We, therefore, conclude that, once the gridding has been performed and at the current level of statistical uncertainty, we are probing
Our local universe contains a number of (seemingly) rare structures such as the Virgo supercluster (Courtois et al., 2013). These massive objects must (to some extent) influence the local velocity field and hence our inferences of cosmological parameters. The extent to which this effect is accounted for via cosmic variance errors is unclear. Note, however, that the proposition that clusters such as Virgo are exceedingly rare, and as a result, will bias inferences made from the peculiar velocity field violates the Copernicus principle. Accordingly, we will not further address such questions in this work.

Following Fig. 2.9 we conservatively fix $k_{\text{max}} = 0.15 h \text{Mpc}^{-1}$ for the $(\Omega_m, \sigma_8)$ fits, given that on smaller scales we observe a slight trend away from the fiducial cosmology (yet still consistent at the $2\sigma$ level). For the power spectrum fits, we note a small amount of correlation exists between the different wavenumber bins. We give a typical example of the correlation coefficients between the bins in Fig. 2.7, determined using the Monte Carlo Markov Chain.

Figure 2.7: Correlation coefficients $r$ between the amplitude parameters $A_i$, and the non-linear velocity dispersion $\sigma_v$. The results here were calculated using an MCMC chain (of length $\sim 10^6$) produced when analysing a single realisation from Mock set (I). We expect very similar correlations to exist between the growth rate measurements and note that the correlations between the different bins are quite weak.
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Figure 2.8: 68% confidence regions for the matter density, $\Omega_m$, and the RMS clustering in $8h^{-1}$Mpc spheres, $\sigma_8$, using mock set (I), including RSD and using the $\delta m$ variable. The transparent contours (dashed outline) give the constraints from some example single survey realisations. The opaque contours (solid outline) give the combined constraints from 8 realisations. For the combined constraints we give 68% and 95% confidence regions. A smoothing length of $20h^{-1}$Mpc is used for all constraints. For each plot we vary the length scale, $k_{\text{max}}$ at which we truncate the integral for the calculation of the covariance matrix, that is the integral given in Eq. (2.31) (i.e., the smallest scales included in the analysis). Varying this scale allows us to test the validity of the constraints as we move into the non-linear regime. From left to right the wavenumbers at which we cut off the integration are $k_{\text{max}} = [0.1, 0.15, 0.175, 0.20]h\text{Mpc}^{-1}$. The black square symbols give the cosmology input into the simulation.
Figure 2.9: 68% confidence intervals for the amplitude parameters $A_i$ describing the mean ‘power’ within each bin using mock set (I). The thick blue (horizontal) lines give the mean power in each bin for the fiducial cosmology calculated using Eq. (2.27). Here we include RSDs, use $\delta m$ and a smoothing length of $10h^{-1}\text{Mpc}$. The blue points are the constraints found for individual mock realisations, while the red points show the constraints found by combining the results from 8 different mocks. Consistency with the assumed fiducial cosmology occurs when the given confidence levels overlap with the mean power; the specific position of the point along the bin length is arbitrary. The green dashed line shows the velocity power spectrum calculated assuming the fiducial cosmology. Section 2.5.3 gives the wavenumber bin intervals used here, with the exception that $k_{\text{min}} = 0.0065h\text{Mpc}^{-1} = 2\pi/L_{\text{box}}$. 
Figure 2.10: 68% confidence intervals for the normalised scale-dependent growth rate $f(z = 0, k) \sigma_8(z = 0)$ in 5 different bins in Fourier space. The thick black line gives the prediction of the input cosmology. For each $k$-bin we plot the results from 6 different realizations from mock set (I). We include RSDs in the mocks, use the variable $\delta m$, and choose a smoothing length of $10h^{-1}\text{Mpc}$. The specific $k$ values within a given bin for the measurements are arbitrary. The bin intervals used here are given in Section 2.5.3, with the one correction that $k_{\text{min}} = 0.0065h^{-1}\text{Mpc}$, corresponding to the size of the simulation.
Figure 2.11: (Left) 68% confidence intervals for the velocity power spectrum amplitude in three Fourier bins. We consider five separate realisations taken from mock set (II). The small blue points show the individual constraints found using the variable $\delta m$, while the small red points show the constraints found using the median of the velocity distributions (viz, $v_i = \text{Median}[P(v_i)]$) (this gives very similar results to the direct method). The larger blue and red points show the results from combining the five realisations. The circle symbols (left panel) give the median value of the probability distributions. (Right) Constraints on the parameters $\Omega_m$ and $\sigma_8$ found from combining the results from 8 different realisations with mock set (II). The contours give 68% and 95% confidence levels. The blue contour shows the result of using the variable $\delta m$. The red and green contours show the result of using the PV as the main variable, where the red contour gives the result from directly calculating the PV from the observable quantity ignoring the Jacobian term, and the green contour gives the constraints from using the mean value of $P(v)$. 
2.4. TESTING WITH SIMULATIONS

When testing the effect of non-Gaussian observational error for PVs, both the sky coverage of the survey and the distance error are relevant, therefore we consider both mock set (I) and (II). We find that for mock set (I) using the velocity not magnitude as the variable in the analysis results in no significant bias. This continues to be true even when we limit the survey to one hemisphere. This can be understood because the degree of departure from Gaussianity of the probability distribution of peculiar velocities, \( P(v) \), is dependent on the magnitude of the distance error. With relatively small distance errors, \( P(v_p) \) is described well by a Gaussian distribution.

In the case of a distance and sky distribution corresponding to 6dFGSv, that is, \( \sigma_d \sim 30\% \) and only considering one hemisphere (i.e., mock set (II)) we find a significant bias is introduced when using PVs\(^{10} \). We use 8 realisations from mock set (II), generate realistic observational errors and perform the likelihood analysis twice using either PV or \( \delta m \) as the variable. For the likelihood analysis using PV one is required to input a single velocity value, which gives us some freedom in how we choose to compress the distribution \( P(v_p) \) into a single value. Here we consider the mean, maximum likelihood (ML) and median. For a detailed investigation into the effect of these choices, in the context of bulk flow measurements, see Scrimgeour et al. (2016). In all prior PV analysis when the full probability distribution of the distance measure (e.g., the absolute magnitude, \( M \), in the case of the Tully-Fisher relation) was not available the PV was calculated directly from this variable. The Jacobian term is ignored in this case, we label this method the ‘direct approach’. To give an example; for the Fundamental Plane relation using this direct method one would determine the ML value of \( x = \log_{10}(D_{z}/D_{H}) \) then using this value calculate the corresponding PV, again ignoring the Jacobian term given in Eq. (2.10).\(^{10}\)

We give the constraints for the amplitude of the velocity power spectrum and the cosmological parameters \( \sigma_8 \) and \( \Omega_m \), found when using the magnitude fluctuation \( \delta m \), in Fig. 2.11. For the fits of \( \sigma_8 \) and \( \Omega_m \) we also use the mean of \( P(v_p) \) and the direct method; while for the velocity power spectrum fits we use the median of \( P(v_p) \) (viz, \( v_i = \text{Median}[P(v_i)] \)). Here we have combined the constraints from different mock realisations. Note for the separate fits using \( \delta m \) and the PV we have used the

\(^{10}\)This also applies for future analyses; a number of Fundamental Plane and Tully-Fisher surveys are forthcoming and will have similar properties.
same mock realisations. We interpret the slight offset from the fiducial model (still within 1σ) of the constraints found using 3m as simply a result of cosmic variance and covariance between mock realisations.

We conclude that for the constraints on σ8 and Ωm using the mean, median and ML of P(v_p) and the direct method in the likelihood analysis all introduce a significant bias (i.e., > 2σ) in the final cosmological parameter values when considering a radial and angular halo distribution similar to 6dFGSv (and averaging over 8 realisations). We find a similar, yet less significant, bias for the velocity power spectrum, given the derived constraints are now consistent at the two sigma level. As shown in the left panel of Fig. 2.11, the result is more power relative to the fiducial cosmology on the largest scales, which is consistent with a negative bias in Ω_m. The non-Gaussian distributions imprint a bias in the mean radial velocity and therefore influence power on the largest scale. Once a full sky survey is considered this effect is less severe as the bias tends to averages out.

We test the sensitivity of the final constraints to the process of marginalising over the zero-point. We find that the final results are reasonably insensitive to this procedure. As expected, the error in measurements on the largest scales is increased, which slightly weakens the constraints in the largest scale bin for the growth rate and velocity power spectrum measurements, and equivalently weakens the constraints on the matter density Ω_m.

2.5 Parameter fits to velocity data sets

In this section we present the results from the analysis of the 6dFGSv and low-z SNe peculiar velocity surveys. Analysing the fluctuations in the measured PVs and their correlations (as a function of their spatial separation) we are able to derive constraints on the following: the cosmological parameters Ω_m and σ8 (Section 2.5.2); the amplitude of the velocity power spectrum, P_{vv}(k) ≡ P_{θθ}(k)/k^2 in a series of (five) Δk ~ 0.03hMpc^{-1} bins (Section 2.5.3); the scale-dependent normalized growth rate of structure, fσ8(z = 0, k), in a series of (five) Δk ~ 0.03hMpc^{-1} bins (Section 2.5.4); and the scale-independent growth rate of structure, fσ8(z = 0) (Section 2.5.4). All the constraints given are at a redshift z ~ 0. We emphasize
that, because we have not included any information from the local density field, as inferred by the local distribution of galaxies, the results presented here do not rely on any assumptions about galaxy bias. Additionally, here we are working solely within the standard ΛCDM model.

For sections 2.5.2, 2.5.3, 2.5.4 we give the results derived when analysing the individual surveys separately. Comparing the results from different PV surveys allows one to check for systematic effects. When combining the PV surveys we consider two different approaches; both introduce extra degrees of freedom that allow the relative ‘weight’ of each sample to vary in the likelihood calculation. Firstly, we introduce a free parameter $\sigma_v$ to each survey, this term accounts for non-linear velocity dispersion. Secondly, we allow the relative weight of each survey to be varied by the use of a matrix hyper-parameter method (introduced in Section 2.3.7). In this case we fix the $\sigma_v$ values of both surveys to the maximum likelihood values found when analysing the surveys separately. The purpose of the hyper-parameter analysis is to check the statistical robustness of our constraints. In the case that the hyper-parameter analysis is statistically consistent with the standard method of combining the surveys we quote the results from the standard method as our final measurement. The two PV samples we use for this analysis have significant overlap, therefore we expect the individual results to be highly correlated, given they share the same cosmic variance. This limits the benefits from combining the samples. In addition complications arise when data points from each survey are placed on the same grid point, as occurs when the velocity surveys are separately smoothed onto grids\(^{11}\).

For all likelihood calculations in the following sections we marginalise over the unknown zero-point\(^{12}\) (i.e., a monopole contribution to the velocity field). The result of this process is that our constraints are not sensitive to the uncertainties present in the determination of the zeropoint in PV surveys and the assumptions required to determine the zeropoint.

\(^{11}\)We treat these data points as if they were perfectly correlated in the full covariance matrix.

\(^{12}\)We allow each survey to have different zero-point offsets for the marginalisation.
2.5.1 MCMC sampling strategy

To sample the posterior distributions we use a python implementation of the affine-invariant ensemble sampler for Markov Chain Monte Carlo (MCMC) \texttt{MCMC-hammer} (Foreman-Mackey et al., 2013). This technique was introduced by Goodman & Weare (2010). We use the \texttt{MCMC-hammer} algorithm because, relative to the standard Metropolis–Hastings (M–H) algorithm the integrated autocorrelation time is lower and less ‘tuning’ is required; specifically, only two parameters are required to tune the performance of the Markov chain, as opposed to $N[N+1]/2$ parameters in M–H, where $N$ is the dimension of the parameter space. Additionally the \texttt{MCMC-hammer} algorithm is trivially parallelized using MPI and the affine invariance (invariance under linear transformations) property of this algorithm means it is independent of covariances between parameters\(^{13}\) (Foreman-Mackey et al., 2013).

We discard the first 20\% of each chain as ‘burn in’ given that the sampling may be non-Markovian, while the convergence of each chain is assessed using the integrated autocorrelation time. From the samples we generate an estimate of the posterior maximum-likelihood (ML) and median; given the posterior distributions of the parameters tend to be non-Gaussian, the 68\% confidence intervals we quote are found by calculating the 34\% limits about the estimated median. In the case where we cannot quote a robust lower bound, when the probability distribution peaks near zero, we quote 95\% upper limits.

2.5.2 Matter density and clustering amplitude

The base set of parameters we allow to vary in this analysis is $[\Omega_m, \sigma_8, \sigma_v]$. In the case where we combine PV surveys we consider two extensions to this base set. Firstly, we include a free parameter modelling the non-linear velocity dispersion $\sigma_v$ for each survey and therefore consider the set of parameters $[\Omega_m, \sigma_8, \sigma_{v1}, \sigma_{v2}]$. Secondly, we fix the values for the velocity dispersion and introduce hyper-parameters, this gives the set $[\Omega_m, \sigma_8, \sigma_6dF, \sigma_{SNe}]$.

\(^{13}\text{No internal orthogonalisation of parameters is required.}\)
the corresponding velocity power spectrum. While the calculation of the velocity power spectrum in velMPTbreeze is significantly faster than previous RPT calculations, it remains too slow to embed directly in MCMC calculations. Therefore, the approach we take here is to pre-compute a grid of velocity power spectra then use a bilinear interpolation between the grid points to estimate the power spectra.

Using velMPTbreeze, we evaluate a grid of velocity power spectra; we use the range $\Omega_m = [0.050, 0.500]$ and $\sigma_8 = [0.432, 1.20]$, which act as our priors. We use step sizes of $\Delta \Omega_m = 0.01$ and $\Delta \sigma_8 = 0.032$. We do not investigate the region of parameter space where $\Omega_m < 0.05$, as here the theoretical modelling of the velocity power spectrum becomes uncertain as it becomes highly non-linear on very large scales. The prior placed on all $\sigma_v$ parameters is $\sigma_v=[0,1000]$ km s$^{-1}$ and $\alpha_i=[0,10]$. For each value of $\Omega_m$ the matter transfer function needs to be supplied, to do this we use the CAMB software package Lewis, Challinor & Lasenby (2000). The numerical integration over the velocity power spectrum requires us to specify a $k$-range. Here we integrate over the range $k = [0.0005, 0.15] h$ Mpc$^{-1}$. We note that integrating to larger scales (i.e. smaller values of $k$) when computing the full covariance matrix has a negligible effect on the derived constraints. Additionally, for the constraints given in this section we smooth the local velocity field with a gridding scale of $20h^{-1}$ Mpc.

The constraints for the parameters are shown in Fig. 2.12 and the best-fit values and 68% confidence regions are given in Table 2.1. Using only the 6dFvGS sample we determine $\Omega_m = 0.136^{+0.07}_{-0.04}$ and $\sigma_8 = 0.69^{+0.18}_{-0.14}$, and for the SNe velocity sample we determine $\Omega_m = 0.233^{+0.14}_{-0.09}$ and $\sigma_8 = 0.86 \pm 0.18$. The results show that the two PV samples are consistent with each other and given the size of the errors we do not find a strong statistical tension (less than $2\sigma$) with the parameter values reported by Planck. Combining the two PV surveys we determine $\Omega_m = 0.166^{+0.11}_{-0.06}$ and $\sigma_8 = 0.74 \pm 0.16$; similarly we find no strong statistical tension with Planck. For the matrix hyper-parameter analysis we find $\alpha_{6dF} = 1.23 \pm 0.05$, $\alpha_{SNe} = 0.87 \pm 0.08$, $\Omega_m = 0.228^{+0.12}_{-0.08}$ and $\sigma_8 = 0.96^{+0.14}_{-0.16}$; although the constraints from the hyper-parameters are best fit with the slightly higher $\sigma_8$ value, we find the results from the hyper-parameter analysis are statistically consistent with the previous constraints, as shown in Fig. 2.12.

The constraints on $\Omega_m$ and $\sigma_8$ outlined in this section, while not competitive in
Figure 2.12: 68 % confidence intervals for the matter density $\Omega_m$, $\sigma_8$ and the non-linear velocity dispersion $\sigma_v$. Results are shown for 6dFGSv (blue), the SN sample (green), the combined analysis (red) and the combined hyper-parameter analysis (black). The $\sigma_v$ constraints from the combined analysis are very similar to the individual constraints hence we do not add them here.

terms of statistical uncertainty to other cosmological probes, do offer some insight. In contrast to most methods to determine the matter density, $\Omega_m$, constraints from PV do not result from determining properties of the global statistically homogeneous universe (geometric probes); the constraints arise from the dependence of the clustering properties of dark matter on $\Omega_m$. The consistency between these probes is a strong test of the cosmological model.

### 2.5.3 Velocity power spectrum

Analysing the surveys individually we consider the base parameter set $\mathbf{p} = [A_1(k_1), A_2(k_2), A_3(k_3), A_4(k_4), A_5(k_5), \sigma_v]$. Each $A_i$ parameter (defined in Eq. (2.25)) acts to scale the amplitude of the velocity power spectrum, $P_{vv}(k)$, over a specified
Table 2.1: Derived cosmological parameter values for $m$ and $8$ plus the derived value for the non-linear velocity dispersion $v$ and the hyper-parameters $6dF$ and $SNe$. Parameters not allowed to vary are fixed at their Planck ML values. Columns 2 and 3 give results from the 6dFGSv survey data alone. Columns 4 and 5 give results from the SNe sample data alone. For columns 6 and 7 we give the results combining both surveys; and for columns 8 and 9 we give the results combining both surveys using a matrix hyper-parameter analysis. Note the hyper-parameters are only given for columns 8 and 9 as they are not included in the other analyses. All varied parameters are given with 68% limits. All prior parameters are given with 68% limits.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>6dF</th>
<th>SNe</th>
<th>6dF + SNe (Norm)</th>
<th>6dF + SNe (Hyp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>0.103 $^{+0.07}_{-0.04}$</td>
<td>0.169 $^{+0.134}_{-0.09}$</td>
<td>0.107 $^{+0.11}_{-0.06}$</td>
<td>0.183 $^{+0.12}_{-0.08}$</td>
</tr>
<tr>
<td>$8$</td>
<td>0.66 $^{+0.18}_{-0.14}$</td>
<td>0.89 $^{+0.18}_{-0.18}$</td>
<td>0.73 $^{+0.16}_{-0.16}$</td>
<td>1.06 $^{+0.14}_{-0.16}$</td>
</tr>
<tr>
<td>$v$ [km/s]</td>
<td>32.7 $^{+245}_{-58}$</td>
<td>388 $^{+76}_{-62}$</td>
<td>205 $^{+37}_{-48}$</td>
<td>500 $^{+109}_{-117}$</td>
</tr>
</tbody>
</table>
Figure 2.13: 68% confidence intervals for the amplitude parameters $A_i$ scaled by the mean power within each bin for the 6dFGSv data, SNe data and the combined constraint. The thick blue lines give the mean power in each bin in the fiducial cosmology calculated using Eq. (2.27). The black dashed line shows the velocity power spectrum $P_{vv}(k)$ calculated assuming the Planck cosmology. The circle symbols here give the median of the posterior distribution.
## Parameter Fits to Velocity Data Sets

### Table 2.2: Constraints on the velocity power spectrum amplitude parameters

The table provides constraints on the velocity power spectrum amplitude parameters along with the value of the non-linear velocity dispersion $v$ and the hyper-parameters $6dF$ and $SNe$. Parameters not allowed to vary are fixed at their Planck ML values. Columns 2 and 3 give results from the 6dFGSv survey data alone. Columns 4 and 5 give results from the SNe sample data alone. For columns 6 and 7 we give the results combining both surveys; and for columns 8 and 9 we give the results combining both surveys using a matrix hyper-parameter analysis. All varied parameters are given at priors.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>ML</th>
<th>Median</th>
<th>ML</th>
<th>Median</th>
<th>ML</th>
<th>Median</th>
<th>ML</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1.98</td>
<td>2.64</td>
<td>1.62</td>
<td>2.50</td>
<td>2.43</td>
<td>3.20</td>
<td>2.22</td>
<td>3.05</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.20</td>
<td>0.74</td>
<td>0.25</td>
<td>0.89</td>
<td>0.14</td>
<td>0.44</td>
<td>0.26</td>
<td>0.65</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.20</td>
<td>0.94</td>
<td>0.57</td>
<td>1.0</td>
<td>0.13</td>
<td>0.50</td>
<td>0.27</td>
<td>0.63</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.27</td>
<td>1.51</td>
<td>0.43</td>
<td>1.34</td>
<td>0.15</td>
<td>0.44</td>
<td>0.27</td>
<td>0.63</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.30</td>
<td>1.36</td>
<td>0.84</td>
<td>2.79</td>
<td>0.38</td>
<td>1.17</td>
<td>0.40</td>
<td>1.45</td>
</tr>
<tr>
<td>$v$</td>
<td>98.4</td>
<td>137.5</td>
<td>91</td>
<td>132.2</td>
<td>372.8</td>
<td>365.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>88.1</td>
<td>137.5</td>
<td>91</td>
<td>132.2</td>
<td>372.8</td>
<td>365.2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: All values, columns 2 and 3 give results from the 6dFGS survey data alone. Columns 4 and 5 give results from the SNe sample data alone. For columns 6 and 7 we give the results combining both surveys; and for columns 8 and 9 we give the results combining both surveys using a matrix hyper-parameter analysis. All varied parameters are given at priors.
wavenumber range given by \( k_1 \equiv [0.005, 0.02] \), \( k_2 \equiv [0.02, 0.05] \), \( k_3 \equiv [0.05, 0.08] \), \( k_4 \equiv [0.08, 0.12] \) and \( k_5 \equiv [0.12, 0.150] \). When combining samples we consider the parameter sets \([A_1, A_2, A_3, A_4, A_5, \sigma_{v,1}, \sigma_{v,2}]\) and \([A_1, A_2, A_3, A_4, A_5, \alpha_{6dF}, \alpha_{SNe}]\). We use a flat prior on the amplitude parameters, \( A_i = [0, 100] \), and the hyper-parameters \( \alpha_i = [0, 10] \).

The constraints for the amplitude of the velocity power spectrum are shown in Fig. 2.13 and the best-fit values and 68% confidence regions are given in Table 2.2. The deviation between the ML values and median values (as shown in Table 2.2) is caused by the skewness of the distributions and the physical requirement that \( A_i > 0 \). This requirement results in a cut-off to the probability distribution that becomes more significant as the size of the errors increases. Therefore we caution that defining a single best-fitting value from the distribution requires subjective choices; note this is not the case for the growth rate constraints as shown in the next section. The fiducial power in each Fourier bin is consistent with that expected in our fiducial cosmological model assuming the best-fitting Planck parameters.

### 2.5.4 Scale-dependent growth rate

We consider the results outlined in this section the most significant component of this work. We present the first measurement of a scale-dependent growth rate which includes the largest-scale growth rate measurement to date (viz., length scales greater than \( 300h^{-1}\text{Mpc} \)). Additionally, we present a redshift zero measurement of the growth rate that is independent of galaxy bias and accurate to \( \sim 15\% \). Comparing this result to that obtained from the RSD measurement of 6dFGS (i.e., Beutler et al., 2012) allows one to test the systematic influence of galaxy bias, a significant source of potential systematic error in RSD analysis.

Analysing the surveys individually we consider two parameter sets: firstly we determine the growth rate in the scale-dependent bins defined above constraining the parameter set \([f\sigma_8(k_i), \sigma_v]\) \((i = 1..5)\); secondly we fit for a single growth rate measurement \([f\sigma_8(z = 0), \sigma_v]\). When combining data sets we consider the extensions to the base parameter set \( +[\sigma_{v,1}, \sigma_{v,2}] \), and \( +[\alpha_{6dF}, \alpha_{SNe}] \) and use a smoothing length of \( 10h^{-1}\text{Mpc} \). We fix the shape of the fiducial velocity power spectrum \( \Omega_m \) to the
2.5. **PARAMETER FITS TO VELOCITY DATA SETS**

*Planck* value. By separating the power spectrum into wavenumber bins we expect that our final constraints are relatively insensitive to our choice of \( \Omega_m \). Varying \( \Omega_m \) generates a \( k \)-dependent variation in the power spectrum over very large scales; considering small intervals of the power spectrum this \( k \)-dependence is insignificant and to first order the correction to a variation in \( \Omega_m \) is simply a change in amplitude of the power spectrum, which we allow to vary in our analysis.

We first consider the scale-dependent constraints which are shown in Fig. 2.14; with the best-fit and 68% confidence intervals given in Table 2.3 and the full probability distributions in Fig. 2.17. For 6dFGSv we determine:

\[
f\sigma_8(k_i) = [0.72^{+0.17}_{-0.23}, 0.38^{+0.17}_{-0.20}, 0.43^{+0.20}_{-0.20}, 0.55^{+0.22}_{-0.23}, 0.52^{+0.25}_{-0.22}]. \tag{2.44}
\]

For the SNe velocity sample we have:

\[
f\sigma_8(k_i) = [0.70^{+0.29}_{-0.22}, 0.42^{+0.23}_{-0.19}, 0.45^{+0.24}_{-0.20}, 0.51^{+0.29}_{-0.29}, 0.74^{+0.41}_{-0.33}]. \tag{2.45}
\]

As shown in Table 2.3 the constraints on \( \sigma_v \) from 6dFGSv are very weak relative to the constraints from the SNe sample. The reason the \( \sigma_v \) parameter is much lower (and has a larger uncertainty) for the 6dFGSv sample relative to the SNe sample is that the gridding has a stronger effect for the 6dFGSv sample given the higher number density. This significantly reduces the contribution of non-linear velocity dispersion to the likelihood and hence increases the final uncertainty. In addition, we note that the magnitude of \( \sigma_v \) will be dependent on the mass of the dark matter halo that the galaxy resides in. The halo mass may vary between PV surveys, therefore, causing \( \sigma_v \) to vary between PV surveys.

The results (again) show that the two survey are consistent with each other, viz., they are within one standard deviation of each other for all growth rate measurements. We detect no significant fluctuations from a scale-independent growth rate as predicted by the standard ΛCDM cosmological model. Although the power in the largest-scale Fourier bin is high, it is consistent with statistical fluctuations. When combining both the 6dFGSv sample and the SNe velocity sample we find (no hyper-parameters): \( f\sigma_8(k_i) = [0.79^{+0.21}_{-0.25}, 0.30^{+0.14}_{-0.19}, 0.32^{+0.19}_{-0.15}, 0.64^{+0.17}_{-0.16}, 0.48^{+0.22}_{-0.21}] \). We find no significant departure from the predictions of the standard model.
We next fit for a scale-independent growth rate by scaling the fiducial power spectrum across the full wavenumber range. The measurements of a scale-independent growth rate of structure are given in Fig. 2.15. Here we also compare with previously published results from RSD measurements and the predictions from the assumed fiducial cosmology. The best-fit values and 68% confidence intervals are given at the bottom of Table 2.3. We also plot the full probability distributions in Fig. 2.16, in addition to the results from the hyper-parameter analysis. For 6dFGSv, the SNe velocity sample and 6dFGSv+ SNe (with no hyper-parameters) we determine, respectively, \( f_s(z) = [0.428^{+0.079}_{-0.068}, 0.417^{+0.097}_{-0.084}, 0.418 \pm 0.065] \). The measurements of the growth rate all show consistency with the predictions from the fiducial model as determined by Planck. Specifically, the best-fitting Planck parameters predict \( f_s(z = 0) = 0.443 \). In addition we find consistency with the measurement of the growth rate of structure from the RSD analysis of the 6dFGS (see Fig. 2.15) (Beutler et al., 2012).

For the hyper-parameter analysis the results for the scale-dependent and scale-independent measurements are indistinguishable. We determine \( \alpha_{\text{6dFGSv}} = 1.189 \pm 0.034 \) and \( \alpha_{\text{SNe}} = 0.980^{+0.104}_{-0.094} \), the results for both analysis have been included in Fig. 2.14 and Fig. 2.15. We find that, while there is a slight shift in the best-fit values, the hyper-parameter analysis gives results statistically consistent with the previous results; for the scale-independent measurements this is best shown in Fig. 2.16.

## 2.6 Discussion and conclusions

We have constructed 2-point statistics of the velocity field and tested the ΛCDM cosmology by using low-redshift 6dFGSv and Type-Ia supernovae data. We summarise our results as follows:

- We introduced and tested a new method to constrain the scale-dependence of the normalized growth rate using only peculiar velocity data. Using this method we present the largest-scale constraint on the growth rate of structure to date. For length scales greater than \( \sim 300 h^{-1}\text{Mpc} \) (\( k < 0.02h\text{Mpc}^{-1} \)) we constrain the growth rate to \( \sim 30\% \). Specifically, we find for 6dFGSv, which
2.6. DISCUSSION AND CONCLUSIONS

Figure 2.14: 68% confidence intervals for the normalized scale-dependent growth rate $f(z=0,k)\sigma(z=0)$ in 5 different bins in Fourier space. The thick black line is the prediction found assuming the fiducial Planck cosmology. For each $k$-bin we plot the results from 6dFGSv, the SNe sample and the combined constraint. The bin intervals used here are given in Section 2.5.3. The largest scale bin corresponds to length scales $> 300 h^{-1}$Mpc. The circle symbols give the ML of the posterior distribution.
Figure 2.15: 68\% confidence intervals for the normalized growth rate $f(\sigma = 0)$ averaging over all scales. The solid black line gives the theoretical prediction for $f\sigma(z)$ assuming the Planck cosmology and the dashed-black line gives the prediction assuming the WMAP cosmology. The redshift separation of the PV measurements (coloured points) is simply to avoid overlapping data points; the redshift of the green data point gives the redshift of all the points. We compare our PV measurements to previous constraints from redshift-space distortion measurements from the 6dFGS, 2dFGRS, GAMA, WiggleZ, SDSS LRG, BOSS CMASS and VIPERS surveys given by the black points (Beutler et al., 2012; Hawkins et al., 2003; Blake et al., 2011a, 2013; Samushia et al., 2013; de la Torre et al., 2013).
Figure 2.16: Posterior distributions for the (scale averaged) growth rate of structure $f \sigma_8(z=0)$ for 6dFGSv (blue), SNe (green), combining samples (red) and for the hyper-parameter analysis (black). The posterior distributions are also given for the hyper-parameters $\alpha_{6dF}$ and $\alpha_{SNe}$. The prediction for the growth rate of structure assuming a fiducial Planck cosmology is given by the solid black line.
Table 2.3: Constraints on the growth rate as a function of scale and independent of scale (final row) plus the value of the non-linear velocity dispersion $\sigma_v$ and the hyper-parameters $\alpha_{6dF}$ and $\alpha_{SNe}$. Columns 2 and 3 give results from the 6dFGSv survey data alone. Columns 4 and 5 give results from the SNe sample data alone. For columns 6 and 7 we give the results combining both surveys; and for columns 8 and 9 we give the results combining both surveys using a matrix hyper-parameter analysis.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>6dFGSv</th>
<th>SNe</th>
<th>6dFGSv + SNe (Norm)</th>
<th>6dFGSv + SNe (Hyp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f\sigma_8(k_1)$</td>
<td>0.68 $^{+0.17}_{-0.23}$</td>
<td>0.34 $^{+0.20}_{-0.20}$</td>
<td>0.21 $^{+0.23}_{-0.19}$</td>
<td>0.31 $^{+0.21}_{-0.19}$</td>
</tr>
<tr>
<td>$f\sigma_8(k_2)$</td>
<td>0.39 $^{+0.17}_{-0.20}$</td>
<td>0.34 $^{+0.23}_{-0.19}$</td>
<td>0.21 $^{+0.14}_{-0.19}$</td>
<td>0.31 $^{+0.17}_{-0.21}$</td>
</tr>
<tr>
<td>$f\sigma_8(k_3)$</td>
<td>0.44 $^{+0.20}_{-0.20}$</td>
<td>0.38 $^{+0.24}_{-0.19}$</td>
<td>0.260 $^{+0.10}_{-0.15}$</td>
<td>0.38 $^{+0.17}_{-0.19}$</td>
</tr>
<tr>
<td>$f\sigma_8(k_4)$</td>
<td>0.57 $^{+0.22}_{-0.20}$</td>
<td>0.52 $^{+0.23}_{-0.19}$</td>
<td>0.69 $^{+0.17}_{-0.16}$</td>
<td>0.66 $^{+0.17}_{-0.19}$</td>
</tr>
<tr>
<td>$f\sigma_8(k_5)$</td>
<td>0.49 $^{+0.25}_{-0.22}$</td>
<td>0.67 $^{+0.41}_{-0.33}$</td>
<td>0.49 $^{+0.22}_{-0.21}$</td>
<td>0.53 $^{+0.15}_{-0.17}$</td>
</tr>
<tr>
<td>$\sigma_v$ [km/s]</td>
<td>98.4 $^{+370}_{-140}$</td>
<td>372.8 $^{+44}_{-45}$</td>
<td>372.8 $^{+44}_{-45}$</td>
<td>372.8 $^{+44}_{-45}$</td>
</tr>
<tr>
<td>$\alpha_{6dF}$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$\alpha_{SNe}$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$f\sigma_8(z = 0)$</td>
<td>0.424 $^{+0.078}_{-0.068}$</td>
<td>0.432 $^{+0.007}_{-0.084}$</td>
<td>0.429 $^{+0.018}_{-0.065}$</td>
<td>0.492 $^{+0.044}_{-0.108}$</td>
</tr>
</tbody>
</table>
provides our best constraints, $f\sigma_8(k < 0.02h\text{Mpc}^{-1}) = 0.72^{+0.17}_{-0.23}$. This result is consistent with the standard model prediction of $f\sigma_8(z = 0) = 0.4439$, albeit higher than expected.

- Examining the scale-dependence of the growth rate of structure at $z = 0$ we find the constraints $f\sigma_8(k_i) = [0.79^{+0.21}_{-0.25}, 0.30^{+0.14}_{-0.19}, 0.32^{+0.10}_{-0.15}, 0.64^{+0.17}_{-0.16}, 0.48^{+0.22}_{-0.21}]$ using the wavenumber ranges $k_1 \equiv [0.005, 0.02]$, $k_2 \equiv [0.02, 0.05]$, $k_3 \equiv [0.05, 0.08]$, $k_4 \equiv [0.08, 0.12]$ and $k_5 \equiv [0.12, 0.15]$. We find no evidence for a scale-dependence in the growth rate, which is consistent with the standard model. All the growth rate measurements are consistent with the fiducial Planck cosmology.

- Averaging over all scales we measure the growth rate to $\sim 15\%$ which is independent of galaxy bias. This result $f\sigma_8(z = 0) = 0.418 \pm 0.065$ is consistent with the redshift-space distortion analysis of 6dFGS which produced a measurement of $f\sigma_8(z) = 0.423 \pm 0.055$ (Beutler et al., 2012), increasing our confidence in the modelling of galaxy bias. In addition this measurement is consistent with the constraint given by Hudson & Turnbull (2012) of $f\sigma_8 = 0.400 \pm 0.07$, found by comparing the local velocity and density fields. In contrast to our constraint this measurement is sensitive to galaxy bias and any systematic errors introduced during velocity field reconstruction.

- We also consider various other methods to constrain the standard model. We directly constrain the amplitude of the velocity power spectrum $P_{vv}(k) \equiv P_{\theta\theta}(k)/k^2$ for the same scale range as specified above; we find that the predictions from two loop multi-point propagators assuming the Planck cosmology gives an accurate description of the measured velocity power spectrum. Specifically, the derived amplitudes $A_i$ of the power spectrum of 4 bins are consistent with the fiducial cosmology at the $1\sigma$ level, and the largest scale bin is consistent at the $2\sigma$ level. We can also compare these constraints to those given by Macaulay et al. (2012). Similarly to our results they found the amplitude of the matter power spectrum, determined using the composite sample of PVs, to be statistically consistent with the standard $\Lambda$CDM cosmology. In addition they also find on the largest scales a slightly higher amplitude of the power
We show that when analysing PV surveys with velocities derived using the Fundamental Plane or the Tully-Fisher relation, one should perform the analysis using a variable that is a linear transformation of \( x = \log_{10} \left( \frac{D_z}{D_H} \right) \). We show the intrinsic scatter is not Gaussian for the PV and this can significantly bias cosmological constraints. We show how the analysis can be reformulated using the variable \( \delta m \), which removes the bias.

With a large number of upcoming PV surveys, the prospect for understanding how structure grows in the low-redshift universe is excellent. Future work will move beyond consistency tests by adopting specific modified gravity models and phenomenological parametrisations, including measurements of redshift-space distortions and by self-consistently modifying the growth and evolutionary history of the universe. This will allow a vast range of spatial and temporal scales to be probed simultaneously, providing a strong and unique test of the standard ΛCDM model, and perhaps even providing some insight on the so-far mysterious dark energy component of the universe.

\(^{14}\)Note we cannot directly compare these sets of results given different bin ranges were used.
2.6. DISCUSSION AND CONCLUSIONS

Figure 2.17: 68% confidence intervals for the normalized growth rate $f(k, z = 0)\sigma(z = 0)$ for the combined constraints (using no hyper-parameters). The prediction for the growth rate of structure assuming a fiducial Planck cosmology is given by the solid black line.

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Chapter 3

Searching for Modified Gravity: Constraints from Galaxy Peculiar Velocities

Johnson et al.


Abstract

We present measurements of both scale- and time-dependent deviations from the standard gravitational field equations. These late-time modifications are introduced separately for relativistic and non-relativistic particles, by way of the parameters $G_{\text{matter}}(k, z)$ and $G_{\text{light}}(k, z)$ using two bins in both scale and time, with transition wavenumber $0.01$ Mpc$^{-1}$ and redshift 1. We emphasize the use of two dynamical probes to constrain this set of parameters, galaxy power spectrum multipoles and the direct peculiar velocity power spectrum, which probe fluctuations on different scales. The multipole measurements are derived from the WiggleZ and BOSS Data Release 11 CMASS galaxy redshift surveys and the velocity power spectrum is measured from the velocity sub-sample of the 6-degree Field Galaxy Survey. We combine with additional cosmological probes including baryon acoustic oscillations, Type Ia SNe, the cosmic microwave background (CMB), lensing of the CMB, and the temperature–galaxy cross-correlation. Using a Markov Chain Monte Carlo likelihood analysis, we find the inferred best-fit parameter values of $G_{\text{matter}}(k, z)$ and
3.1. OUTLINE

$G_{\text{light}}(k,z)$ to be consistent with the standard model at the 95% confidence level. We expand this analysis by performing Bayesian model selection between our phenomenological model and general relativity. Using the evidence ratio we find “no support” for including modifications to general relativity. Furthermore, accounting for the Alcock-Paczynski effect, we perform joint fits for the expansion history and growth index $\gamma$; we measure $\gamma = 0.665 \pm 0.067$ (68% C.L) for a fixed expansion history, and $\gamma = 0.73^{+0.08}_{-0.06}$ (68% C.L) when the expansion history is allowed to deviate from $\Lambda$CDM. With a fixed expansion history the inferred value is consistent with GR at the 95% C.L; alternatively, a 2$\sigma$ tension is observed when the expansion history is not fixed, this tension is worsened by the combination of growth and SNe data.

3.1 Outline

In Section 3.2 we summarise the adopted phenomenological models and further motivate their use. Then in Section 3.3 we outline the primary datasets used along with the methodology we use to analyse them. Section 3.4 then presents the secondary datasets we employ. The results and interpretations of the MCMC analysis are presented in Section 3.5, and the conclusions are outlined in Section 3.6.

3.2 Modified Growth & Evolution

3.2.1 Introduction

Working within the conformal Newtonian gauge, perturbations to the Robertson-Walker metric can be characterised by two scalar potentials. One scalar potential describes a temporal perturbation to the metric $\psi$, the other a spatial perturbation, $\phi$. The line element in this case is given by

$$ds^2 = a^2[-(1 + 2\psi)d\tau^2 + (1 - 2\phi)d\bar{x}^2] ,$$

(3.1)

where $a$ is the scale factor, $\tau$ is the conformal time – related to the proper time of comoving observers by $\tau = \int dt/a(t)$ – and $\bar{x}$ the spatial coordinate. A non-relativistic fluid within this space-time is characterised in terms of a velocity divergence $\theta(\bar{x}, \tau)$.
and a density perturbation $\delta \rho(\bar{x}, \tau)$. The cosmic evolution of this fluid is then determined by its coupling to the metric potentials.

We concentrate on modifying two of the four gravitational field equations, by requiring energy-momentum conservation ($\nabla^\mu T_{\mu\nu} = 0$), or equivalently, by requiring the contracted Bianchi identity to hold, i.e., $\nabla^\mu G_{\mu\nu} = 0$. Enforcing either constraint one finds the relativistic continuity and Euler equations in Fourier space:

$$\dot{\delta}_m = -\theta_m + 3\phi,$$  \hspace{1cm} (3.2)

$$\dot{\theta}_m = -\mathcal{H}\theta_m + k^2 \psi,$$  \hspace{1cm} (3.3)

where $\delta_m \equiv \delta \rho_m/\bar{\rho}_m$ and $\mathcal{H} \equiv \dot{a}/a = (da/d\tau)/a$, and $\bar{\rho}_m$ is the background matter density. This system of four variables can then be closed by specifying the gravitational field equations; in particular, by defining the relationship between the two metric potentials, and the coupling between the metric potentials and the matter over-density. In GR these relationships are given by

$$\nabla^2 \psi = 4\pi G_N a^2 \bar{\rho}_m \Delta_m,$$  \hspace{1cm} (3.4)

$$\phi = \psi,$$  \hspace{1cm} (3.5)

where $G_N$ is Newton’s gravitational constant, and the equations are defined in terms of the comoving-gauge density perturbation $\Delta_m = \delta_m + (3\mathcal{H}/k^2) \theta_m$.

### 3.2.2 $G_{\text{light}}(k, z)$ and $G_{\text{matter}}(k, z)$

We now introduce two dimensionless free parameters $G_{\text{light}}$ and $G_{\text{matter}}$ that we use to model deviations to the field equations. Our model is now specified as (Daniel & Linder, 2013)

$$\nabla^2 \psi = 4\pi G_N a^2 \bar{\rho}_m \Delta_m \times G_{\text{matter}},$$  \hspace{1cm} (3.6)

$$\nabla^2 (\phi + \psi) = 8\pi G_N a^2 \bar{\rho}_m \Delta_m \times G_{\text{light}}.$$  \hspace{1cm} (3.7)

The first equation governs the motion of non-relativistic particles, while the second controls the propagation of light along null geodesics. As a result, $G_{\text{matter}}$ can be
3.2. **MODIFIED GROWTH & EVOLUTION**

measured using RSDs and direct PVs, and $G_{\text{light}}$ can be measured using weak lensing. Because of this distinction the two parameters are significantly less correlated than models involving a ‘slip’ relation (e.g., Bean & Tangmatitham, 2010). Note that the variables $\{\Sigma, \mu\}$ in Simpson et al. (2013) and Zhao et al. (2012) are equivalent to $\{G_{\text{light}}, G_{\text{matter}}\}$. There is also a trivial re-mapping to the $\{Q, R\}$ parameters used by Bean & Tangmatitham (2010), through $G_{\text{matter}} = QR$, $G_{\text{light}} = Q(1 + R)/2$.

To ensure our model can test for a variety of deviations from GR we allow for both scale- and redshift-dependence: that is, $G_{\text{light}} = G_{\text{light}}(z, k)$ and $G_{\text{matter}} = G_{\text{matter}}(z, k)$. To specify these parameters we use a high vs. low-redshift, large vs. small scale binning approach introduced by Daniel & Linder (2010). Note, however, that very general functional forms for these parameters (including scale-dependent terms) have been developed (Silvestri, Pogosian & Buniy, 2013; Baker et al., 2014). We leave such investigations to future work.

Our adopted model introduces 8 free parameters and requires one to specify a redshift and wavenumber transition scale, $z_t$ and $k_t$. We set $z_t = 1$ and $k_t = 0.01$ $\text{Mpc}^{-1}$; therefore, we have two redshift bins (viz., $0 < z < 1$ and $1 < z < 2$) and two wavenumber bins ($10^{-4} \text{Mpc}^{-1} < k < 10^{-2} \text{Mpc}^{-1}$ and $0.01 \text{Mpc}^{-1} < k < 0.1 \text{Mpc}^{-1}$), while for $z > 2$ and $k < 10^{-4} \text{Mpc}^{-1}$ GR is restored. The transition between bins is implemented using an arctan function of width $\Delta z = 0.05$ and $\Delta k = 0.001 \text{Mpc}^{-1}$.

The choice of the bin transitions $k = 0.01 \text{Mpc}^{-1}$ and $z = 1$ follows Daniel & Linder (2010, 2013). These values are motivated by the sensitivity in both redshift and wavenumber of the available data; for example, the PV measurements add constraints at $k < 0.01 \text{Mpc}^{-1}$ while RSDs add constraints at $k > 0.01 \text{Mpc}^{-1}$. Furthermore, the redshift deviation is limited to lower redshifts (i.e., $z < 1$) given the precision of current CMB constraints.

For our first model we choose to leave the cosmic expansion unmodified at the $\Lambda$CDM prediction, and concentrate on the growth of structure. Henceforth, we will refer to this model as model I. To calculate the relevant observables (to be discussed in the next section) we use camb and CosmoMC. The modified field equations (Eq 3.7) are incorporated into camb using the publicly available code ISITGR (Dossett, Ishak & Moldenhauer, 2011), and the exact equations implemented in camb are given by Dossett, Ishak & Moldenhauer (2011). Note the only significant difference between
the equations employed in \texttt{camb} and Eq (3.7) is that the latter are written within the synchronous gauge (Ma & Bertschinger, 1995).

A few technical comments on the model are unavoidable: Firstly, super-horizon curvature perturbations need to be conserved independent of the form of field equations (Bertschinger & Zukin, 2008). This condition was shown to be satisfied for this model by Pogosian et al. (2010). Additionally, it is natural to include a smoothness theory prior on these parameters, however, given the large distance between the centre of our bins we choose not to include such a prior (Silvestri, Pogosian & Buniy, 2013). With more accurate data, and hence a larger number of bins, this argument will no longer be valid. Finally, the accuracy of any mapping from our model to physical models (i.e., those derived from an action) relies on the validity of the quasi-static approximation (QSA). Following the arguments presented in Silvestri, Pogosian & Buniy (2013) it is reasonable to include a theoretical prior to ignore such deviations.

3.2.3 Varying Growth and Expansion: \{\gamma, w_0, w_a\}

As more freedom is introduced to model deviations from GR the precision of the inferred parameters degrades. We must decide then which features of the standard model to preserve; for example, to what extent does the expansion history dictate the growth history. This presents a balancing problem with no clear solution. To partially circumvent this issue we adopt a second model (which we label model II). In contrast to our first model, this model includes only minimal extensions to the standard model. As a result there are fewer free parameters and more precise tests are possible (although we nonetheless introduce deviations to both the expansion and growth history).

This minimal extension to the standard model using the parameters \{w_0, w_a, \gamma\} has been advocated by Linder & Cahn (2007); Linder (2005), and Simpson & Peacock (2010), and applications have been presented, for example, by Huterer & Linder (2007). To expand on this, we introduce deviations to the expansion history through a time-dependent equation of state \(w(z)\), which is expressed in terms of two free parameters: \(w_0 = w(a = 0)\) and \(w_a = -(dw/da)|_{a=1}\), as a function of the
redshift \( w(z) = w_0 + w_0 z/(1 + z) \). Note the expansion history is still governed by
the Friedman equation, there is simply more freedom in the properties of the dark
energy component. We introduce deviations in the growth history by parameter-
ing the growth rate as \( f(z) \equiv \Omega_m(z) \gamma \), where \( \gamma \) is the growth index; within GR
one expects \( \gamma \sim 0.55 \). The growth rate is defined by \( f(a) \equiv d \ln D(a)/d \ln a \), and
\( D(a) \equiv \delta(a)/\delta(a = 1) \).

### 3.3 Primary Datasets: Methodology

Below we will outline the measurements we use in Sec. 3.5, in addition to the tools
we use to analyze them. A general summary is provided in Table 3.1 where the
datasets, the measured quantities, and the fitting ranges adopted are specified. The
focus will be on introducing extensions to the public MCMC code CosmoMC (Lewis
& Bridle, 2002) and camb (Lewis, Challinor & Lasenby, 2000) to update the range
of datasets one can analyze.

#### 3.3.1 Velocity Power Spectrum

The radial PVs of galaxies in the local universe induce a fluctuation in the apparent
magnitude \( m \), defined as (Hui & Greene, 2006)

\[
\delta m(z) = [m(z) - \bar{m}(z)].
\] (3.8)

The over-bar indicates that the variable is being evaluated within a homogeneous
universe, namely, a universe with no density gradients and therefore no peculiar
velocities. Recall the apparent magnitude is defined as

\[
m = M + 5 \log_{10}(D_L(z)) + 25.
\] (3.9)

Here \( M \) is the absolute magnitude, and \( D_L(z) \) the luminosity distance. The presence
of large scale clustering induces fluctuations in \( \delta m(z) \) from galaxy to galaxy (this is
equivalent to a peculiar velocity), furthermore, these fluctuations are correlated for
nearby galaxies (Hui & Greene, 2006; Gordon, Land & Slosar, 2007). The magnitude
Table 3.1: Summary of the datasets used in this analysis. Given model I includes scale-dependent terms, we divide our measurements into three separate groups: those used to constrain model I & II, only model I, and only model II. This division is indicated by the horizontal lines, and follows the order in which the categories were introduced.

<table>
<thead>
<tr>
<th>Cosmological Probe</th>
<th>Dataset</th>
<th>Measured quantity</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMB temperature</td>
<td>Planck</td>
<td>( C_{TT}^T )</td>
<td>Planck Collaboration et al. (2013)</td>
</tr>
<tr>
<td>CMB polarization</td>
<td>WMAP-9</td>
<td>( C_{EE}^E )</td>
<td>Bennett et al. (2013)</td>
</tr>
<tr>
<td>CMB-Lensing</td>
<td>Planck</td>
<td>( C_\phi^\phi )</td>
<td>Planck Collaboration et al. (2014b)</td>
</tr>
<tr>
<td>BAOs</td>
<td>6dFGS</td>
<td>( r_s/D_v(z) )</td>
<td>Beutler et al. (2011)</td>
</tr>
<tr>
<td></td>
<td>BOSS DR11 LOWZ</td>
<td>( D_V(r_s^{6d}/r_s) )</td>
<td>Anderson et al. (2014b)</td>
</tr>
<tr>
<td></td>
<td>BOSS DR11 QSA-Lyα</td>
<td>( H(z)r_s, D_A/r_s )</td>
<td>Font-Ribera et al. (2014)</td>
</tr>
<tr>
<td></td>
<td>BOSS DR11 Lyα</td>
<td>( H(z)r_s, D_A/r_s )</td>
<td>Delubac et al. (2014)</td>
</tr>
<tr>
<td>Type Ia Supernovae</td>
<td>SNLS</td>
<td>( \mu(z) )</td>
<td>Conley et al. (2011)</td>
</tr>
<tr>
<td>Dataset extension I</td>
<td>WMAP3</td>
<td>( C_{TT}^T )</td>
<td>Ho et al. (2008)</td>
</tr>
<tr>
<td>Velocity Power Spectrum</td>
<td>6dFGSv</td>
<td>( P_{\text{re}}(k) )</td>
<td>Johnson et al. (2014)</td>
</tr>
<tr>
<td>BAO (reconstructed)</td>
<td>WiggleZ</td>
<td>( D_V(r_s^{6d}/r_s) )</td>
<td>Kazin et al. (2014)</td>
</tr>
<tr>
<td></td>
<td>DR11 CMASS</td>
<td>( D_A(z)(r_s^{6d}/r_s), H(z)(r_s/r_s^{6d}) )</td>
<td>Anderson et al. (2014a)</td>
</tr>
<tr>
<td>Power Spectrum Multipoles</td>
<td>DR11 CMASS</td>
<td>( P_0(k), P_2(k) )</td>
<td>Beutler et al. (2014)</td>
</tr>
<tr>
<td></td>
<td>WiggleZ (( z_{\text{eff}} = 0.44 ))</td>
<td>( P_0(k), P_2(k), P_4(k) )</td>
<td>Blake et al. (2011a)</td>
</tr>
<tr>
<td></td>
<td>WiggleZ (( z_{\text{eff}} = 0.73 ))</td>
<td>( P_0(k), P_2(k), P_4(k) )</td>
<td>Blake et al. (2011a)</td>
</tr>
</tbody>
</table>

| Dataset extension II | WMAP3 | \( C_{TT}^T \)    | Ho et al. (2008) |
| RSDs               | 6dFGS  | \( f_{\sigma_s}(z) \) | Beutler et al. (2012) |
| RSD-BAO-AP         | WiggleZ | \( A(z), F_{\text{AP}}(z), f_{\sigma_A}(z) \) | Blake et al. (2012) |
| RSD-BAO-AP         | BOSS CMASS | \( D_v/r_s(z), F_{\text{AP}}(z), f_{\sigma_A}(z) \) | Beutler et al. (2014) |

\( a \) Both the reconstructed BAO measurements (CMASS and WiggleZ) have been calculated by marginalising over the general shape of the correlation function. Marginalising over the shape decorrelates the BAO measurement with the power spectrum multipole measurement, allowing one to fit for both measurements simultaneously.

\( b \) Note, however, these measurement have been updated in this work using an improved methodology.
3.3. PRIMARY DATASETS: METHODOLOGY

of both effects can be described by a covariance matrix which we define as $C_{ij}^m \equiv \langle \delta m_i(z_i) \delta m_j(z_j) \rangle$. Once a model is specified this covariance matrix can be calculated as

$$C_{ij}^m = G(z_i, z_j) \int \frac{dk}{2\pi^2} k^2 \mathcal{P}_{vv}(k, a = 1) W(k, \alpha_{ij}, r_i, r_j).$$

(3.10)

Where $\mathcal{P}_{vv}(k) = \mathcal{P}_{\theta\theta}(k)/k^2$ is the velocity power spectrum, and $\theta = \nabla \cdot \vec{v}$ is the velocity divergence. Moreover, the window function is defined as

$$W(k, \alpha_{ij}, r_i, r_j) = 1/3 [j_0(kA_{ij}) - 2j_2(kA_{ij})] \cdot \hat{r}_i \cdot \hat{r}_j + \frac{1}{A_{ij}^2} j_2(kA_{ij}) r_i r_j \sin^2(\alpha_{ij}),$$

and,

$$G(z_i, z_j) \equiv \left( \frac{5}{\ln 10} \right)^2 \left( 1 - \frac{(1 + z_i)^2}{H(z_i) D_L(z_i)} \right) \left( 1 - \frac{(1 + z_j)^2}{H(z_j) D_L(z_j)} \right),$$

where $\alpha_{ij} = \cos^{-1}(\hat{r}_i \cdot \hat{r}_j)$, $A_{ij} \equiv |r_i - r_j|$ and $r_i$ is the position vector of the $i$th galaxy. This analytic solution for the window function was presented by Ma, Gordon & Feldman (2011). For further details on this calculation we refer the reader to Johnson et al. (2014).

For this analysis we perform a full likelihood calculation using the 6dFGSv peculiar velocity sample. This is done using the covariance matrix Eq. (3.10). The velocities in this sample are derived by Springob et al. (2014) using the Fundamental Plane (FP) relation. To calculate the covariance matrix we integrate over the wavenumber range $k = 0.0005 - 0.15 h$ Mpc$^{-1}$. For this calculation we neglect velocity bias because large-scale information currently dominates in peculiar velocity measurements.

Joachimi, Singh & Mandelbaum (2015) recently reported a detection of spatial correlations among offsets in galaxy size from a FP. Moreover, they highlight that this trend will bias measurements of the velocity power spectra at the 10% level for $k > 0.04 h$ Mpc$^{-1}$. In relation to this potential source of systematic bias, we note that Johnson et al. (2014) demonstrated that the velocity power spectra measurements were consistent when using PVs derived from different distance indicators; as such, we argue that currently this trend does not significantly influence our results.
In order to minimise the influence of poorly understood non-linear effects a non-linear velocity dispersion component $\sigma_{PV}$ is introduced into the diagonal elements of the covariance matrix (Silberman et al., 2001). This nuisance parameter is marginalised over in the analysis. The covariance matrix is thereby updated:

$$\Sigma_{ij} \equiv C_{ij}^{m} + \sigma_{PV}^{2}\delta_{ij}.$$  \hspace{1cm} (3.11)

One can now define the posterior distribution as

$$P(\Sigma|\delta m) = |2\pi \Sigma|^{-1/2} \exp \left( -\frac{1}{2} \delta m^T \Sigma^{-1} \delta m \right),$$ \hspace{1cm} (3.12)

where $\delta m$ is a vector of the observed apparent magnitude fluctuations. Note the dependence on the cosmological model is introduced through the covariance matrix.

The model velocity power spectrum is generated using a transfer function. This can be defined starting from the peculiar velocity in the synchronous gauge $v_p^{(s)}$ (cf., Ma & Bertschinger, 1995)\(^1\). As this gauge is defined in the dark matter rest frame, i.e., there are no temporal $g_{\mu\nu}$ perturbations, a gauge transformation is necessary. Using the convention of Ma & Bertschinger (1995) we define $\epsilon$ and $\eta$ as the metric perturbation in the synchronous gauge. Now by moving into the Newtonian gauge one finds the appropriate transfer function:

$$T_v(k) = \frac{c}{k^2} \left( k\alpha + \rho_b v_p^{(s)}/(\rho_b + \rho_c) \right),$$ \hspace{1cm} (3.13)

where $k^2\alpha = \dot{\epsilon}/2 + 3\eta$.

In Fig. 3.1 we plot the measurements of $P_{vv}(k)$ by Johnson et al. (2014), here the blue (green) points were measured using the 6dFGSv (low-z SNe) sample. For this plot the black line shows the power spectrum prediction assuming GR, while the red and orange lines show the predictions for different values of the post-GR parameters. For these calculations the Planck best-fit parameters are assumed. Additionally, the green line shows the prediction when using our best-fit parameter values (see sect. 3.5 for details). Note the time evolution of the density perturbation $\Delta_m$ is set by a friction term $2\mathcal{H}\Delta_m$ and a source term $k^2\dot{\psi}$. Therefore, by modifying $G_{\text{matter}}$ one

\(^1\)Our starting point is set by variables used within camb.
changes the source term to $k^2 \psi \sim a^2 G_{\text{matter}}(k, z) \Delta_m$; hence, with $G_{\text{matter}}(k, z) > 1$ both the late-time clustering and the amplitude of the velocity power spectrum are enhanced.

3.3.2 Power Spectrum Multipoles

We measured the multipole power spectra of the WiggleZ Survey data using the direct estimation method introduced by Yamamoto et al. (2006) and extended by Blake et al. (2011a) and Beutler et al. (2014). We provide a brief summary of the technique here, referring the reader to the above papers for a full description.

The redshift-space 2D galaxy power spectrum $P_s^g(k, \mu)$, where $\mu$ is the cosine of the angle of the wavevector $\vec{k}$ with respect to the line-of-sight, may be expressed in terms of multipole moments $P_\ell(k)$ using a basis of Legendre polynomials $L_\ell(\mu)$:

$$\sum_{\text{even } \ell} P_\ell(k) L_\ell(\mu),$$

where

$$P_\ell(k) = \frac{2\ell + 1}{2} \int_{-1}^{1} d\mu P_s^g(k, \mu) L_\ell(\mu).$$

The power spectrum multipoles provide a form of data compression; in linear theory all the information is contained in the $\ell = 0, 2, 4$ terms, with the first two multipoles dominating the observed signal.

The rapid estimation technique of using Fast Fourier Transform (FFT) methods to measure $P_s^g(k, \mu)$ in bins of $k$ and $\mu$, where $\mu$ is defined with respect to a fixed axis parallel to the line-of-sight of the field centre, and then estimating $P_\ell(k)$ by a direct sum over the binned results using Equation 3.15, has two difficulties. First, for a wide-area survey the line-of-sight direction with respect to which $\mu$ should be measured will not be fixed. Secondly, at low $k$ the sum over $\mu$ bins is problematic to evaluate due to the limited number of modes available in Fourier space. The Yamamoto et al. (2006) method estimates $P_\ell(\vec{k})$ using a sum over all galaxies for each wavevector $\vec{k}$ on the FFT grid, allowing the line-of-sight vector to vary for each object and without binning in $\mu$. Window function effects are included using a similar sum over unclustered objects. Additive corrections are included for shot
Figure 3.1: The velocity power spectrum $P_{vv}(k)$ at $z = 0$ for different parameter combinations of the adopted phenomenological model. The black line shows the prediction assuming General Relativity, and the orange and red lines illustrate the effect of varying the low-$z$ and high-$k$ bin for $G_{\text{matter}}$. For the red line $G_{\text{matter}}(z < 1; k > 0.01) = 1.8$ and for the orange line $G_{\text{matter}}(z < 1; k > 0.01) = 0.3$: for these predictions the standard cosmological parameters are fixed at the Planck best-fit values, and unless specified otherwise all non-GR parameters are set to be consistent with GR (i.e., set equal to 1). Moreover, the green line shows the prediction found using the best-fit parameter values found using set 4 (see sect. 3.5 for details). The best-fit values here correspond to the parameter values that maximise the likelihood. The blue and green data points correspond to the 68% confidence intervals for the mean power within each bin for the 6dFGSv data and the low-$z$ SNe data set constructed in Johnson et al. (2014). The thick black line indicates the the mean power predicted by GR in each k-bin, this is calculated assuming a Planck cosmology.
noise and for the discreteness of the grid. The measurements are then binned by wavenumber $k = |\vec{k}|$.

Following the analysis of the WiggleZ baryon acoustic oscillations (Blake et al., 2011b), we estimated the $\ell = 0, 2, 4$ multipole power spectra in the (9, 11, 15, 22, 1, 3)-hr survey regions in the overlapping redshift ranges $0.2 < z < 0.6$, $0.4 < z < 0.8$ and $0.6 < z < 1.0$. We measured the spectra in 14 wavenumber bins of width $\Delta k = 0.02\ h\ \text{Mpc}^{-1}$ in the range $0.02 < k < 0.3\ h\ \text{Mpc}^{-1}$. For this analysis, however, we only use the non-overlapping redshift ranges that we label low-$z$ and high-$z$. The results for the monopole and quadrupole are given in Fig. 3.2.

We determined the covariance matrix of each vector $[P_0(k), P_2(k), P_4(k)]$ by repeating the measurements in each survey region for a series of 600 mock catalogues, built from N-body simulations generated by the method of COnmoving Lagrangian Acceleration (COLA; Tassev, Zaldarriaga & Eisenstein, 2013). As described by Kazin et al. (2014) we produced a halo catalogue by applying a friends-of-friends algorithm to the dark matter particles, and populated the haloes with mock galaxies using a Halo Occupation Distribution such that the projected clustering matched that of the WiggleZ galaxies. The mocks were sub-sampled using the selection function of each region, and galaxy co-ordinates converted to redshift-space.

We also determined the convolution matrix for each region and redshift slice, which should be used to project a model multipole vector to form a comparison with the data given the survey window function. For a wide-angle survey such as the BOSS, determination of the convolution involves a numerically-intensive double sum over randomly-distributed objects (Beutler et al., 2014). However, for the more compact WiggleZ Survey geometry, we found that it was acceptable (in the sense that any offset was far smaller than the statistical error) to use a flat-sky approximation, in which FFT methods were used to convolve a series of unit multipole vectors, generating each row of the convolution matrix in turn.

In addition to the WiggleZ multipole measurements, we include the monopole and quadrupole measurements from the BOSS-DR11 CMASS sample presented in Beutler et al. (2014); the reader is referred to this paper for technical details on the calculation. From the CMASS sample the $l = 0, 2$ multipole power spectrum are calculated for the wavenumber range $k = 0.01 - 0.20\ h\ \text{Mpc}^{-1}$ with a spacing of
$\Delta k = 5 \times 10^{-3} h \text{ Mpc}^{-1}$. These measurements are presented for both the North and South Galactic Cap regions at an effective redshift of $z_{\text{eff}} = 0.57$.

We plot the CMASS multipole measurements in Fig. 3.3. For this plot the blue-dashed (red-dashed) lines show the multipole predictions when setting $G_{\text{matter}}(k > 0.01; z < 1) = 1.8$ ($G_{\text{matter}}(k > 0.01; z < 1) = 0.3$), while the black lines show the prediction assuming GR. For these predictions the best-fit parameters from Planck are assumed, in addition we set the bias to $b = 1.85$, the non-Poisson contribution to shot noise to $N = 1800 h^{-3} \text{Mpc}^3$, and the velocity dispersion to $\sigma_v = 4 h^{-1} \text{Mpc}$.

Moreover, the orange lines give the prediction when using our best-fit model parameters (see sect. 3.5 for details). Note, for simplicity the theory predictions have only been convolved with the NGC window function.

Modelling the Power Spectrum Multipoles

To model the redshift-space 2D galaxy power spectrum $P_g^s(k, \mu)$ we use linear theory plus an empirical Gaussian damping term (Hatton & Cole, 1998); the resulting model is given by

$$P_g^s(k, \mu) = \left[ P_{gg}(k) - 2\mu^2 P_{g\theta}(k) + \mu^4 P_{\theta\theta}(k) \right] D(\mu, k),$$

(3.16)

where $D(\mu, k) = \exp\left[-(kf\mu\sigma_v)^2\right]$. The standard interpretation of this damping, which is clearly observed in redshift surveys, is the uncorrelated pairwise velocity dispersion of galaxies. We absorb our ignorance by treating $\sigma_v$ as a free parameter to be marginalised over for each survey.

Assuming linear theory the continuity equation (eq. 3.3) can be written in Fourier space as

$$\theta(k) = -f(a)\delta(k).$$

(3.17)

However, we are modifying the gravitational field equations, so one needs to be self-consistent, given that the modifications (Eq. 3.7) will change the growth rate in a scale-dependent manner. We calculate this modified scale-dependent growth rate
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Figure 3.2: The monopole, quadrupole and hexadecapole measurements from the WiggleZ survey, for both the high-$z$ and low-$z$ samples ($z = 0.44, 0.74$, respectively). For simplicity we combine the results from the 6 different survey regions; however, note this is not the format of the data we use, each survey region has a different window function and therefore is analysed separately.
as
\[ f(k, a) = \frac{d \ln \Delta_v(k, a)}{d \ln a}. \] (3.18)

This is self-consistent given \texttt{camb} contains all the relevant physics, i.e., the density and velocity variables are evolved according to the modified field equations. As a reminder of the potential scale-dependence we write the growth rate as \( f(k) \). Note, for both the CMASS and WiggleZ multipoles the standard Poisson shot noise (\( 1/n \)) has been subtracted. However, for the CMASS multipoles (reflecting the approach taken by Beutler et al. (2014)) we include a free parameter \( N \) to account for non-Poisson contributions to the shot noise (Baldauf et al., 2013). Now assuming a local, scale-independent linear bias (\( \delta_g = b \delta \)) and no velocity bias (\( \theta_g = \theta \)) Eq. (3.16) reduces to
\[ P_s^g(k, \mu) = b^2 (P_{\delta \delta}(k) + N) \left( 1 + f(k) \mu^2 / b \right)^2 D(\mu, k), \] (3.19)

To justify the previous assumptions we truncate the fit for both the WiggleZ and CMASS multipoles to relatively large scales; to wit, we set \( k_{\text{CMASS}}^{\text{max}} = 0.10 h \text{ Mpc}^{-1} \) and \( k_{\text{WiggleZ}}^{\text{max}} = 0.15 h \text{ Mpc}^{-1} \). The WiggleZ measurements are used to a higher wavenumber because of the smaller bias of the sample (\( b \sim 1 \)), in addition to the larger error bars\(^2\). The satellite fraction of the sample will also influence the wavenumber at which non-linear (1-halo) contributions are significant. On this point, we note that Halo Occupation Distribution fits to the projected clustering of WiggleZ galaxies (Koda et al. 2016) show that WiggleZ galaxies have a satellite fraction consistent with zero, and the projected clustering can be produced by central galaxies alone.

The matter power spectrum is calculated within \texttt{camb} using only linear theory: we choose not to incorporate non-linear corrections via \texttt{HALOFIT}. The use of \texttt{HALOFIT} presents an issue as the corrections have not been shown to be valid for general modified gravity models.

In order to correctly interpret RSD measurements one is required to consistently incorporate our ignorance of the expansion history of the universe (viz., \( H(z) \)), bear-\footnote{With a lower biased tracer, for example, the effect of non-local halo bias is less significant (Chan, Scoccimarro & Sheth, 2012).}
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ing in mind that these measurements are performed assuming a fiducial cosmological model. As a result, in a trial cosmology, the growth rate measurements should be adapted using the covariance with the Alcock-Paczynski (AP) distortion. Any discrepancy between the chosen fiducial expansion history \((\tilde{D}_A(z), \tilde{H}(z))\) and the physical expansion history \((D_A(z), H(z))\) can be accounted for by scaling the true (physical) radial and tangential wavenumbers \((k_{\parallel}^{\text{true}}, k_{\perp}^{\text{true}})\). The amplitude of the wavenumber scalings is determined by

\[
\alpha_{\parallel} = \frac{H^\text{fid}(z)}{H(z)}, \quad \alpha_{\perp} = \frac{D_A(z)}{D^\text{fid}_A(z)}
\]

Hence the observed wavenumbers are given by \(k_{\parallel}^{\text{obs}} = \alpha_{\parallel} k_{\parallel}^{\text{true}}\), and \(k_{\perp}^{\text{obs}} = \alpha_{\perp} k_{\perp}^{\text{true}}\). Including this scaling in Eq (3.19) one finds (Ballinger, Peacock & Heavens, 1996; Matsubara & Suto, 1996; Simpson & Peacock, 2010)

\[
P_\delta^4(k', \mu') = \frac{b^2}{\alpha_{\perp}^2 \alpha_{\parallel}} \left[ 1 + \mu'^2 \left( \frac{1 + \beta}{\alpha_{\parallel}^2 / \alpha_{\perp}^2 - 1} \right) \right]^2 \times \left[ 1 + \mu'^2 \left( \frac{\alpha_{\perp}^2}{\alpha_{\parallel}^2} - 1 \right) \right]^{-2} \times
\]

\[
P_\delta \left[ \frac{k'}{f_{\perp}} \sqrt{1 + \mu'^2 \left( \frac{\alpha_{\perp}^2}{\alpha_{\parallel}^2} - 1 \right)} \right] \times D(\mu, k)
\]

where \(k' = \sqrt{(k_{\perp}^{\text{obs}})^2 + (k_{\parallel}^{\text{obs}})^2}\), \(\mu' = k_{\parallel}^{\text{obs}} / k'\), and \(\beta = f / b\). This scaling introduces a new source of anisotropy in the clustering of galaxies, making it partially degenerate with redshift-space distortion effects, accordingly it is important to account for this effect in this type of analysis (Blake et al., 2012; Beutler et al., 2014).

Two components must be included to compare our theoretical predictions with observations: the window function and integral constraint effect, both of which result in a distortion to the measured power spectrum relative to the true power spectrum. Window function effects are induced by the complex geometry of the survey (viz, a non-cubical geometry); and the integral constraint effect occurs as the condition \(\delta_{k=0} = 0\) is applied to the data: this imposed normalization for the \(k = 0\) mode is invalidated by super-survey modes. Both effects induce a suppression of power at

A consistent comparison between our model and the observations therefore requires us to include the window function effects in our modelling. Following Beutler et al. (2014) the convolved multipoles $P_{l}^{\text{conv}}(k)$ are calculated for the CMASS sample as

$$P_{l}^{\text{conv}}(k) = 2\pi \int dk' k'^2 \sum_{L} P_{L}^{\text{theory}}(k')|W(k, k')|^2_{lL} - P_{l}^{\text{ic}}(k),$$

where

$$|W(k, k')|^2_{lL} = 2\ell(-i)^L (2\ell + 1) \sum_{i \neq j} w_{FKP}(\vec{x}_i) w_{FKP}(\vec{x}_j) j_{\ell}(k|\Delta \vec{x}|) j_{L}(k'|\Delta \vec{x}) L_{\ell}(\hat{x}_h \cdot \hat{x}) L_{L}(\hat{x}_h \cdot \hat{x}),$$

and the integral constraint term is given by

$$P_{l}^{\text{ic}}(k) = 2\pi \frac{|W(k)|^2_{lL}}{|W(0)|^2_{0}} \int dk' k'^2 \sum_{L} P_{L}^{\text{theory}}(k')|W(k')|^2_{lL} \frac{2}{2L + 1}.$$

Here $j_{L}$ are spherical Bessel functions of order $L$, $N_{\text{ran}}$ is the number of galaxies in the synthetic catalogue, and we sum over the monopole and quadrupole ($L = 0, 2$).

Each survey region has a different window function and hence needs to be treated separately. To compute the CMASS likelihood we use the publicly available CMASS window functions\(^3\). For example, the WiggleZ likelihood is computed as

$$-2 \ln(\mathcal{L}^{\text{WiggleZ}}) = \sum_{i=1}^{12} (\tilde{P}^{\text{WiggleZ}}_i - \tilde{P}^{\text{Conv}}_i)^T \hat{C}^{-1}_{\text{Wig},i} (\tilde{P}^{\text{WiggleZ}}_i - \tilde{P}^{\text{Conv}}_i),$$

The $i$ indices specify the two redshift bins and six survey regions for WiggleZ, and $\tilde{P}^{\text{conv}}_i = [P_{0}^{\text{conv}}(k), P_{2}^{\text{conv}}(k), P_{4}^{\text{conv}}(k)]$. The hat on the covariance matrix indicates that we are using a statistical estimator for the inverse covariance matrix. This estimator is determined by the covariance matrix measured from mock catalogues: typically one would use $\hat{C}^{-1} = C^{-1}_{\text{mock}}$, however, the noise in the derived covariance

\(^3\)https://sdss3.org/science/boss_publications.php
matrix \((C_{\text{mock}}^{-1})\) makes this estimator biased (Hartlap, Simon & Schneider, 2007). We correct this bias using the estimator\(^4\)

\[
\hat{C}^{-1} = \frac{N_s - n_b - 2}{N_s - 1} C_{\text{mock}}^{-1},
\]

(3.23)

where \(n_b\) is the number of power spectrum bins, and \(N_s\) the number of mock realisations used to construct the covariance matrix.

### 3.3.3 BAOs

Acoustic oscillations in the photon-baryon plasma, prior to recombination, imprint a series of fluctuations in large-scale structure: in configuration-space one finds a preference for galaxies to be distributed with a given comoving separation \((\sim 105 \, h^{-1}\text{Mpc})\).

This excess in clustering (the BAO feature) functions as a cosmic yard-stick allowing the cosmic expansion history to be mapped out. By measuring the spherically averaged BAO position one determines

\[
D_V(z) = \left[c z (1 + z)^2 D_A(z)^2 / H(z)\right]^{1/3}.
\]

(3.24)

Here \(D_A(z)\) is the angular diameter distance. With higher signal to noise measurements one can extract more information by isolating the transverse and line-of-sight BAO positions, determining

\[
\alpha_{\text{perp}} = \frac{D_A(z) r_{s,\text{fid}} / D_A^\text{fid}(z) r_s}{D_A(z) r_{s,\text{fid}} / D_A^\text{fid}(z) r_s} \quad \text{(3.25)}
\]

\[
\alpha_{\text{par}} = \frac{H_{\text{fid}}(z) r_{s,\text{fid}} / H(z) r_s}{H_{\text{fid}}(z) r_{s,\text{fid}} / H(z) r_s} \quad \text{(3.26)}
\]

By including the dependence on \(r_s\) (the sound horizon at the drag epoch), and expressing the measured quantity as a ratio of the fiducial prediction, the dependence on CMB physics and the assumed cosmology has been made explicit.

To constrain the expansion history we use the following BAO measurements:

---

\(^4\)Note, we neglect the secondary correction to this term introduced by Percival et al. (2014). We estimate the magnitude of this effect, which is an error-in-the-error, to be around 5%.
Figure 3.3: The monopole and quadrupole power spectrum for both the BOSS-DR11 CMASS survey regions (NGC and SGC). The blue-dashed line shows the prediction with $G_{\text{matter}}(k > 0.01; z > 1) = 1.8$, the red-dashed lines $G_{\text{matter}}(k > 0.01; z < 1) = 0.3$; for these predictions the best-fit parameters from Planck are assumed and we set the bias to $b = 1.85$, the non-Poisson contributions to the shot noise to $N = 1800 \, h^{-3}\text{Mpc}^3$, and the velocity dispersion to $\sigma_v = 4 \, h^{-1}\text{Mpc}$. In this plot, for simplicity, the theory predictions have only been convolved with the NGC window function. The orange lines gives the prediction from the best-fit model parameters (see sect. 3.5 for details), convolved with the NGC window function. Note, for the final analysis we only fit our model to $k_{\text{max}} = 0.10 h \, \text{Mpc}^{-1}$. 

\[ P_0(k) = \left( \frac{\text{Mpc}}{h} \right)^3 \]

\[ P_2(k) = \left( \frac{\text{Mpc}}{h} \right)^3 \]
3.3. PRIMARY DATASETS: METHODOLOGY

WiggleZ reconstructed from Kazin et al. (2014), reconstructed DR11–CMASS and DR11–LOWZ from Anderson et al. (2014a), and the 6dFGS measurement from Beutler et al. (2011). By ‘reconstructed’ we are referring to the process of sharpening the acoustic peak by using information from the local density field (cf. Padmanabhan et al., 2012). The above measurements (excluding CMASS) can be incorporated into a likelihood given by

\[
-2 \ln \mathcal{L} = (x - S)^T C^{-1}(x - S),
\]

with the theory vector

\[
x = [D_V(0.44)(r_{\text{fid}}/r_s), D_V(0.6)(r_{\text{fid}}/r_s), D_V(0.73)(r_{\text{fid}}/r_s), D_V(0.32)/r_d, r_s/D_V(0.106)],
\]

the data vector

\[
S = [1716, 2221, 2516, 8.25, 0.336],
\]

and the covariance matrix\(^5\)

\[
C_{\text{BAO}}^{-1} = \begin{pmatrix}
2.17898 & 1.11633 & 0.46982 & 0 & 0 \\
1.11633 & 1.70712 & 0.71847 & 0 & 0 \\
0.46982 & 0.71847 & 1.65283 & 0 & 0 \\
0 & 0 & 0 & 36.025 & 0 \\
0 & 0 & 0 & 0 & 4444.4
\end{pmatrix}
\]

The CMASS measurements are in the form of probability distributions for \(P(\alpha_{\text{perp}})\) and \(P(\alpha_{\text{par}})\) evaluated at \(z_{\text{eff}} = 0.57\). These measurements are therefore analysed separately, for details see Anderson et al. (2014a). A number of these BAO measurements have been calculated using the approximate fitting formula for \(r_s(z_d)\) from Eisenstein & Hu (1998); hence throughout, where appropriate, the BAO measurements derived using this approximation are scaled to be consistent with the result from camb (cf., Mehta et al., 2012).

\(^5\)We have scaled the WiggleZ elements for clarity; the true covariance matrix is obtained by scaling the WiggleZ elements by \(10^{-4}\): \(C_{\text{BAO}}^{-1\text{True}} = 2.17898 \times 10^{-4}\).
To further improve the redshift range of our expansion history measurements we extend this ‘base’ sample by including the Lyman-α BAO measurements from Delubac et al. (2014), and the Quasar-Lyα cross-correlation measurement from Font-Ribera et al. (2014). The measurements are $D_H(z = 2.34)/r_s = 9.18 \pm 0.28$, $D_A(z = 2.34)/r_s = 11.28 \pm 0.65$, $D_H(z = 2.36)/r_s = 9.0 \pm 0.3$, $D_A(z = 2.36)/r_s = 10.8 \pm 0.4$, where $D_H = c/H$. Both common cosmic variance or a common source for the measurement error would induce correlations between the Lyman-α measurement. Fortunately, the origin of the dominant error components for these measurements are distinct, and hence the measurements are uncorrelated (Font-Ribera et al., 2014). Additionally, we treat any correlations between the BOSS and WiggleZ surveys as insignificant, given the small overlapping area ($\sim 550$ deg$^2$) and the significance of shot noise in WiggleZ measurements (Beutler et al., 2016).

3.3.4 Growth Rate and Alcock-Paczynski Measurements

The growth rate measurements presented in this section will be used to constrain $\gamma$. Following the arguments presented in subsection 3.3.2 we only include growth rate constraints that have consistently incorporated the Alcock-Paczynski effect. The exception to this point is for very low-redshift observations, which are effectively insensitive to changes in the expansion history.

In order to self-consistently express the degeneracy with the expansion history we chose to fit to joint 3D posterior distributions from AP, BAO and RSD measurements: as opposed to marginalized 1D constraints on $f\sigma_8(z)$. The growth rate measurements we utilize are measured from BOSS-DR11 survey, the WiggleZ Dark Energy Survey, and the 6dF Galaxy survey (Beutler et al., 2014; Blake et al., 2012; Beutler et al., 2012). For the CMASS sample we use the data vector\(^6\)

$$S_{k_{\text{max}}=0.20}^{\text{BOSS}} = [D_V(0.57)/r_s(z_d), F_{\text{AP}}(0.57), f(0.57)\sigma_8(0.57)]$$

$$= [13.88, 0.683, 0.422]. \quad (3.30)$$

Where the AP effect translates into a geometric constraint on $F_{\text{AP}}(z) = (1 + \ldots$

\(^6\)This result is found fitting the power spectrum multipoles to $k_{\text{max}} = 0.20 \, h$ Mpc$^{-1}$. 109
3.3. PRIMARY DATASETS: METHODOLOGY

The WiggleZ survey measurements are performed within three overlapping, hence correlated, redshift bins at $z_{\text{eff}} = 0.44, 0.60, 0.73$. We first split the data vector into redshift bins, namely $S^{\text{WiggleZ}}_{k_{\text{max}}=0.30} = (S_{z_1}, S_{z_2}, S_{z_3})$. In each of these redshift bins Blake et al. (2012) measure the parameter combination

$$S_{z_i} = [A(z_i), F_{\text{AP}}(z_i), f(z_i)\sigma_8(z_i)] ,$$

where $A(z)$, the acoustic parameter, is given by

$$A(z) \equiv \frac{100D_V(z)\sqrt{\Omega_m h^2}}{cz} .$$

The measured values are now $S_{z_1} = (0.474, 0.482, 0.413)$, $S_{z_2} = (0.442, 0.650, 0.390)$, and $S_{z_3} = (0.424, 0.865, 0.437)$. Table 2 in Blake et al. (2012) gives the full covariance matrix for $S^{\text{WiggleZ}}$. The final measurement we use is $f(0.067)\sigma_8(0.067) = 0.423 \pm 0.55$ from Beutler et al. (2012). As noted previously, the AP effect is not significant for this measurement given the low-redshift nature of the sample. All of the introduced measurements are now incorporated using the likelihood

$$-2 \ln \mathcal{L} = (x - S)^T C^{-1} (x - S) ,$$

here $x$, $S$ and $C$ are the appropriate theory vector, data vector and covariance matrix. Note that BAO information is included in both Sec. 3.3.3 and Sec. 3.3.4 and we do not double-count this information.
3.4 Secondary Datasets

A brief introduction and motivation is given for the additional datasets we use.

3.4.1 Type-Ia SNe

Sample variance effectively imposes a minimum volume limit for BAO detection. Accordingly, large volumes and hence higher redshift observations are preferable. Type-Ia SNe measurements do not have this restriction and hence can provide very accurate constraints on the low-redshift expansion rate: an epoch where the presence of “dark energy” appears to dominate.

Therefore we include the distance modulus measurements for 473 type Ia SNe presented in Conley et al. (2011). The ”SNLS” sample is a combination of a number of previous surveys combining supernova legacy survey results with other low-z and high-z observations. These measurements are included in our analysis using the cosmomc likelihood module provided by Conley et al. (2011)\(^7\). This likelihood is evaluated by (firstly) calculating the model apparent magnitudes (or more accurately, the rest-frame peak B-band magnitude):

\[
m_{\text{model}} = 5 \log_{10} D_L(z_{\text{CMB}}, z_{\text{Hel}}, \ldots) - \alpha(S - 1) + \beta C + M_B.
\]

Here \(D_L\) is luminosity distance with the dependence on the Hubble constant removed (it’s dimensionless). And \(z_{\text{CMB}}\) and \(z_{\text{Hel}}\) are the CMB frame and heliocentric frame redshifts of the SN. \(M_B\) is a parameter which controls the zero-point and is a function of both the absolute magnitude of the SN and \(H_0\), this parameter is marginalised over. The brightness of each SN is ‘standardised’ using observations of the shape of the light curve, \(s\), and the colour \(C\); in addition to the empirical relationship of these parameter with the luminosity of the object: these dependences are characterised by the parameters \(\alpha\) and \(\beta\).

Writing the model predictions as a vector \(\vec{m}_{\text{model}}\) the likelihood is given by

\[
-2 \ln \mathcal{L} = (\vec{m}_{\text{obs}} - \vec{m}_{\text{model}})^T C^{-1} (\vec{m}_{\text{obs}} - \vec{m}_{\text{model}}),
\]

\(^7\text{https://tspace.library.utoronto.ca/handle/1807/25390}\)
where $\vec{m}_{\text{obs}}$ is a vector of the observed B-band magnitudes. The elements of the non-diagonal covariance matrix $C$ includes contributions from the following effects: the intrinsic-scatter of type Ia SN, the errors on the fitted light curve parameters, the redshift error, a host correction error, and the covariance between $s$, $C$ and $m_{\text{obs}}$. There are additional corrections for the local peculiar velocity field, for further details see Conley et al. (2011).

In Section 3.5.5 we adopt a second SNe dataset, namely the JLA sample (Betoule et al., 2014). This sample is composed of recalibrated SN Ia light-curves and distances for the SDSS-II and SNLS samples; this sample can be distinguished from the SNLS sample by the treatment of systematic effects, the end result is a 1.8$\sigma$ shift from the SNLS 3-year results.

### 3.4.2 CMB

For the models we adopt GR is restored at the time of the last scattering surface; accordingly, the components of the temperature fluctuations, unmodified by large-scale structure, provide a powerful tool to both constrain the physical components of the universe and the initial conditions which seed large-scale structure.

The likelihood code for the power spectrum $C_{TT}^l$ from Planck is a hybrid: it is divided into high-$l$ and low-$l$. For high-$l$ ($l > 50$) we use the likelihood code CamSpec described by Planck Collaboration et al. (2014a). This algorithm uses temperature maps derived at 100, 143 and 217 GHz. Once both diffuse Galactic emission and Galactic dust emission are masked, 57.8% of the sky remains for the 100 GHz map and 37.3% for the remaining maps. At low multipoles ($2 < l < 49$) the likelihood is computed using the Commander algorithm (Eriksen et al., 2008) using the frequency range 30-353 GHz over 91% of the sky.

Sub-Hubble modes near reionization are damped by Thomson scattering, thus obscuring our view of the primordial power spectrum: We observe a fluctuation amplitude $A_s e^{-2\tau}$. The degeneracy between the optical depth $\tau$ and the amplitude of the primordial power spectrum $A_s$ can be partially broken by including polarization data: the relative amplitude of the polarization and temperature power spectrum constrain $\tau$. For this purpose we include the large-scale polarization measurements.
(\(C_l^{EE}\)) from \(WMAP-9\) (Bennett et al., 2013). We use the likelihood code from \(Planck\) which fits to the \(l\)-range \((2 < l < 32)\).

**CMB Lensing**

Photons travelling from the last scattering surface to our satellites encounter a number of over- and under-densities along the way. The intersected structure deflects the photon paths and the large-scale clustering of matter causes these deflection paths to be correlated over the sky (Blanchard & Schneider, 1987). The combined effect of this CMB lensing is a re-mapping of the CMB temperature fluctuations (cf., Lewis & Challinor, 2006):

\[
T(\hat{n}) = T^{\text{unlensed}}(\hat{n} + \nabla \Phi(\hat{n})).
\]  

(3.36)

Where \(\Phi(\hat{n})\) is the CMB lensing potential given by

\[
\Phi(\hat{n}) = - \int_0^{\chi^*} d\chi G(\chi, \chi^*) \left[ \phi(\chi \hat{n}; \eta_0 - \chi) + \psi(\chi \hat{n}; \eta_0 - \chi) \right].
\]  

(3.37)

Here \(\chi\) is the conformal distance, \(\eta\) is the conformal time (\(\eta_0\) is the time today), and \(G(\chi, \chi^*)\) is a weighting function. The integration is taken from the last scattering surface (\(\chi^*\)) to today (\(\chi = 0\)); hence this term represents the integrated effect of structure on photon paths, or more accurately, since we are interested in testing GR, the integrated effect of spatial and curvature perturbations.

The lensing power spectrum \(C_l^{\phi\phi}\) can be extracted from CMB maps; here we use the results from Planck Collaboration et al. (2013) for the \(l\)-range \(40 < l < 400\) (with the bin size \(\Delta l = 64\)): this \(l\)-range is chosen as it encompasses the majority of the lensing signal (\(\sim 90\%\)) and is likely less influenced by systematic effects (cf., Planck Collaboration et al., 2013). Given the lensing kernel peaks at \(z \sim 2\) and we are only using \(l < 400\), the lensing power spectrum measurements used are only probing linear scales. Accordingly, we use linear theory to predict the lensing power spectrum and expect no systematic errors to be introduced from this modelling.
3.5. MCMC ANALYSIS

Temperature-Galaxy Cross-Correlation

At late times the accelerating cosmic expansion dictates the evolution of density perturbations, one consequence is time-dependent metric potentials. This time-dependence is apparent in the CMB as it generates a net energy loss for CMB photons as they propagate through these potential wells (Sachs & Wolfe, 1967). This feature is known as the integrated Sachs-Wolfe (ISW) effect. The influence on the CMB power spectrum is given by

$$C_l \sim (\phi + \psi).$$

(3.38)

The ISW effect induces a correlation between the CMB (low-$l$) and large-scale structure probes: this is measured using the temperature-galaxy cross-correlation power spectrum $C^g_T l$ (cf. Ho et al., 2008). For our analysis we use the measurement of $C^g_T l$ presented in Ho et al. (2008), and the likelihood code described in Dossett, Ishak & Moldenhauer (2011). This likelihood code expands on that presented in Ho et al. (2008) by including the effects of modified gravitational field equations.

The density field for the cross-correlation is approximated by the following measurements: the 2MASS Two Micron All Sky Survey, the Sloan-Digital Sky Survey Luminous Red Galaxy Sample, the Sloan-Digital Sky Survey Quasars, and the NRAO VLA Sky Survey. And the CMB temperature data is taken from WMAP-5\(^8\). The final $l$-range we adopt is $6 < l < 130$: this range is taken to ensure linear theory is valid, specifically, this $l$-range is imposed to ensure a wavenumber cutoff of $k \leq 0.05 \ h \text{ Mpc}^{-1}$.

3.5 MCMC Analysis

We sample the parameter space of cosmological parameters using Markov Chain Monte Carlo techniques with the CosmoMC package. The MCMC algorithm implement-

\(^8\)Note, the NVSS radio survey is the best tracer of large-scale structure at a high-redshift: this survey provides the most significant detection of a cross-correlation. Furthermore, the ISW effect is only dominant at low-$l$ and hence is limited by cosmic variance. For both reasons, the measurement of $C^g_T l$ has not been significantly improved from Ho et al. (2008), hence justifying our use of this data.
mented within this code is an adaptive Metropolis-Hastings method which utilizes a number of techniques to ensure fast convergence times. The definitions and adopted priors of each parameter are given in Table 3.2. Our results are derived using 8 separate chains which are run until convergence is achieved. The convergence of the Markov chains is determined using the Gelman and Rubin convergence criteria, for which chains require $R - 1 < 0.02$ to be satisfied for the least-converged orthogonalized parameter; $R$ being the ratio of the variance of the chains’ mean and the mean of the chains’ variances (Gelman & Rubin, 1992). The posterior mean and 68% confidence intervals are then computed using thinned Markov chains.

There is currently no consensus on the $H_0$ value as measured from Cepheid data. The most up-to-date measurements are presented by Efstathiou (2014), Riess et al. (2011), and Humphreys et al. (2013): they measure $H_0 = 70.6 \pm 3.3, 73.8 \pm 2.4, 72.0 \pm 3$ km/s/Mpc, respectively. Note both Efstathiou (2014) and Humphreys et al. (2013) have used the revised geometric maser distance to NGC 4258 (as presented in Humphreys et al., 2013), however their measurements still do not agree: the disagreement can be traced to different outlier rejection criteria being applied. For this analysis we adopt two approaches, because of this tension. When the expansion history is described by ΛCDM we do not include any $H_0$ prior as the model-dependent constraints from the CMB are sufficient. When we do include deviations from ΛCDM in the expansion history we add an $H_0$ prior using the measurement by Efstathiou (2014).

### 3.5.1 Parameter Fits: Model I

Using different combinations of the measurements outlined in the previous sections, we performed fits to the base ΛCDM parameters $(\omega_b, \omega_c, \theta_{MC}, \tau, n_s, A_s)$ and the modified gravity parameters $G_{\text{matter}}(k, z)$ and $G_{\text{light}}(k, z)$. Recall each modified gravity parameter is binned in both redshift and scale.

In addition to the physical parameters, a number of nuisance parameters are introduced to account for unknown astrophysical effects. For the WiggleZ multipole calculation for each redshift bin we include the galaxy bias and velocity dispersion as nuisance parameters, that is, $b_{\text{lin}}(z = 0.44), \sigma_v(z = 0.44), b_{\text{lin}}(z = 0.73),$ and
### Cosmological Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Prior Range</th>
<th>Baseline</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0$</td>
<td>Current expansion rate in km s(^{-1}) Mpc(^{-1})</td>
<td>[70, 100]</td>
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</tr>
<tr>
<td>$\Omega_b h^2$</td>
<td>Baryon density today</td>
<td>[0.01, 0.1]</td>
<td>...</td>
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<tr>
<td>$\Omega_c h^2$</td>
<td>Cold dark matter density today</td>
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<td>...</td>
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<tr>
<td>$\Omega_m$</td>
<td>Dark energy density today</td>
<td>[0.01, 0.1]</td>
<td>...</td>
</tr>
<tr>
<td>$n_s$</td>
<td>Scalar spectrum power-law index ((k = 0.05 \text{ Mpc}^{-1}))</td>
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</tr>
<tr>
<td>$\ln(10^{10} A_s)$</td>
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<td>...</td>
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<tr>
<td>$\theta_s$</td>
<td>Neutrino mass sum in eV</td>
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<tr>
<td>$N_{\text{eff}}$</td>
<td>Effective number of relativistic degrees of freedom</td>
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<td>$w_0$</td>
<td>Dark energy equation of state, (w = w_0 + (1 - a)w_a)</td>
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<td>$w_a$</td>
<td>Redshift-dependent modification to the equation of state</td>
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<td>$\Gamma$</td>
<td>Transition redshift for GR modifications</td>
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</tr>
<tr>
<td>$z_T$</td>
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<td>Sum of neutrino masses in eV</td>
<td>[0.01, 0.5]</td>
<td>...</td>
</tr>
</tbody>
</table>

The parameters introduced to allow modifications from General Relativity. Note a prior is included on the derived parameters to allow modifications from General Relativity. No prior is included on the derived parameters to allow modifications from General Relativity. The parameters with a specified prior range are treated as free parameters in the NGC analysis, while the remaining parameters are fixed at their fiducial values. The parameters without a specified prior range are treated as free parameters in the NGC analysis, while the remaining parameters are fixed at their fiducial values. The parameters without a specified prior range are treated as free parameters in the NGC analysis, while the remaining parameters are fixed at their fiducial values. The parameters without a specified prior range are treated as free parameters in the NGC analysis, while the remaining parameters are fixed at their fiducial values. The parameters without a specified prior range are treated as free parameters in the NGC analysis, while the remaining parameters are fixed at their fiducial values. The parameters without a specified prior range are treated as free parameters in the NGC analysis, while the remaining parameters are fixed at their fiducial values.
The uniform priors imposed on these parameters are $b_{\text{lin}} \in [0.5, 3]$ and $\sigma_v \in [0, 10]h^{-1}\text{Mpc}$. For the DR11-BOSS CMASS multipole measurement we also include galaxy bias and velocity dispersion as free parameters, $b_{\text{lin}}(z = 0.57)$, $\sigma_v(z = 0.57)$. Additionally for BOSS, we include a free parameter to account for the non-Poisson contributions to shot noise $N$, this is given the prior $(N \in [0, 2000]h^{-3}\text{Mpc}^3)$. For the velocity power spectrum measurement we include a velocity dispersion parameter $\sigma_{PV}(z = 0) \in [0, 500]\text{km/s}$.

In order to understand the sensitivity of each cosmological probe to the physical parameters, and test for residual systematics, we analyse different combinations of cosmological probes. The different combinations are defined and labeled in Table 3.3 (henceforth we will use these definitions). The final results of this section are displayed in Fig. 3.4 and Fig. 3.5, and further information is provided in Table 3.4.

The first figure shows the constraints on $G_{\text{matter}}(k, z)$ and the second on $G_{\text{light}}(k, z)$. The black-dashed lines in both figures show the predictions from General Relativity. We do not plot the 2D contours between $G_{\text{matter}}(k, z)$ and $G_{\text{light}}(k, z)$ as their correlations are small, i.e., $\langle |\rho_c| \rangle \sim 0.15$. Here $\rho_c$ is the cross-correlation coefficient, and $\langle \rangle$ indicates the average over all the possible values between $G_{\text{matter}}(k, z)$ and $G_{\text{light}}(k, z)$. Similarly, we do not plot the inferred constraints on the base CDM parameters as, with two exceptions, the base CDM parameters are not highly correlated with the post-GR parameters, the exception being $\sigma_8$ and $\Omega_m$ with $G_{\text{matter}}$. When averaging over the four $G_{\text{matter}}$ parameters we find $\langle |\rho_c| \rangle \sim 0.77, 0.39$, respectively. The remainder of this section will involve a discussion of the content of these plots, in addition to some comments on potential systematics effects and the derived astrophysical parameters.

As shown in Fig. 3.4, we observe very little variation in $G_{\text{light}}(k, z)$ as we add extra datasets to the base sample (the green contour): this is because the ISW effect on the T-T power spectrum is dominating the fit; additionally, galaxy velocities have no sensitivity to $G_{\text{light}}(k, z)$, so we expect the benefit of including them to be minimal. The grey contours in Fig. 3.4 are derived by adding the T-g measurements.

---

---

We note that this shot noise contribution can potentially be negative (Baldauf et al., 2013), therefore one may question whether our imposed prior is overly constraining. However, we find that the posterior probability distribution for the CMASS shot noise parameter is well localized within the range of our chosen prior for all cases considered.
Table 3.3: The dataset combinations we use for fits to Model I, in addition to the labels we adopt to refer to them. The corresponding datasets should be clear from the information given in Table 3.1. We define Base as the combination High−ℓ + low−ℓ + WP + BAO + SNe. Below CMASS refers to the monopole and quadrupole multipole measurements from the BOSS-CMASS sample. And WiggleZ refers to the monopole, quadrupole and hexadecapole measurements from WiggleZ (as presented above).

<table>
<thead>
<tr>
<th>Label</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>SET 1</td>
<td>Base</td>
</tr>
<tr>
<td>SET 2</td>
<td>Base + Direct PV</td>
</tr>
</tbody>
</table>
| SET 3 | Base + CMASS ($k_{\text{max}} = 0.10 h \text{ Mpc}^{-1}$)  
      | WiggleZ ($k = 0.15 h \text{ Mpc}^{-1}$) + Direct PV |
| SET 4 | Base + CMASS ($k_{\text{max}} = 0.10 h \text{ Mpc}^{-1}$)  
      | WiggleZ ($k = 0.15 h \text{ Mpc}^{-1}$) + Direct PV  
      | + ISW-Density + CMB Lensing |
| SET 5 | Base + ISW-Density |
| SET 6 | Base + CMASS ($k_{\text{max}} = 0.10 h \text{ Mpc}^{-1}$) |
| SET 7 | Base + CMASS ($k_{\text{max}} = 0.15 h \text{ Mpc}^{-1}$) |
| SET 8 | Base + WiggleZ ($k_{\text{max}} = 0.15 h \text{ Mpc}^{-1}$) |
| SET 9 | Base + WiggleZ ($k_{\text{max}} = 0.19 h \text{ Mpc}^{-1}$) |
Figure 3.4: 68% and 95% confidence regions for the four $G_{\text{light}}(k, z)$ bin parameters. Here $z > 1$ is referring to the redshift range $2 > z > 1$. Note all of the parameters specified in Table 2 are being varied in this analysis, however for clarity we only plot the constraints on $G_{\text{light}}(k, z)$ in this plot. Recall we have defined Base as High$-t + $low$-t + $WP + BAO + SNe.
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Figure 3.5: 68% and 95% confidence regions for the four $G_{\text{matter}}(k, z)$ bin parameters. Here $z > 1$ here is referring to the redshift range $2 > z > 1$. Note all of the parameters specified in Table 2 are being varied in this analysis yet for clarity we only plot the constraints on $G_{\text{matter}}(k, z)$. Recall we have defined Base as High$-l$ + low$-l$ + WP + BAO + SNe.
Table 3.4: Cosmological parameter constraints for Model I. The constraints are derived from four different groups of cosmological probes, we label these groups Set 1 to 4 and define each in Table 3.3. For each parameter in each group we provide the 68% confidence levels. To keep the table a reasonable size we only consider the parameters most relevant to our analysis.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>SET 1</th>
<th>SET 2</th>
<th>SET 3</th>
<th>SET 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>68% limits</td>
<td>68% limits</td>
<td>68% limits</td>
<td>68% limits</td>
</tr>
<tr>
<td>$G_{\text{matter}}(z &lt; 1; k &gt; 0.01)$</td>
<td>$0.48^{+0.50}_{-0.52}$</td>
<td>$0.66 \pm 0.47$</td>
<td>$0.65 \pm 0.43$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.96^{+1.14}_{-0.44}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{\text{matter}}(z &lt; 1; k &lt; 0.01)$</td>
<td>$1.32^{+0.42}_{-0.29}$</td>
<td>$1.32^{+0.41}_{-0.30}$</td>
<td>$1.22^{+0.39}_{-0.34}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.81^{+0.59}_{-0.46}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{\text{matter}}(z &gt; 1; k &gt; 0.01)$</td>
<td>$1.12^{+0.84}_{-0.33}$</td>
<td>$0.54 \pm 0.35$</td>
<td>$0.53 \pm 0.32$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.23^{+0.71}_{-0.28}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{\text{matter}}(z &gt; 1; k &lt; 0.01)$</td>
<td>$0.88 \pm 0.37$</td>
<td>$0.82 \pm 0.32$</td>
<td>$0.87 \pm 0.30$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.95^{+0.42}_{-0.36}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{\text{light}}(z &gt; 1; k &gt; 0.01)$</td>
<td>$1.06^{+0.063}_{-0.046}$</td>
<td>$1.07^{+0.063}_{-0.043}$</td>
<td>$1.057^{+0.053}_{-0.045}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1.076^{+0.063}_{-0.046}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{\text{light}}(z &lt; 1; k &lt; 0.01)$</td>
<td>$1.044 \pm 0.050$</td>
<td>$1.048 \pm 0.048$</td>
<td>$1.048 \pm 0.048$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1.038 \pm 0.048$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{\text{light}}(z &lt; 1; k &gt; 0.01)$</td>
<td>$1.12^{+0.10}_{-0.078}$</td>
<td>$1.14^{+0.10}_{-0.077}$</td>
<td>$1.153^{+0.089}_{-0.098}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1.13^{+0.098}_{-0.084}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{\text{light}}(z &gt; 1; k &lt; 0.01)$</td>
<td>$1.015 \pm 0.026$</td>
<td>$1.016 \pm 0.027$</td>
<td>$1.016 \pm 0.026$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1.016 \pm 0.027$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Omega_bh^2$</td>
<td>$0.02228 \pm 0.00025$</td>
<td>$0.02227 \pm 0.00025$</td>
<td>$0.02226 \pm 0.00025$</td>
<td>$0.02230 \pm 0.00025$</td>
</tr>
<tr>
<td>$\Omega_ch^2$</td>
<td>$0.1172 \pm 0.0013$</td>
<td>$0.1172 \pm 0.0013$</td>
<td>$0.1168 \pm 0.0013$</td>
<td>$0.1163 \pm 0.0013$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$0.087 \pm 0.013$</td>
<td>$0.089^{+0.012}_{-0.014}$</td>
<td>$0.089 \pm 0.013$</td>
<td>$0.086 \pm 0.012$</td>
</tr>
<tr>
<td>$\ln(10^{10} A_s)$</td>
<td>$3.076 \pm 0.026$</td>
<td>$3.081 \pm 0.025$</td>
<td>$3.080 \pm 0.025$</td>
<td>$3.073 \pm 0.025$</td>
</tr>
<tr>
<td>$\Omega_L$</td>
<td>$0.7011 \pm 0.0077$</td>
<td>$0.7090 \pm 0.0078$</td>
<td>$0.7031 \pm 0.0074$</td>
<td>$0.7060 \pm 0.0074$</td>
</tr>
<tr>
<td>$\Omega_m$</td>
<td>$0.2989 \pm 0.0077$</td>
<td>$0.2990 \pm 0.0078$</td>
<td>$0.2969 \pm 0.0074$</td>
<td>$0.2940 \pm 0.0074$</td>
</tr>
<tr>
<td>$\sigma_8$</td>
<td>$0.851^{+0.14}_{-0.091}$</td>
<td>$0.783^{+0.095}_{-0.063}$</td>
<td>$0.717^{+0.018}_{-0.022}$</td>
<td>$0.711^{+0.017}_{-0.020}$</td>
</tr>
<tr>
<td>$H_0$</td>
<td>$68.49 \pm 0.61$</td>
<td>$68.46 \pm 0.63$</td>
<td>$68.61 \pm 0.59$</td>
<td>$68.83 \pm 0.61$</td>
</tr>
</tbody>
</table>
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to the base sample, and the red contours are derived by adding the multipole and velocity measurements to the base sample. And the blue contours show the main results which are derived using Set 4. From these measurements for $G_{\text{light}}(k, z)$ we infer (in terms of 68% CLs)

\[
\begin{align*}
G_{\text{light}}(z > 1; k > 0.01) &= 1.057^{+0.053}_{-0.045}, \\
G_{\text{light}}(z < 1; k < 0.01) &= 1.048 \pm 0.048, \\
G_{\text{light}}(z < 1; k > 0.01) &= 1.153^{+0.080}_{-0.068}, \\
G_{\text{light}}(z > 1; k < 0.01) &= 1.016 \pm 0.026,
\end{align*}
\]

These measurements are compatible at the 95% CL with GR.

For $G_{\text{matter}}(k, z)$ we observe a significant amount of variation as new measurements are added to the base sample. In Fig. 3.5 the green, grey, red and blue contours correspond respectively to measurements using the dataset combinations Set 1, Set 2, Set 3, Set 4 (sets 6 to 9 are used for systematics checks to be discussed in the next section). As derived from Set 4 (i.e. using all the datasets) the 1D marginalised results for $G_{\text{matter}}$ (in terms of 68% CLs) are

\[
\begin{align*}
G_{\text{matter}}(z < 1; k > 0.01) &= 0.65 \pm 0.43, \\
G_{\text{matter}}(z < 1; k < 0.01) &= 1.22^{+0.39}_{-0.34}, \\
G_{\text{matter}}(z > 1; k > 0.01) &= 0.53 \pm 0.32, \\
G_{\text{matter}}(z > 1; k < 0.01) &= 0.87 \pm 0.30.
\end{align*}
\]

Similarly to above, these results are consistent with GR at the 95% CL, while at the 68% CL level we observe a tension with GR in the high-redshift and large-wavenumber bin. Furthermore, the constraints from Set 4 on the 2D CLs of the low-$z$ high-$k$ and high-$z$ high-$k$ bins of $G_{\text{matter}}$ show a tension with the standard model at greater than $2\sigma$. For the 1D marginalised results this tension is significantly reduced as the high-$z$ and low-$z$ $G_{\text{matter}}$ bins are highly correlated, as can be seen in Fig. 3.5. This degeneracy occurs as some probes, such as the CMB, are sensitive to integrated quantities over redshift, such that higher growth at high-$z$ can be compensated for by lower growth at low-$z$. 

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Introducing direct PV measurements the constraints shift from the green to the grey contours. The most prominent shift occurs in the low-$z$ and low-$k$ $G_{\text{matter}}$ bin, as expected: we find a shift from $G_{\text{matter}}(z < 1; k < 0.01) = 0.81^{+0.59}_{-0.46}$ to $G_{\text{matter}}(z < 1; k < 0.01) = 1.32^{+0.42}_{-0.29}$. We find further improvements in the constraints for the high-wavenumber and low-redshift bin. Future PV surveys should be able to considerably improve on this situation (cf. Koda et al., 2014). Using the best-fit parameters from Set 4, we measure $\chi^2_{6dFGSv} = 778$ with 979 data points: the full 6dFGSv velocity field is smoothed onto a grid with 979 non-empty elements (cf. Johnson et al., 2014).

Including RSD measurements results in the shift from the grey to red contours, for which we find a significant improvement in the constraint on the high-$z$ and high-$k$ $G_{\text{matter}}$ bin. Moreover, we find that the RSD measurements have more influence on the high-$z$ bin than the low-$z$ bin: this is an further consequence of measuring integrated quantities. As a systematic check we isolate the measurements from WiggleZ and BOSS and perform separate fits, we find that the two separate constraints on $G_{\text{matter}}$ are consistent. We can also assess how well our model fits the observations. By adding the multipole likelihoods we find $\Delta\chi^2 = 322$, for a total of 324 measurement points. Individually, for the fit to the WiggleZ multipoles, with 126 data points per redshift bin we measure $\chi^2_{\text{WiggleZ}} = 129.88$ for the low-$z$ region, and $\chi^2_{\text{WiggleZ}} = 121.6$ for the high-$z$ region. Finally, for BOSS, given we are fitting to $k_{\text{max}} = 0.10 h$ Mpc$^{-1}$, there are 72 measurement points and we find $\chi^2_{\text{CMASS}} = 72.6$.

3.5.2 Model Comparison: Bayesian Evidence

To evaluate the statistical significance of deviations from GR we previously used the marginalized posterior distributions of $G_{\text{matter}}$ and $G_{\text{light}}$. An alternative approach is to use model selection. This allows one to rank the viability of a series of models based on a measure (Jeffreys, 1961). For cosmological applications see Liddle, Mukherjee & Parkinson (2006); Trotta (2008). The philosophy behind model selection is as follows: Simple models with a high degree of predictability are favored, equivalently, complex models with a large number of highly tuned parameters are penalized. The relevant measure weighs both the ability of the model to fit observations and its degree of simplicity.
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To discriminate between models one first computes the evidence for each model, \(E\), which is defined as the probability of the data \(D\) given the model \(\mathcal{M}\) (defined by a set of model parameters \(\theta\)):

\[
E = P(D|\mathcal{M}) = \int P(D|\theta,\mathcal{M})\pi(\theta|\mathcal{M}).
\] (3.39)

In our case we will consider two models, \(\mathcal{M}_1\) and \(\mathcal{M}_2\), which represent GR and our phenomenological modified gravity model, respectively. The ratio of the evidence values is called the Bayes factor, \(B\), and the value of this indicates a degree of preference for one model over the other:

\[
B_{12} = \frac{\int P(D|\theta_1,\mathcal{M}_1)\pi(\theta_1|\mathcal{M}_1)}{\int P(D|\theta_1,\mathcal{M}_2)\pi(\theta_1|\mathcal{M}_2)}.
\] (3.40)

One can extend this calculation by incorporating the posterior probabilities of the models by adding model priors of the form \(\pi(\mathcal{M})\). For our case, for simplicity we set \(\pi(\mathcal{M}_1) = \pi(\mathcal{M}_2) = 1\). This choice suggests we have no preference (based on physical intuition) between the models 10.

To compute the multidimensional integrals in Eq (3.40) we use the software package Multinest, as presented by Feroz & Hobson (2008). This program uses the Monte Carlo technique importance nested sampling to compute the relevant integrals. For further details we refer the reader to Feroz & Hobson (2008). Moreover, for the results that follow, we adopt a conservative threshold for interpreting the Bayes factor, specifically, we use the scale suggested by Kass & Raftery (1995).

Given evidence calculations are more computationally demanding than Markov chains, we do not include the velocity power spectrum likelihood in this calculation. Following the results above, we observe there is no strong tension between the velocity measurements and the \(\Lambda\)CDM model, hence we expect our conclusions (i.e., the evidence ratio) will not be sensitive to this likelihood. With the exclusion of the velocity likelihood, we use the remaining measurements, as introduced above.

10Recent work by Gubitosi et al. (2015) suggest a method to incorporate the philosophical notion of falsifiability into model selection. This additional selection criteria arises when paradigms are being compared, and results in a more severe penalty for unfalsifiable paradigms. We will not include this additional selection effect here.
For a ΛCDM cosmology we find a global log-evidence of \( \ln(\mathcal{E}_{\Lambda\text{CDM}}) = -5423.75 \pm 0.15 \). While allowing \( G_{\text{matter}} \) and \( G_{\text{light}} \) to vary we find, \( \ln(\mathcal{E}_{\text{MG}}) = -5715.41 \pm 0.14 \). Thus we find an evidence ratio of \( 2\log(\mathcal{E}_{\text{MG}}/\mathcal{E}_{\Lambda\text{CDM}}) = -5.83 \times 10^2 \). This ratio suggests “strong evidence” in favor of the ΛCDM model. Or, equivalently, “no support” for our phenomenological model. One can interpret this result as the extra parameters not significantly improving the the fit to data, yet increasing the complexity of the model; accordingly, the model is strongly disfavored.

For this calculation we include a theory prior on the parameterized deviations to GR. This prior ensures we only consider parameter combinations that produce positive \( C_l \)'s. We expect this prior to have a small impact on the final results for the following reasons. Firstly, the posterior distributions are localized with the prior volume. And, secondly, reducing the prior volume with this theory prior one would improve the evidence towards the modified gravity model. This model is already strongly disfavored, hence any shift would not influence our conclusions. Had the modified gravity model been favored we would have been required to carefully consider the influence of this prior.

### 3.5.3 Checking Systematics and Astrophysical Parameter constraints

The main potential sources of systematic errors for RSDs and PV measurements are as follows:

- Scale-dependent galaxy bias, \( b_g(k) \), for RSD.
- Non-linear contributions to the galaxy power spectrum for RSD.
- Non-linear contributions to the velocity power spectrum for PV.
- Scale-dependent velocity bias, \( b_v(k) \), for PV.
- Malmquist bias for PV.
- Non-Gaussian PV distributions.

Regarding the PV measurements, for full details, we refer the reader to Johnson et al. (2014). In short, however, we address the above issues below. First, we use
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a series of synthetic galaxy catalogues to determine the significance of non-linear errors, we find the influence to be neglectable. To avoid non-Gaussian errors, we transform the PV measurements into log-space. We argue that the effects of velocity bias are neglectable, given simulations and theory suggest that the velocity bias is only significant at very small scales. Finally, we account for Malmquist bias during the fitting of the Fundamental Plane, see Springob et al. (2014).

When calculating the power spectrum multipole predictions, we assumed a linear bias factor and linear perturbation theory. The validity of both assumptions may be questioned. We examine, albeit crudely, the importance of these assumptions by determining the sensitivity of the parameter fits to the small-scale cut-off $k_{\text{max}}$. For our model fits using the CMASS and WiggleZ multipole likelihood calculations we ran new Markov chains using different cut-off values $k_{\text{max}}^{\text{CMASS}} = 0.10, 0.15 h \text{ Mpc}^{-1}$ and $k_{\text{max}}^{\text{WiggleZ}} = 0.15, 0.19 h \text{ Mpc}^{-1}$. The results showed no statistically significant shift when the fitting range was changed. However, we note that a detailed investigation of mock catalogues is needed to fully validate these assumptions.

The astrophysical parameters for the multipole and direct PV fits only vary slightly when using different dataset combinations, hence we choose to only present results from Set 4 (given in terms of 68% CLs). For the fit to the WiggleZ multipole we find $\sigma_v(z = 0.73) = 2.30^{+1.2}_{-1.8} h^{-1}\text{Mpc}$, $\sigma_v(z = 0.44) = 4.468^{+1.8}_{-1.0} h^{-1}\text{Mpc}$, $b_1(z = 0.44) = 1.089 \pm 0.042$, and $b_1(z = 0.73) = 1.207 \pm 0.059$. For the fit to CMASS we find $\sigma_v(z = 0.57) = 2.44^{+0.68}_{-1.2} h^{-1}\text{Mpc}$, $b_1(z = 0.57) = 2.055 \pm 0.084$, and $N(\text{Shot Noise}) = 705 \pm 200 h^{-3}\text{Mpc}^3$. Finally, from the fit to the velocity power spectrum we determine the 95% upper limit $\sigma_{PV}(z = 0) < 334.6 \text{km/s}$. With different $k_{\text{max}}$ values adopted, one should not necessarily compare our results for the shot noise and velocity dispersion with previous analysis; however, we find our bias measurements to be consistent with previous analysis.

3.5.4 Previous Measurements: Summary and Comparisons

Below we briefly summarise recent work in this field, with a focus on results that adopt a similar parameterisation.

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• **Daniel & Linder (2010)** present measurements of \( \{G, V \} \), our \( \{G_{\text{light}}, G_{\text{matter}}\} \), in bins of time and wavenumber. To constrain these parameters they used the following probes: WMAP7, supernova Union2, CFHTLS weak lensing data, temperature-galaxy cross correlation, and the galaxy power spectrum. They identify the CFHTLS survey as responsible for a 2\( \sigma \) tension with GR in the high-\( k \) and low-\( z \) bin for \( V \). This feature is not observed when using the COSMOS data or in subsequent analysis of the final CFHTLenS catalogue (Heymans et al., 2012). Note RSD information was not included, therefore the final constraints on \( V \) are of order \( \sim 1 \).

• **Simpson et al. (2013)** measure the parameters \( \{\Sigma, \mu\} \) (i.e., \( \{G_{\text{light}}, G_{\text{matter}}\} \)) using tomographic weak lensing measurements from CFHTLenS and RSD measurements of \( f\sigma_8 \) from 6dFGS and WiggleZ, in addition to WMAP7 (including low-\( l \)) and geometric information (see also Dossett et al. (2015) and Zhao et al. (2012)). Their measurements are consistent with GR: they find \( \mu = 1.05 \pm 0.25 \) and \( \Sigma = 1.00 \pm 0.14 \). For this fit they assumed \( \Sigma, \mu \) are scale-independent and adopt a specific functional form for their temporal evolution: this effectively confines deviations to very low-redshifts. Measurements of the T-G cross correlations, CMB-lensing, and the growth rate measurement from CMASS were not included in these fits.

• **Planck Collaboration et al. (2015)** have recently provided the state-of-the-art measurements of post-GR parameters, placing constraints on an extensive range of specific and phenomenological models. For the phenomenological model they adopt the parameters \( \{\mu, \eta\} \), as implemented in **MGCAMB**. Motivated by \( f(R) \) models, a specific functional form for the redshift and scale dependence of these parameters is assumed. As appropriate to their aim, they ensure their angular cuts to the tomographic shear-shear measurements from CFHTLenS isolate the linear signal (see their Fig 2). This approach is not adopted throughout, however. Their adopted \( f\sigma_8(z = 0.57) \) measurement (by Samushia et al. (2014)) was derived by fitting to the monopole and quadruple of the correlation function on scales larger than \( 25h^{-1}\text{Mpc} \). As highlighted by the authors (see their Fig. 7) non-linear terms are significant on these length scales, the result is a dependence on non-linear physics.
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In relation to the most up-to-date measurements, our results can be distinguished in two main ways: Firstly, the inclusion of the velocity power spectrum measurements, which improve low-\(k\) constraints; secondly, the methodology we use to analyse RSD measurements, and the range of RSD measurements analysed. We argue that the methodology of directly analysing the power spectrum multipoles allows constraints to be derived that are more widely applicable to non-standard cosmological models. This is because it allows one to restrict the analysis to scales within the linear regime, where the phenomenological models we use describe physical models (see Sect. 3.1). Moreover, the multipoles contain scale-dependent information, which is necessary if scale-dependent terms are introduced.

3.5.5 Parameter Fits: Model II

We now explore fits to a new parameter space that is more rigid regarding the allowed deviation to the growth history. Two scenarios will be considered when fitting for these parameters, firstly, an expansion history fixed to \(\Lambda\)CDM; and secondly, an expansion history than can deviate from \(\Lambda\)CDM via a time-dependent equation of state. We define the two parameter spaces as \(p_1 = \{\gamma, \omega_b, \omega_c, \theta_{\text{MC}}, \tau, n_s, A_s\}\), and \(p_2 = \{\gamma, w, w_a, \omega_b, \omega_c, \theta_{\text{MC}}, \tau, n_s, A_s\}\). We choose not to include the influence that deviations in the expansion history have on the expected growth (that is, the relation \(\gamma = f(\gamma_0, w_0, w_a)\)) as the corrections are currently small.

Note that by changing the growth rate we modify \(\sigma_8\), this effect is included by altering the growth history well into the matter dominated regime. The modified growth factor is calculated as

\[
D(a_{\text{eff}}) = \exp \left( - \int_{a_{\text{eff}}}^{1} da \Omega_m(a)^{0.55}/a \right),
\]

now we scale the fiducial prediction \(\sigma_8^{\text{Fid}}(z_{\text{high}})\) to find the modified amplitude \(\sigma_8^*(z_{\text{eff}})\):

\[
\sigma_8^*(z_{\text{eff}}) = \frac{D(a_{\text{eff}})}{D(z_{\text{high}})} \sigma_8^{\text{Fid}}(z_{\text{high}}).
\]
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The first set of results, which assume a $\Lambda$CDM expansion history are shown in Fig. 3.6. This plot shows the 68% and 95% 2D likelihood contours for the parameter combinations $\{\Omega_m, \gamma\}$ and $\{\tau, \gamma\}$. The expected value of $\gamma$ from GR is given by the grey-dashed line. In addition to the growth rate and AP constraints, these measurements are inferred using high-$l$ + WP + low-$z$ BAO (which we label in this section as base). For fits in this section we do not use the low-$l$ CMB T-T data or CMB lensing, since we have not included the dependence of these signals on $\gamma$. For the final constraint we measure $\gamma = 0.665 \pm 0.067$, which is consistent with GR at the 95% C.L.

The results for $p_2$ are presented in Fig. 3.7, where we plot the 2D likelihood contours (68% and 95%), and the marginalised 1D probability distributions for $\gamma, w_0, w_a, \Omega_m, \tau$. Again, the black-dashed lines indicate the values expected from the standard model; namely, $\gamma = 0.55, w_0 = -1$, and $w_a = 0$. The degraded constraint on $\gamma$ is a direct result of the degeneracy between the expansion and growth histories: this is the reason we consider both a fixed and non-fixed expansion history.

We use four different dataset combinations to constrain these parameters, they are defined as follows: fit 1 is the base sample, fit 2 is base + WP, fit 3 is base + WP + SNLS, and fit 4 is base + WP + JLA. We define base here as the combination High-$l$ + $H_0$ + RSD/AP + low-$z$ BAO. We use two SN samples in order to understand how sensitive the growth index is to our choice of adopted dataset.

We will first discuss the main results, which are found using fit 3 and 4 (the red and blue contours in Fig. 3.7) and then consider how the constraints are influenced by the different probes. Using fit 3 we infer (in terms of 68% CL) the marginalised constraints

$$w_0 = -0.98^{+0.13}_{-0.15},$$
$$w_a = -0.42^{+0.62}_{-0.47},$$

which are consistent with the standard model. In terms of deviation to the growth history we measure (in terms of 68% CL)

$$\gamma = 0.76^{+0.089}_{-0.087}.$$
Figure 3.6: 2D marginalized posterior distributions for \( \{ \Omega_m, \gamma \} \) and \( \{ \tau, \gamma \} \), assuming a \( \Lambda \)CDM expansion history. The contours are the 68% and 95% CL. Below we refer to base as the dataset combination high-\( l \) + WP + low-\( z \) BAO. The green contours are found using base + CMASS, the grey contours are found using base + WiggleZ, the red contours are found using base + 6dFGS, and the blue contours show the combined fit to all the growth rate measurements plus the base measurements. Moreover, we include the AP and BAO information with the growth rate constraints, without double counting BAO measurements.
Figure 3.7: 68% and 95% confidence regions for the most relevant parameters describing model II. The base sample of datasets, as referred to above, represents the combination High-$l$ + $H_0$ + RSD/AP + low-z BAO.
3.5. MCMC ANALYSIS

This result is at tension with GR at a level greater than 2\(\sigma\). Changing our SN sample to the JLA sample we find this tension is slightly reduced. Using fit 4 we now measure

\[
\gamma = 0.73^{+0.08}_{-0.10},
\]  

which is just consistent at the 2\(\sigma\) level, and for the expansion history we find \(w_0 = -0.89^{+0.12}_{-0.12}\) and \(w_a = -0.63^{+0.36}_{-0.48}\). Note, without including any SN data, using fit 2 (the grey contour in Fig. 3.7) we measure \(\gamma = 0.69^{+0.09}_{-0.11}\), which is consistent at the 95\% C.L. This may suggest there exists a mild tension between the growth rate and the SN measurements. Finally, we note our measurements of the growth index are relatively insensitive to the polarization data, as can be observed in Figure 3.7 by comparing the green (no WP) and grey (including WP) contours.

Comparing the best-fit values for the expansion history using only BAO measurements with the BAO + SN fit (which is driven by SN) is interesting as it provides a test of the significance of non-linear structure on SNe distance measurements (Clarkson et al., 2012). With fit 2, which only uses the low-redshift BAO measurements to constrain the expansion history, we infer (in terms of 68\% C.L) \(w_0 = -0.68^{+0.29}_{-0.26}\) and \(w_a = -1.27^{+0.92}_{-0.97}\). These measurements are consistent at the 95\% CL with the standard model and the constraints from the SN + BAO fit; moreover, they highlight the current necessity of type Ia SN in placing tight constraints on the redshift evolution of the equation of state. By introducing the Lyman-\(\alpha\) BAO measurements into this fit we measure \(w_0 = -0.58^{+0.27}_{-0.22}\) and \(w_a = -1.55^{+0.74}_{-0.89}\), which indicates a tension with the standard model predictions at a level > 2\(\sigma\), in agreement with the results by Font-Ribera et al. (2014). Further checks for systematics will be required to confirm this result given its significance and the complexity of the measurement.

3.5.6 Comparison with Previous Results

Below we summarise a subsample of previous measurements of the parameters \(\{w_0, w_a, \gamma\}\).
• Beutler et al. (2014) measure $\gamma = 0.772^{+0.124}_{-0.097}$ using the power spectrum multipoles from the DR11 CMASS sample and Planck: this fit includes the AP effect, but does not allow for deviation in the expansion history. This value is consistent with the measurement by Sánchez et al. (2013) of $\gamma = 0.64 \pm 0.26$ found using the clustering wedges of CMASS combined with BAO and SNe measurements.

• Rapetti et al. (2013) perform fits to $\{w_0, \gamma\}$ and $\gamma$. For a fixed expansion history, using WMAP combined with galaxy cluster data from ROSAT and Chandra, they measure $\gamma = 0.415^{+0.128}_{-0.126}$. When adding further data from RSD measurements (WiggleZ and 6dFGS) they find $\gamma = 0.570^{+0.064}_{-0.063}$.

• Beutler et al. (2012) measure $\gamma = 0.547 \pm 0.088$ using WMAP7 and the two-point correlation function measured from 6dFGS. For this fit the expansion history is fixed, as the AP effect is not relevant. Note, there is a small difference between our measurement of $\gamma$ from 6dFGS and this result. This change is driven by the preference for a higher $\Omega_m$ in Planck compared to WMAP.

For this analysis we extend the range of RSD measurements used to constrain $\gamma$ relative to Sánchez et al. (2013); Beutler et al. (2014) and Rapetti et al. (2013). Moreover, relative to Rapetti et al. (2013) we also use the updated Planck measurements as opposed to WMAP. The final accuracy of our measurement of the growth index improves upon Sánchez et al. (2013) and Beutler et al. (2014), given the additional measurements we analyze. Note, our constraint on the growth index disagrees with Rapetti et al. (2013) as we use different datasets, and the two measurements have similar accuracy as we choose to focus only on growth rate measurements from RSD: we do not include additional probes sensitive to the growth rate. This position is motivated by recent suggestions that there exists some tension between the predictions from a Planck cosmology and RSD measurements (e.g. Macaulay, Wehus & Eriksen, 2013).
3.6 Conclusions And Discussion

In search of departures from the standard cosmological model and clues towards possible extensions, we have measured time- and scale-dependent deviations to the gravitational field equations of General Relativity. We model these deviations using the time and scale-dependent parameters \( \{G_{\text{matter}}, G_{\text{light}}\} \). These parameters are defined using 2 bins in time and 2 bins in scale. \( G_{\text{matter}} \) modifies the gravitational interaction for non-relativistic particles, and hence alters structure formation, while \( G_{\text{light}} \) acts equivalently for relativistic particles, thus affecting how light propagates through the universe.

To measure the eight parameters describing this model, plus the six describing the standard model, we utilize a range of cosmological probes including BAOs, Type Ia SNe, the CMB, CMB lensing, and the cross-correlation of the CMB with large-scale structure probes. In addition, we include measurements of the power spectrum multipoles from the WiggleZ and CMASS galaxy redshift samples, and the velocity power spectrum from 6dFGSv. Our motivation for adopting a phenomenological model is to provide a set of results that can self-consistently be used to test the widest possible range of models. To this end, we have focused on only analyzing measurements on scales within the linear regime. We summarise our main results as follows:

- We perform a new measurement of the power spectrum multipoles of the WiggleZ survey, featuring a new calculation of the window function convolution effects and an improved determination of the covariance from N-body simulations.

- Modeling deviation from General Relativity in terms of the growth of large-scale structure, we find the following results, given in terms of 68% CLs: \( G_{\text{matter}}(z < 1; k > 0.01) = 0.65 \pm 0.43 \), \( G_{\text{matter}}(z < 1; k < 0.01) = 1.22^{+0.39}_{-0.34} \), \( G_{\text{matter}}(z > 1; k > 0.01) = 0.53 \pm 0.32 \), \( G_{\text{matter}}(z > 1; k < 0.01) = 0.87 \pm 0.30 \). These constraints are consistent with GR (i.e., \( G_{\text{matter}} = 1 \)) at the 95% confidence level. We observe a small tension (> 1\( \sigma \)) for the high-wavenumber and high-redshift bin.
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- Modeling deviation from General Relativity in terms of light propagation, we derive the following constraints, given in terms of 68% CLs: 
  \[ G_{\text{light}}(z > 1; k > 0.01) = 1.057^{+0.053}_{-0.045}, \]
  \[ G_{\text{light}}(z < 1; k < 0.01) = 1.048 \pm 0.048, \]
  \[ G_{\text{light}}(z < 1; k > 0.01) = 1.153^{+0.080}_{-0.068}, \]
  \[ G_{\text{light}}(z > 1; k < 0.01) = 1.016 \pm 0.026. \]
  These constraints are consistent with General Relativity at the 95% confidence level: the significant improvement in constraining power, relative to \( G_{\text{matter}} \), is due to the the sensitivity of the ISW effect and CMB lensing to deviations in \( G_{\text{light}} \).

- We preform Bayesian model comparison between general relativity and our phenomenological model. To do this we compute the evidence for both models and take the ratio. Following the scale suggested by Kass & Raftery (1995), the ratio of evidence values suggests “no support” for modifying general relativity. This is consistent with the results from the posterior distributions.

- Adopting an alternative model, we introduce deviation in the expansion and growth histories simultaneously by varying the growth index and two parameters describing a redshift-dependent equation of state. For this fit we utilize, among other probes, recent growth rate constraints from RSDs, as measured from the WiggleZ, CMASS, and 6dF surveys. Our final result assuming a \( \Lambda \)CDM expansion history (in terms of 68% CL) is \( \gamma = 0.665 \pm 0.067 \), while allowing the expansion history to deviate from \( \Lambda \)CDM we measure \( \gamma = 0.69^{+0.09}_{-0.11} \). Both these results are consistent with the standard model; however, introducing SN measurements to this fit (either SNLS or JLA) we find a \( \sim 2\sigma \) tension with \( \Lambda \)CDM.

Probes of the velocity field of galaxies have an indispensable role to play in addressing questions of the nature of dark energy as they are uniquely sensitive to only temporal perturbations. The observational datasets we have analyzed are consistent with a vacuum energy interpretation of dark energy; however, due to the magnitude of current uncertainties any final conclusions drawn from these, and other current, observations would be premature. In future analysis tomographic weak lensing and galaxy-galaxy lensing measurements will be included to improve our constraints; furthermore, we will begin assessing the viability of specific models using the inferred parameter constraints.
3.6. CONCLUSIONS AND DISCUSSION
Chapter 4

Determining Source Redshift Distributions With Cross Correlations

Abstract

We present a new quadratic estimator that determines the redshift probability distribution of a photometric sample of galaxies. Our estimator recovers this unknown probability distribution from the angular auto- and cross-correlations between the photometric sample and an overlapping spectroscopic sample. This estimator is derived by extending the analysis by McQuinn & White (2013). In particular, we suggest a number of corrections to their original derivation of an optimal quadratic estimator. We show without these corrections the original estimator is biased, when keeping all relevant terms. To determine the accuracy of this estimator we use a series of N-body simulations to construct mock catalogues. These mocks span $0.1 < z < 0.9$ and cover $60 \, \text{deg}^2$, while the simulations we use are the Scinet L1ght Cone Simulations (SLICS) (Harnois-Déraps & van Waerbeke, 2015). For each mock we construct a photometric and spectroscopic sample from input redshift distributions. To describe the redshift probability distribution of the photometric sample we use a single Gaussian distribution. At the level of statistical error present in individual mock catalogues, we find the reconstructed distributions to be consistent with the input distribution, at a level of 2-$\sigma$. Averaging over the results from all 20 mock catalogues, we find some discrepancy with the input redshift distribution. In particular,
four redshift bins deviate from the model at a level greater than 2-σ. These discrepancies likely result from biases in the modelling of the angular correlation function. In future work we will apply this methodology to infer the distribution of galaxies, in tomographic bins, present in the Kilo-degree lensing imaging survey using the spectroscopic 2-degree Field Lensing Survey.

4.1 Introduction

Current and forthcoming photometric surveys aim to image a significant fraction of the sky\(^1\). Doing so, they will obtain the angular positions of millions of galaxies. To fully exploit the potential of these surveys an estimate of the redshift distribution of the galaxies in the sample is required. One is the most important examples of this, for understanding dark energy, is tomographic weak lensing (Hu, 1999; Huterer, 2002). For tomographic weak lensing to compute the expected lensing signal, the distribution of galaxies, within each tomographic bin, must be measured very accurately. Further examples include the integrated Sachs-Wolfe effect, and angular power spectrum measurements. Motivated by this, in this work we investigate a method to measure galaxy redshift distributions using angular cross-correlations.

We outline this approach as follows. Consider two galaxy samples: a spectroscopic sample of galaxies, with a known redshift distribution, and a photometric sample of galaxies with an unknown redshift distribution. The samples overlap on the sky and in redshift-space. Therefore, they were formed from the same underlying density field, and hence share a common sample variance. Because of this we expect a positive angular cross-correlation function between the samples. Importantly, this signal occurs independent of the galaxy populations (e.g., the galaxies’ color, luminosity, etc.) within each sample.

The amplitude of the angular cross-correlation will depend on the degree of overlap of the two samples (e.g., Ho et al., 2008; Erben et al., 2009; Newman, 2008): where the larger the spatial overlap the larger the correlation. Therefore,

\(^1\)Surveys currently being completed are the Kilo Degree Survey (de Jong et al., 2015), the Dark Energy Survey (The Dark Energy Survey Collaboration, 2005), and the HyperSuprimeCam. Future surveys include the Large Synoptic Sky Telescope (LSST), and Euclid.
to localise the photometric sample one can divide the spectroscopic sample into adjacent redshift bins, and then for each bin measure the angular cross-correlation with the photometric sample\(^2\). The amplitude of the correlation within each bin tells how much of the photometric sample falls within that redshift region.

In this work we develop a statistical estimator that uses angular cross-correlations to estimate the redshift distribution of a sample of galaxies. We do this with the application to weak lensing in mind, where, as stated above, knowledge of the redshift distribution of galaxies in the sample is a critical component.

Weak gravitational lensing induces an angular deflection of light rays from the source galaxy population. Using the observed (distorted) shapes of these galaxies one can measure the spatial gradient of this deflection field, which we label the lensing convergence \(\kappa\). Because the geometry of the source-lens system changes with redshift, the deflection field and hence the convergence field also change with redshift, viz., \(\kappa = \kappa(\theta, z)\). Rather than measuring the shear field at a specific redshift, galaxy shapes probe the integrated effect of the shear field. Therefore, the observable of interest is the effective convergence (Bartelmann & Schneider, 2001), viz.,

\[
\kappa^\text{eff}(\theta) = \int dz P(z) \kappa^\text{fixed}(\theta, z). 
\]

\(P(z)\) is the normalized probability distribution function for galaxies in the photometric sample (the quantity we wish to measure). Transforming to harmonic space, the shear field can be characterized by the convergence power spectrum, \(P_\kappa(l) = \langle |\kappa^\text{eff}(\ell)|^2 \rangle\). Knowledge of this power spectrum allows one to compute the two-point shear correlation function, the quantity most commonly derived from galaxy ellipticity measurements:

\[
\xi^+(\theta) = \frac{1}{2\pi} \int d\ell \, \ell \, P_\kappa(l) J_+ (\ell \theta),
\]

where \(J_\pm (\ell \theta)\) is the zeroth (fourth) order Bessel function of the first kind for \(\xi^+ (\xi^-)\).

From Eq. (4.1), uncertainty in \(P(z)\) will directly propagate into the statistical

\(^2\)As we will discuss below, there are also additional effects which can correlate the two samples.
uncertainty of the shear field, and hence cosmological constraints. Similarly, any bias in \( P(z) \) (for example, a bias in a mean redshift) will introduce a bias into cosmological inferences made from lensing. To be more quantitative, for “Stage IV” experiments – examples include, BigBOSS, LSST, Euclid, and WFIRST, see Weinberg et al. (2013) – to avoid a degradation of constraints by more than 50%, the mean and standard deviation of the photometric redshift distribution \((\phi(z))\) need to be measured to \(\sim 0.002(1+z)\); failing this high level of calibration, photo-z errors will be the dominant systematic error (Huterer et al., 2006; Newman et al., 2013a).

We define \textit{direct calibration} methods as those that attempt to calibrate a mapping from the flux in photometric bands to a galaxy’s redshift. Template-based approaches and machine learning algorithms both lie in this category. These approaches have many factors that make the above level of accuracy difficult to obtain: examples include, catastrophic photometric errors, completeness requirements for spectroscopic training samples, and sample variance (see, Bernstein & Huterer, 2010; Cunha et al., 2014, 2012; Newman et al., 2013a). As we will discuss in the following section, cross-calibration methods offer the potential to avoid the more severe of these complications, motivating our investigation of this technique. In particular, we will focus on the extension and application of the \textit{optimal quadratic estimation} method proposed by McQuinn & White (2013).

The outline of this chapter is as follows. In Sec. 4.2 we introduce the strengths and weaknesses of calibration via cross-correlations, and discuss the previous work in the field. We introduce background theory in Sec. 4.3. In Sec. 4.4 we derive a new quadratic estimator for the redshift distribution of galaxies. We outline the construction of our mock galaxy catalogues in Sec. 4.5, and then discuss the results in Sec. 4.6. Finally, we present our conclusions and discuss future applications in Sec. 4.7.
4.2 Calibration Using Cross-Correlations

4.2.1 Motivations...

Photometric redshift calibration can be divided into two categories: ‘indirect calibration’ and ‘direct calibration’. Machine learning and template-based methods are examples of direct photometric calibration, while using cross-correlations is an example of indirect calibration. Machine learning methods use spectroscopic training samples to calibrate an empirical mapping from the magnitude and colour of galaxies to their redshift. While template methods use theoretical models to predict spectral energy distributions, which are then convolved with the filters being adopted by the photometric survey. Redshifts are then estimated by minimising the scatter between the model and observed photometry.

The key point of distinction between these approaches is that direct calibration requires spectroscopic redshifts for a subsample of the full photometric sample; importantly, this subsample needs to be representative in both color and magnitude space, relative to the full sample (see, Sánchez et al., 2014; Sadeh, Abdalla & Lahav, 2015). This requires the targeted spectroscopic sample to be highly complete, i.e., a secure redshift needs to be measured for > 99% of targets. To achieve this level of completeness, spectroscopic redshifts are required for very faint and high-redshift galaxies that are abundant in deep imaging surveys. For cross-correlation analysis one is free to target any tracer of large-scale structure (most usefully, very bright galaxies), thus avoiding this difficulty.

Achieving a high level of completeness presents a significant observational challenge as the chance of obtaining a successful redshift is very dependent on the object’s magnitude. Therefore, spectra are typically obtained for a non-random subsample of the target catalog. A useful example is the DEEP2 survey conducted on the DEIMOS spectrograph at Keck Observatory (Newman et al., 2013b). For the highest redshift quality class, secure redshifts were only obtained for 60% of the galaxies (Newman et al., 2013b). Considering future surveys, the severity of this problem is

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3 Examples of machine learning algorithms include SkyNet (Graff et al., 2014), TPZ (Carrasco Kind & Brunnner, 2013), and ANNz2 (Sadeh, Abdalla & Lahav, 2015). And examples of template-based methods are BPZ (Benítez, 2000), and EAZY (Brammer, van Dokkum & Coppi, 2008).
demonstrated by the requirement for spectroscopic follow-up suggested by Newman et al. (2013a); namely, to obtain > 90% redshift completeness at $i = 25.3$ (LSST depth) would require more than 100 nights on Keck.

Note, the requirements for spectroscopic follow-up can be reduced by assigning weights to galaxies during the training phase of photometric redshift calibration (Cunha et al., 2012; Sadeh, Abdalla & Lahav, 2015). These weights are assigned based on the color-magnitude phase space distribution of both the parent photo-z sample and the follow up spectroscopic sample. The result is that the weighted spec-z sample more closely matches the photo-z sample in color-magnitude space. The extent to which this method allows one to reduce the necessary completeness is unclear.

The above challenges are relevant for machine learning algorithms, as they require separate training, testing, and validation samples. However, similar requirements exist for template-based approaches. In this case the accuracy of the photometric redshift (“photo-z”) estimate is still highly dependent on the completeness of the follow-up spec-z sample. This occurs for the following reasons. To develop accurate spectral templates one requires a representative sample of objects. Deriving a Bayesian prior used in the fitting process (e.g., BPZ Benítez, 2000) requires a spectroscopic redshift (“spec-z”) sample: this prior can strongly influence the final results (see, Sánchez et al., 2014). Finally, to accurately validate the template predictions requires a representative sample.

Catastrophic errors present an additional issue present with direct calibration methods, these are photo-z estimates ($z_p$) with $|z_p - z_{\text{true}}| \sim O(1)$. This occurs because with only broad band flux information there are degeneracies in galaxy colors – common examples include the confusion between the Lyman-α break and the Balmer break, and when Balmer break moves out of the optical filter set, for galaxies at $z > 1$. General studies of the consequences of catastrophic errors are presented by Bernstein & Huterer (2010) and Hearin et al. (2010).
4.2.2 Challenges...

As we emphasize above, there are specific advantages to calibrating photo-z’s by cross-correlation, rather than direct methods. Nonetheless, there are significant challenges to this approach, the most pertinent of which we outline below:

- **Degeneracies**: The quantity measured via cross-correlations is, unavoidably, the combination \( P(z) b(z) \), where \( b \) is the source galaxy bias factor. Therefore, an evolving galaxy bias \( b(z) \) is completely degenerate with a fluctuation in number density (see, Newman, 2008). To calibrate the redshift-dependent galaxy bias extra probes (e.g., galaxy-galaxy lensing) or assumptions (e.g., the bias varies smoothly with redshift) will be required. Very likely this will be the dominant source of statistical error in constraining the redshift distribution.

- **Cosmological Dependence**: To calibrate the number density one needs to compare the observed and predicted angular cross-correlation functions. The amplitude of the predicted cross-correlation function depends on both our guess of \( P(z) \) and the cosmological model. This introduces a worrying circularity, as our aim is to test the cosmological model with lensing measurements derived using the measured \( P(z) \).

- **Extra Source of Correlations**: Magnification bias can introduce an additional source of correlations. As a result, if this effect is unaccounted for, it can bias measurements of the redshift distribution. In addition to changing the shape of galaxies, lensing changes the size and as a result the flux of observed galaxies. Therefore, galaxies normally outside the flux limit of a survey can be magnified enough to be included. This effect, *magnification bias*, correlates source and lens galaxies because the source galaxies are magnified by the lens galaxies (Bernstein & Huterer, 2010; McQuinn & White, 2013; Duncan et al., 2014)
4.2.3 Developments

We highlight recent work on photometric calibration with angular cross-correlations.

- **Estimators**: A number of independent estimators have been suggested in the literature for inferring the redshift distribution of a photometric sample using an overlapping spec-z sample (Newman, 2008; Matthews & Newman, 2010; Ménard et al., 2013; Schulz, 2010; McQuinn & White, 2013). To give some examples of the types of analysis, McQuinn & White (2013) develop a quadratic estimator, while Ménard et al. (2013) and Schulz (2010) use maximum likelihood approaches to infer the input distribution.

- **Self-Calibration**: Dividing a photometric sample into redshift bins allows one to cross-correlate between bins. This correlation allows one to determine the contamination fraction for the sample and potentially other systematic errors. Work on this calibration approach is presented by Padmanabhan et al. (2007); Erben et al. (2009); Benjamin et al. (2013).

- **Applications**: The methodology presented by Ménard et al. (2013) has been applied to estimate the redshift distributions of the LRG sample from the SDSS, infrared sources from WISE, and radio sources from FIRST (Ménard et al., 2013; Rahman et al., 2015). In contrast, the approach suggested by McQuinn & White (2013) has yet to be applied to data.

For a comparison between the quadratic estimator methodology and the methodology developed by Ménard et al. (2013) we refer the reader to McQuinn & White (2013). In short, the scales analysed by these methods are quite distinct. In particular, the measurements from Ménard et al. (2013) predominantly derive from very small scales, viz., $< 300$ proper kpc (Schmidt et al., 2013). This complicates the measurement of redshift distributions for the following reasons. First, these length scales can be inside individual haloes and in this case the correlation signal will depend on how the samples inhabit the same haloes. Second, on these length scales
scale-dependent bias terms may be significant, as well as the effects of baryonic physics.

The quadratic estimator is effectively an application of the Newton-Raphson optimizing method to $\log(\mathcal{L})$, where $\mathcal{L}$ is a multi-variate Gaussian likelihood function for the observed angular cross-correlation function. Therefore, we expect the constraints derived from this method and from the maximum likelihood approach developed by Schulz (2010) to be the same. However, we argue that the quadratic estimator has some advantages. In particular, the time taken for the quadratic estimator to converge will be faster (typically only $\sim 20$ steps are needed), the method is simpler to implement, and the inclusion of magnification bias for the quadratic estimator is trivial.

4.3 Parameterization and Modeling

We first consider the simplistic case of a uniform distribution of galaxies divided into non-overlapping redshift bins and introduce the relevant covariances. Unless stated otherwise, throughout this work we will assume the fiducial parameters of the Scinet LIght Cone Simulations (SLICS) (Harnois-Déraps & van Waerbeke, 2015). The SLICS simulations adopts a WMAP9+BAO+SN cosmological parameter set: matter density $\Omega_m = 0.2905$, baryon density $\Omega_b = 0.0473$, Hubble parameter $h = 0.6898$, spectral index $n_s = 0.969$, and normalization $\sigma_8 = 0.826$. Using the small angle (or ‘Limber’) approximation (Limber, 1954) the cross power spectrum between redshift slices is given by

$$C_{z_i z_j}(\ell) = \delta^K_{ij} \int_0^\infty dz \ p_{z_i}(z)^2 \frac{b_i^2 \mathcal{P}(k, z)}{r(z)^2 r_H(z)}.$$  \hspace{1cm} (4.3)

Where $\mathcal{P}(k, z)$ is the matter power spectrum, $r_H(z) \equiv \partial r(z)/\partial z$, and $k = (\ell + 1/2)/r$. Importantly, with the Limber approximation, because we are ignoring long wavelength modes, the covariance between non-overlapping bins is zero. This relation (Eq. 4.3) further assumes a linear relationship between the dark matter and galaxy density field, i.e., $\delta_g = b \delta_{DM}$. Note, the galaxy bias can vary between redshift bins. Finally, $p_{z_i}(z)$ is the galaxy probability distribution function for the $i$th redshift bin.
This term weights the contribution from each redshift to the integral. For our initial case it is given by a top hat window function:

\[ P_{z_j}(z) = \begin{cases} 
1/\Delta_j & \text{for } |z - z_j| < \Delta_j/2 \\
0 & \text{otherwise.}
\end{cases} \] (4.4)

Where \( \Delta_i \) is the size of the \( i^{th} \) redshift bin.

For the case we consider in this paper we have two samples of galaxies (a photometric and spectroscopic sample) with non-uniform number densities. The distribution of the photometric galaxies is unknown, while the distribution of the spectroscopic sample is well known.

We parameterize the redshift distribution of each sample in \( N^{\text{bin}} \) step-wise bins in redshift space, such that,

\[ N^{(A)}(z) = N_i \quad \text{for } |z - z_j| < \Delta_j/2 \] (4.5)

Applying these parameterizations, the probability distribution function for either sample can be written as \( \phi^{(A)}(z) = \sum_i N_i^{(A)} p_{z_i}(z) \); where \( (A) \) can take the value of \( p \) or \( s \), referencing either the photometric or the spectroscopic sample. For a single bin in the spec-z sample, \( \phi_i^{(s)}(z) = N_i^{(s)} p_{z_i}(z) \). Following from these definitions and Eq. (4.3), the auto and cross-correlations between the full photo-z sample and the individual spec-z bins can be written as (McQuinn & White, 2013)

\[
\begin{align*}
\langle p s_i \rangle(l) &= N_i^{(s)} b_i^{(s)} N_i^{(p)} b_i^{(p)} C_{z_i z_i}(\ell) + \omega^{p,i}, \\
\langle s_i s_j \rangle(l) &= \delta_{ij} \left[ \left( N_i^{(s)} b_i^{(s)} \right) 2 C_{z_i z_j}(\ell) + \omega^{s,i} \right], \\
\langle p p \rangle(l) &= \sum_{j=1}^{N^{\text{bin}}} \left( N_j^{(p)} b_j^{(p)} \right) 2 C_{z_i z_j}(\ell) + \omega^p.
\end{align*}
\] (4.6-4.8)

Note, we have introduced a shot noise component to the angular power spectra. To model the shot noise we assume Poisson statistics – hence we are assuming the non-Poisson contributions (Baldauf et al., 2013) are negligible. Following this assumption the shot noise components are \( \omega^{A,i} = N_i^{(A)} \), and \( \omega^{p,i} = f_{\text{over}} N_i^{(s)} \); where \( f_{\text{over}} \) is the
overlap fraction between the photo-z and spec-z sample. To compute the correlation statistics we first compute Eq. (4.3) for each redshift bin via the public software CHOMP, as introduced by Morrison & Schneider (2013)\(^4\). This calculation requires as input the matter power spectrum for each redshift bin, we model this using the HaloFit code (Smith et al., 2003), where the HaloFit parameters adopted are those fit by Takahashi et al. (2012). Given the redshift bins are narrow, CHOMP computes the redshift evolution in each bin from linear theory, viz., using \( P(k, z) = D(z)^2 P(k) \), where \( D(z) \) is the growth function.

### 4.3.1 Complications

A number of systematic biases could potentially make our cross-correlation predictions inaccurate, and hence bias our final constraints. We outline the most pertinent of these below. But first we transform Eqns. (4.6,4.7,4.8) into configuration space, where the angular auto- and the cross-correlation statistics are given by

\[
\begin{align*}
    w_{ss}(\theta) &= b^{(s)}_i b^{(s)}_j P^{(s)}(z_i) P^{(s)}(z_j) w_{ss}(\theta) \\
    w_{sp}(\theta) &= b^{(s)}_i P^{(s)}(z_i) b^{(p)}_j P^{(p)}(z_j) w_{ss}(\theta) \\
    w_{pp}(\theta) &= \sum_i \left[ b^{(p)}_i P^{(p)}(z_i) \right]^2 w_{ss}(\theta).
\end{align*}
\]

We introduce these formulae because we will analyse our simulations in configuration space in section 4.5. For simplicity we write the angular correlation functions in terms of step-wise redshift probability distributions \( P^{(s)}(z_i) \) and \( P^{(p)}(z_i) \), rather than the number count distributions \( N^{(p)}_i, N^{(s)}_i \): the two are related by \( N^{(p)}(z_i) = N^{(p)}_i / \sum_i N^{(p)}_i \). The modelling complications follow:

- **Non-linear effects.** We will measure the angular-correlation function down to scales \(~ 1 \text{ [arcmin]}\). On such scales non-linear effects become significant and the halo-fit model we adopt may become inaccurate.

\(^4\)We check the accuracy of this code by comparing its output with our own calculations. For both the angular power spectrum and correlation function, the calculations agree. We adopt CHOMP because of its useful class based structure.
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- **Bias evolution.** We will present measurements of the redshift probability distribution $P^{(p)}(z_i)$. To determine these constraints we need to account for the redshift evolution of galaxy bias, for both samples. An incorrect bias evolution model will bias the recovered $P^{(p)}(z_i)$: as the true observable is the combination $P^{(p)}(z_i)b_i^{(p)}$.

- **Flat $N(z)$ approximation.** To derive the above equations we must approximate the redshift distributions using a step-wise parametrisation. So within each bin we consider the redshift distribution to be constant. This approximation will break down if there are steep gradients in the redshift distribution. As a result, the effective redshift at which the auto-correlation functions $(w_{s_i s_i}(\theta))$ are calculated will be incorrect.

### 4.4 Optimal Quadratic Estimation

We outline the construction and properties of the quadratic estimator we employ to measure the redshift distribution of galaxies. This work extends that presented by McQuinn & White (2013, henceforth, McQ+13).

#### 4.4.1 Background

At this point we consider the multipole coefficients for both samples as the observable quantities, and combine these coefficients into a single data vector $\mathbf{x} = (\hat{p}(\ell, m), \hat{s}(\ell, m))$: Where $(s)_i = s_i$ are the coefficients for the $i$th redshift bin, and the hat ($\hat{..}$) indicates a statistically estimated quantity. We now construct a quadratic estimator for the redshift distribution of the photo-z sample from the data $\mathbf{x}$.

Our estimator will depend on an initial guess of the number distribution, which we label $[\hat{N}]_{\text{last}}$. This guess is then updated after each iteration, where the estimated $\hat{N}_i$ becomes the old guess $[\hat{N}]_{\text{last}}$. This proceeds iteratively, until convergence is reached. The general form of this estimator is (see, Bond, Jaffe & Knox, 1998; Tegmark, 1997; Dodelson, 2003)

$$\hat{N}_i = [\hat{N}]_{\text{last}} + \mathbf{x}' \mathbf{E}' \mathbf{x} - \mathbf{b}_i.$$  \hspace{1cm} (4.12)
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Where $E^i$ is a symmetric matrix and $b_i$ is a constant. Requiring that the estimator is unbiased (it converges to the input distribution) and optimal (it minimizes the variance) one can solve for both variables, the result is (Tegmark, 1997)

$$\hat{N}_i = [\hat{N}_i]_{\text{last}} + \frac{1}{2} \sum_j [F^{-1}]_{ij} \left[ x^T Q_j x - \text{tr}(Q_j A) \right], \quad (4.13)$$

where

$$Q_j = A^{-1} A_j A^{-1}. \quad (4.14)$$

And $A = \langle xx \rangle$ is the covariance matrix of the data and its derivative is $A_{,\alpha} = \partial A / \partial N_\alpha$. Note, implicitly $A$ is a function of both $\ell$ and $m$, i.e., $A = A(\ell, m)$. Assuming many modes are included one can approximate $F$ as the Fisher matrix where\(^5\)

$$F_{ij} = \frac{1}{2} \sum_{\ell,m} \text{Tr} \left[ A^{-1} A_j A^{-1} A_{,i} \right]. \quad (4.15)$$

When this assumption is violated, the Fisher matrix will be biased by sample variance, we do not expect this assumption to have a significant effect on our results (for an explanation see, Bond, Jaffe & Knox, 1998, McQ+13). Most importantly, the variance of this estimator is equal to the Fisher matrix, thus the estimator is optimal.

4.4.2 Revised Analytic Form Of Estimator

In this section we re-derive the analytic form of the estimator, Eq. (4.13), during which we suggest a number of corrections to the original derivation presented by McQ+13. As will be shown below, all of our corrections propagate from a single assumption.

To understand our revision, note that our Eq. (4.13) is more general than Eq. (16) in McQ+13. The two results agree as $\text{Tr}(Q^{(\alpha)} A)$ reduces to $\text{Tr}(A^{-1} A_{,\alpha})$. However, for the derivation of Eq 31 McQ+13 neglect all derivatives of $A_{,00}$, thus changing

\(^5\)The Fisher matrix is an approximation of the curvature matrix. Furthermore, the Fisher matrix is equal to the ensemble average of the curvature matrix, for details see Bond, Jaffe & Knox (1998).
4.4. **OPTIMAL QUADRATIC ESTIMATION**

\( Q \). This invalidates the above simplification, so \( \text{Tr}(Q^\alpha A) \neq \text{Tr}(A^{-1} A_\alpha) \). Therefore, when neglecting the derivatives of \( A_{00} \) one should keep the term \( \text{Tr}(Q^\alpha A_\alpha) \) and then re-derive the full expression for the estimator.

By definition the components of the covariance matrix \( A \) are \( A_{00} = \langle pp \rangle \), \( A_{0i} = \langle ps_i \rangle \), and \( A_{ij} = \langle s_i s_j \rangle \); hence, from the formula in the previous section (Eqs. 4.6, 4.7, 4.8) the elements of this matrix are known. Moreover, note, from Eq. 4.6 the derivative of the off-diagonal terms is \( \partial A_{0i}/\partial N_i = b_i^{(p)} b_i^{(s)} N_i^{(s)} C_{zi}(\ell) \).

To simplify the expressions that follow, using the notation from McQ+13, we define the ‘Schur’ parameter \( S \) as

\[
S \equiv A_{00} \left( A_{00} - \sum_{i=1}^{N_{\text{bin}}} \frac{A_{0i}^2}{A_{ii}} \right)^{-1} = \left( 1 - \sum_{i=1}^{N_{\text{bin}}} r_i^2 \right)^{-1}.
\]

Here \( r_i(\ell) \) is the cross correlation coefficient between the photo-z sample and spec-z sample in the \( i \)th z-bin: \( r_i(\ell) \equiv A_{0i}/(A_{00} A_{ii})^{1/2} \). Adopting these definitions, the estimator can be written as

\[
\hat{N}_i^{(p)} = \left[ \hat{N}_i^{(p)} \right]_{\text{last}} + \sum_j (\mathcal{F}^{-1})_{ij} \sum_{\ell,m} \left( \frac{S}{A_{00} A_{jj}} \right) \frac{\partial A_{0j}}{\partial p_j} \\
- \frac{SC_{0j}}{A_{00}} \hat{p} \hat{p} + \sum_k \left( \delta_{jk} + 2Sr_j r_k \sqrt{\frac{A_{jj}}{A_{kk}}} \right) \hat{p} \hat{s}_k \\
+ \sum_k \frac{A_{0k}}{A_{kk}} \left( \delta_{jk} + Sr_j r_k \sqrt{\frac{A_{jj}}{A_{kk}}} \right) \hat{p}_k \hat{p}_k - A_{0j} - 2Sr_j r_k \sqrt{\frac{A_{jj}}{A_{kk}}} A_{0k},
\]

where we have included both the auto-correlation and cross-correlation terms. Note, the importance of these terms in different scenarios is discussed in McQ+13.

One can check this expression converges to the input theory as follows (Bond, Jaffe & Knox, 1998). Firstly, we write the correction term as \( \delta n_i \equiv \hat{N}_i^{(p)} - \left[ \hat{N}_i^{(p)} \right]_{\text{last}} \). Now assuming the input theory is correct \( \langle \hat{p} \hat{s}_j \rangle = A_{0j}, \langle \hat{p} \hat{p} \rangle = A_{00}, \) and \( \langle \hat{s}_i \hat{s}_j \rangle = A_{ij} \). Then, following some algebra, Eq. 4.17 implies \( \langle \delta n_i \rangle = 0 \), proving the estimator will converge. For the equivalent equation from McQ+13 \( \langle \delta n_i \rangle \neq 0 \).
The analytic form of the Fisher matrix (Eq. 4.15) remains unchanged from McQ+13, where

$$F_{ij} = \sum_{l,m} \frac{S}{A_{00}} \left( \frac{\delta^K_{ij}}{A_{ii}} + 2 S \sqrt{\frac{r^2_{ij}}{A_{ii} A_{jj}}} \right) [A_{0i}]_i [A_{0j}]_j. \quad (4.18)$$

In the limit where shot noise dominates (i.e., where $r_i(\ell) \approx 0$ and $S \approx 1$), and neglecting auto-correlations, our result (Eq. 4.17) agrees with Eq. (36) from McQ+13.

### 4.4.3 Converting To Configuration Space

When analyzing observational data we will work exclusively in configuration space, this allows us to avoid difficulties with complex survey geometries: however, in the future this may not be necessary (see, Alsing et al., 2016; Köhlinger et al., 2015). Thus in order to match our theory with observations, we need to convert our estimator (Eq. 4.13) from harmonic to configuration space. This conversion is straightforward given the following relation (McQ+13):

$$\sum_{l,m} v_i(l) \left[ \hat{p}(l,m) \hat{s}_i(l,m) - N^{p,si} \right] = 8\pi^2 \int dx v_i(x) \hat{w}_{ps,i}(x)$$

$$\approx 8\pi^2 \sum_\alpha \Delta\theta_i \theta_\alpha v_i(\theta_\alpha) \hat{w}_{ps,i}(\theta_\alpha). \quad (4.19)$$

Where $\hat{w}_{ps,i}(\theta)$ is the observed angular cross-correlation, $\theta$ is the angular separation scale, and $x = \hat{n} \cdot \hat{n}' \equiv \cos(\theta)$. Here we have explicitly subtracted the shot-noise component\(^6\). Eq. 4.19 is valid for an arbitrary function $D(l)$, which is related to $D(\theta)$ by

$$D_i(\theta) = \sum_\ell \left( \frac{2\ell + 1}{4\pi} \right) D_i(\ell) P_\ell(\cos \theta). \quad (4.20)$$

Our measurements of the angular correlation functions are made in bins of $\Delta \theta_\alpha$.

\(^6\)Note, incorrectly modeling the shot-noise component will introduce a bias into the final measurements.
4.4. OPTIMAL QUADRATIC ESTIMATION

with central values $\theta_\alpha$. These values set the properties of the summation in Eq. 4.19. Note, because the kernel ($= \theta_\alpha v_i(\theta_\alpha) \tilde{w}_{ps_j}(\theta_\alpha)$) is not a slowly varying function, a narrow $\theta$ spacing ($\Delta \theta_\alpha$) is needed to accurately approximate the integral.

Now, to convert Eq. (4.13) into configuration space we first rewrite the estimator in terms of four weighting function defined as

$$D_j(l) \equiv \left( \frac{S}{C_{00} C_{jj}} \right) \frac{\partial C_{0j}}{\partial p_j},\quad (4.21)$$

$$E_{jk}(l) \equiv D_j(l) \times 2Sr_jr_k \sqrt{\frac{C_{jj}}{C_{kk}}},\quad (4.22)$$

and,

$$H_j(l) \equiv D_j(l) \times \frac{C_{0j}}{C_{jj}}, G_i(l) \equiv D_j(l) \times \frac{C_{0j}}{C_{00}}.\quad (4.23)$$

Using Eq. 4.19 one finds that our estimator (Eq. 4.13) in configuration space takes the form

$$\hat{N}_i^{(p)} = [\hat{N}_i^{(p)}]_{last} + 8\pi^2 \sum_j (F^{-1})_{ij}$$

$$\sum_\alpha \Delta \theta_\alpha \theta_\alpha \left[ D_j(\theta_\alpha) \left\{ \tilde{w}_{ps_j}(\theta_\alpha) - w_{ps_j}(\theta_\alpha) \right\} ight.$$  

$$+ H_j(\theta_\alpha) \tilde{w}_{s_j,s_j}(\theta_\alpha) - G_j(\theta_\alpha) \tilde{w}_{pp}(\theta_\alpha)$$  

$$+ \sum_k E_{jk}(\theta_\alpha) \left\{ \tilde{w}_{ps_k}(\theta_\alpha) - w_{ps_k}(\theta_\alpha) + \frac{1}{2} \tilde{w}_{s_k,s_k}(\theta_\alpha) \right\}$$  

$$]$$

Where $\{D, E, G, H\}$ have been transformed to configuration space using Eq. (4.20). Ignoring both the bin-to-bin correlations, such that the Fisher Matrix is diagonal, and the auto-correlation terms, the second term in Eqn. (4.24) becomes (McQ+13)

$$8\pi^2(F^{-1})_{ii} \sum_\alpha \Delta \theta_\alpha \theta_\alpha \left[ D_j(\theta_\alpha) \left\{ \tilde{w}_{ps_j}(\theta_\alpha) - w_{ps_j}(\theta_\alpha) \right\} \right]$$

This result is much more intuitive. The $N(z)$ is reconstructed from a weighted minimization of $\sim \{ \tilde{w}_{ps_j}(\theta_\alpha) - w_{ps_j}(\theta_\alpha) \}$, where the weights are given by $D_j(\theta)$. Note, the scales that contribute most to the estimator are more accurately represented by
the combination $D_j(\theta)w_{psj}(\theta)$ (for an explanation see, McQ+13). To illustrate the angular scales this estimator is sensitive to, in figure (4.1) we plot these weights for the mocks introduced in the subsequent section. We find the weights peak at $\theta \sim 2$ arcmins and scales above $\sim 30$ arcmins are significantly down-weighted.

### 4.5 Simulations

To determine the reliability of our estimator we construct *synthetic galaxy catalogues*. Briefly, we construct a series of overlapping photometric and spectroscopic samples. Each spectroscopic sample is divided into adjacent redshift bins. And using these galaxy samples we measure the relevant angular auto- and cross-correlations statistics—that is, between the full photometric sample and each redshift bin of the spectroscopic sample. Using these measurements as inputs, we apply our algorithm to reconstruct the redshift distribution of the photometric sample. Finally, to test
our algorithm we compare the reconstructed redshift distributions to the input distribution.

To expand, we generated mock catalogues using the Scinet LIght Cone Simulations (SLICS) series of N-body simulations (Harnois-Déraps & van Waerbeke, 2015) which have been produced using the CUBEP$^3$M code (Harnois-Déraps, Vafaei & Van Waerbeke, 2012) using a WMAP9+BAO+SN cosmological parameter set: matter density $\Omega_m = 0.2905$, baryon density $\Omega_b = 0.0473$, Hubble parameter $h = 0.6898$, spectral index $n_s = 0.969$, and normalization $\sigma_8 = 0.826$. The box-size of the simulations is $L = 505$ $h^{-1}$ Mpc. The simulations follow the non-linear evolution of $1536^3$ particles inside a $3072^3$ grid cube.

For each simulation, the density field is output at 18 redshift snapshots in the range $0 < z < 3$. The gravitational lensing shear and convergence is computed at these multiple lens planes using the Born approximation in the flat-sky approximation, and a survey cone spanning $60 \, \text{deg}^2$ is constructed by pasting together these snapshots.

For the purposes of these tests, we created mock source catalogues by Monte-Carlo sampling sources from the density field with bias $b = 1$, and lens catalogues by randomly sampling dark matter haloes with a fixed mass. For each $60 \, \text{deg}^2$ simulation we generated a uniform distribution of lens galaxies within the range $0.1 < z < 0.9$, adopting a density of 1000 sources deg$^{-2}$. In addition, we distribute the source galaxies using a single Gaussian with mean 0.5 and standard deviation 0.1. We measured the angular auto-correlation function of the sources and of the lenses in $16 \, \Delta z = 0.05$ redshift bins, and the source-lens angular cross-correlation in the same redshift bins. The correlation functions are all measured using 30 equally logarithmically-spaced angular bins between 0.01 and 1 deg. The input dispersion of the source galaxies models the scatter of galaxies within a single tomographic bin: we are assuming the photometric sample has been divided into tomographic bins using photometric redshifts.

We note these mock catalogues significantly expand on the work presented by McQ+13. As such, they allow a much more significant test of the quadratic estimator methodology. Specifically, McQ+13 generate angular power spectra measurements using Gaussian random fields. This treatment neglects non-linear evolution and
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the redshift evolution of large-scale clustering, both of which are included in the simulations we adopt. Moreover, their tests were performed in harmonic space, while current measurements are given in configuration space. We perform all tests in configuration space. This is an improvement because transforming the estimator from harmonic to configuration space is non-trivial for real data, as discussed in section 4.4.3.

4.5.1 Auto- and Cross-Correlation Measurements

To measure the angular auto- and cross-correlation functions we use the Landy-Szalay estimator (Landy & Szalay, 1993). The cross-correlation between samples $i$ and $j$ is

$$w_{i,j}(\theta) = \frac{(D_i D_j)_\theta N_{R,i} N_{R,j}}{(R_i R_j)_\theta N_i N_j} - \frac{(D_i R_j)_\theta N_{R,j}}{(R_i R_j)_\theta N_i} - \frac{(D_j R_i)_\theta N_{R,i}}{(R_i R_j)_\theta N_j} + 1.$$ (4.25)

$(D_i D_j)_\theta$ is the number count of pairs between the two data samples as a function of the angular separation $\theta$. Similarly, $(D_i R_j)_\theta$ is the number count of pairs between the data and random sample. Note, the random sample is constructed to match the selection function and geometry of the corresponding data sample, $R_j$ matches $D_j$. $(R_i R_j)_\theta$ is the pair count between the random samples. Finally, the number counts of the data sample $i$, and the random sample $j$ are $N_{D,i}$ and $N_{R,j}$. For our calculations we use a random catalogue that is 10 times larger than the data catalogue.

4.6 Results

In this section we present the results of applying our quadratic estimator to the mock catalogues. We will present results in terms of probability distribution functions, $P_i^{(p)}(z)$, rather than the number counts, $\tilde{N}_i^{(p)}$, as angular correlation functions are directly sensitive to the former.
4.6. RESULTS

For each mock catalogue we measure the following statistics: The auto-correlation of the source galaxies, \(w_{pp}(\theta)\); 16 lens auto-correlations, \(w^{s_i,s_i}(\theta)\); and, 16 source-lens cross-correlations, \(w^{ps_i}(\theta)\), where the index \(i\) runs across the 16 redshift bins of the spectroscopic sample. However, for these tests we will concentrate on the redshift distributions we can infer using only the cross-correlation statistics. Thus, we apply equation (4.24) but neglect the auto-correlation terms\(^7\). This assumption significantly simplifies the analysis, and does not degrade the constraints significantly: the auto-correlation measurements probe the same degeneracy direction as the cross-correlation measurements.

In figure (4.2) we present the analysis of a single mock catalogue (henceforth, mock 1). For this test, we adopt a mock with an underlying Gaussian redshift distribution, as is evident from the amplitude of the angular cross-correlation peaking at \(z \sim 0.5\). In this figure we plot the angular cross-correlation as a function of separation angle \(\theta\), and as a function of the spectroscopic redshift bin, of which there are 16. The blue points show the cross-correlation functions measured from a single mock catalogue. The errors we plot are derived using Jackknife re-sampling. Hence, as the mocks are only 60 deg, we expect the size of the errors to be underestimated; in particular, we expect the cosmic variance component to be underestimated. Note, the quadratic estimation algorithm does not use these error estimates, therefore, this limitation has no effect on our analysis.

In addition, in figure (4.2) we over-plot the theory predictions for the angular cross-correlations \(w_{ps_i}(\theta)\) (black line). The theory predictions are derived via Eq (4.10). To estimate the reliability of this model, we compute the average of the simulation predictions (dashed-green line). From this measurement, one can observe that, for a number of redshift bins, the simulation measurements differ significantly from the average predictions. This is because, as stated above, we expect the errors to be underestimated.

The red-dashed lines in figure (4.2) show the reconstructed angular cross-correlation functions. To expand, we apply the quadratic estimator to mock 1 and infer a measurement of the photo-z distribution, \(\hat{P}^{(p)}(z_i)\). The reconstructed angular cross-

\(^7\)Note, in this limit the estimator remains unbiased. This is because the auto-correlation terms cancel each other out.
correlations are then compute as $\hat{w}_{ps}(\theta) = b_i^{(s)} b_i^{(p)} P^{(s)}(z_i) P^{(p)}(z_i) w_{ps}(\theta)$. As a test of the convergence of the estimator, in fig. (4.2) we observe that the reconstructed predictions accurately match the simulation measurements. Thus, the combination $\sim \{ \hat{w}_{ps}(\theta_\alpha) - w_{ps}(\theta_\alpha) \}$ is being minimised by the estimator. One should keep in mind the weights being applied to this minimization, as shown by figure (4.1). This result is representative of all the mocks. This is a useful test of the estimator as it is less sensitive to inaccuracies in modelling the correlation functions, relative to plotting $P^{(s)}(z_i)$.

To further examine the differences between the model predictions and the simulation measurements, in figure (4.3) and (4.4) we plot the average simulation predictions (blue points) against our model predictions. In figure (4.3) the errors are equivalent to that expected for a single mock catalogue, while for figure (4.4) we divide the errors by $\sqrt{N}$, where $N = 20$ is the number of mock catalogues. From figure (4.3) we conclude that our modelling of the angular cross-correlation function is sufficient to analyse the results from a single mock catalogue.

However, in figure (4.4) we observe significant deviations from the theory predictions. This occurs for thee reasons. First, we observe that on scales below $\theta = 2$ [arcmins] our model over-predicts the amplitude of the angular cross-correlations by as much as $\sim 30\%$. This is an artefact that results from the construction of the mock catalogues, where the sources are sampled from the projected density field in each snapshot rather than the full halo catalogue. When truncating the fit to only scales $\theta > 2$ [arcmin] we observe only a small change in the results. Therefore, we expect this bias will not significantly influence our results.

Second, the model assumes that the galaxy biases all equal one, for both samples. However, we observe that the bias of the lenses—which we select by populating a fixed HOD at all redshifts—increases with redshift. This is expected because we have selected galaxies at a fixed halo mass. To account for this we will give constraints on the parameter combination $b(z) \times P(z)$ and model the galaxy bias evolution as $b(z) = 1 + \alpha \times (z - z_0)$, that is, as opposed to only measuring $P(z)$, the input Gaussian distribution. For this model $\alpha$ and $z_0$ are free parameters, which we fit

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8To consider an example, in the situation where the model over-predicts the amplitude of the correlation function, the estimator will respond by reducing $P^{(s)}(z_i)$ relative to the true value. However, in this case, $\hat{w}_{ps}$ should still agree with the simulation prediction.
4.6. RESULTS

Figure 4.2: The measured angular cross-correlation functions for a single mock and for all spectroscopic bins (blue points). The theory predictions for the angular cross-correlation based on the true $P(z)$ (black line), which is a Gaussian with mean 0.5 and standard deviation 0.1. Note, the model predictions assume that all bias values are 1, i.e., $b_i^{(p)} = b_i^{(s)} = 1$. The average of the simulation measurements (dashed-green line). The angular cross-correlation predictions based on the inferred $\hat{P}(z)$ (dashed-red line).
for using the results for \( b(z) \times P(z) \). Third, as outlined in section 4.3, we expect some bias to be introduced because we are approximating the redshift distribution as a step-wise expansion. Currently, for this analysis we are unable to account for this effect. In future work will aim to improve this modelling. We will now try to understand whether such biases are significant for current surveys.

We now apply the quadratic estimator to all 20 mocks catalogues. We present a subset of these results in figure (4.5). Here we plot the measured redshift probability distribution functions for 9 mock catalogues (blue points). In addition, we plot \( P(z) \times b(z) \) for two different bias evolution models, where \( P(z) \) is the input Gaussian distribution. First, a constant bias \( b = 1 \) model (black line), and second, a redshift dependent bias \( b(z) = 1 + \alpha \times (z - z_0) \) (green line), with \( \alpha = 1.5 \) and \( z_0 = 0.4 \). The parameters describing the bias evolution are found by fitting to the measured \( \hat{P}(z) \times b(z) \) distributions. Therefore, the green line represents our model predictions after galaxy bias evolution is taken into account.

From figure (4.5) we observe that the reconstructed distributions are statistically consistent with the model predictions, i.e., all of the measurements are within 2-\( \sigma \). We emphasize that similar statistical error levels will be obtained for current generation spectroscopic surveys, which overlap with lensing surveys\(^9\). However, above a redshift of 0.5, we observe a preference for a high amplitude of \( \hat{P}(z_i) \). This preference is made more apparent by averaging over the mocks, as shown in figure (4.5). Here the blue points are the average of the reconstructed measurements. In addition, we plot the theory predictions for \( P(z) \times b(z) \) using three separate bias evolution models. Specifically, we plot a constant bias model with \( b(z) = 1 \) (orange), \( \alpha = 1.5 \) and \( z_0 = 0.4 \) (green), and \( \alpha = 1.3 \) and \( z_0 = 0.35 \) (black). We plot the black distribution to give a second example of a redshift-dependent bias model. From these measurements we conclude the following. First, below \( z = 0.5 \) the measured \( \hat{P}(z_i) \) values are consistent with the input distribution at the 2\( \sigma \) level, with the exception of the bin at \( z = 0.475 \). Alternatively, above \( z = 0.5 \), there are a number of deviations from input distribution at a level greater than 2\( \sigma \). We interpret these discrepancies as a limitation of our modelling of the correlation function. Specifically, we expect

\(^9\)The mocks contain \( \sim 60,000 \) galaxies each, while the largest existing spectroscopic survey, which overlaps with a lensing survey, contains \( \sim 40,000 \) galaxies.
Figure 4.3: The angular cross-correlation function predicted by theory using the fiducial parameters of the SLICS simulations, and fixing the lens and sources bias values to 1 (black line). To reduce cosmic variance, the positions of the blue points are found by averaging over all 20 mock catalogues. However, the corresponding error bars are taken from a single mock catalogue. The purpose of this plot is to compare our model angular correlation functions with the predictions from our mocks, at the level of statistical error consistent with a single mock catalogue.
Figure 4.4: The angular cross-correlation function predicted by theory (black line). For this example all bias values are assumed to be one. The angular cross-correlation function found by averaging over 20 mock catalogues (blue points). Here the errors plotted are those found by dividing the error on a single mock catalogue by $\sqrt{N}$, where $N = 20$ is the number of mock catalogues.
that the approximation of a step-wise probability distribution is breaking down, and that our model for the redshift evolution of the bias, for the lens galaxies, is not sufficient.

4.7 Conclusion and Discussion

Motivated by the potential of future weak lensing experiments to further our understanding of dark energy, we have presented an investigation into one of the key systematic uncertainties in weak lensing science: the calibration of galaxy redshift distributions. In particular, we have focused on calibration via cross-correlation.

Expanding on the analysis presented by McQuinn & White (2013), we derive a new quadratic estimator that determines the redshift probability distribution of a photometric galaxy sample. To determine the unknown redshift distribution, this estimator uses cross-correlation measurements with an overlapping spectroscopic survey. We show that the quadratic estimator presented by McQuinn & White (2013) requires a number of corrections to ensure it is unbiased, which we outline.

We test how accurately the estimator can measure the underlying redshift distribution using mock catalogues constructed from the Scinet Light Cone Simulations (SLICS). From these simulations we generate mock overlapping spectroscopic and photometric samples. For the spectroscopic samples we adopt a uniform redshift distribution over the range $0.1 < z < 0.9$, and use a density of 1000 sources deg$^{-2}$. For the photometric samples we adopt a single Gaussian distribution with mean 0.5 and standard deviation 0.1. We then measure the angular cross-correlations between these two galaxy samples, and use them as inputs to the quadratic estimator.

From the analysis of the mock catalogues we conclude the following. The reconstructed redshift distributions, for individual mocks, are consistent with the input redshift distribution at a level of 2σ, once we have accounted for the bias evolution of the lens sample. Moreover, we expect this level of statistical error to be representative of existing spectroscopic surveys that overlap with photometric surveys. Reducing the statistical error by averaging over the results from the mock catalogues, we find that above a redshift of 0.5 there are statistically significant
Figure 4.5: The reconstructed redshift probability distribution function times a redshift dependent bias, for 9 random mock catalogues (blue points). These measurements are derived using a quadratic estimator using only the cross-correlation measurements. The plotted errors are derived from the relevant Fisher matrix. The input redshift probability distribution times a constant bias (black line), where $b(z) = 1$. The input redshift probability distribution times a redshift dependent bias (green line), where $b(z) = 1 + \alpha \times (z - z_0)$, with $\alpha = 1.5$ and $z_0 = 0.4$. We expect the latter bias evolution model to more accurately describe the bias evolution present in the mocks.
Figure 4.6: The average, over 20 mock catalogues, of the reconstructed redshift probability distribution functions (blue points). We plot three different bias evolution models times the input redshift probability distribution. We model the redshift dependent bias evolution via $b(z) = 1 + \alpha \times (z - z_0)$, with $\alpha$ and $z_0$ as free parameters. Adopting a constant bias model, where $b(z) = 1$ (orange). Adopting a redshift dependent bias model with $\alpha = 1.5$ and $z_0 = 0.4$ (green). Adopting a redshift dependent bias model with $\alpha = 1.3$ and $z_0 = 0.35$ (black). We note the green line represents our best fit bias model.
deviations from the underlying distribution (i.e., greater than 2\sigma). We interpret these deviations as the modelling of the correlation function breaking down and introducing a systematic bias.

Looking towards the future—in particular, to the application to the Kilo-degree lensing imaging survey—we hope to advance this methodology in the following ways. Firstly, the modelling of the angular correlation functions could be improved by adopting a more sophisticated model, or interpolating measurements from very large and high number density simulations. Secondly, the mock catalogues could be improved by using realistic redshift distributions for the spectroscopic and photometric samples, and incorporating a realistic galaxy bias evolution for the photometric sample. To conclude, calibration via cross-correlation is a very promising tool to measure galaxy redshift distributions. However, further work is required in order to make this approach competitive with more traditional direct calibration methods.
In this thesis we have attempted to illuminate the physics behind Dark Energy. In this pursuit we have concentrated on three separate directions, we briefly outline the main results of each below.

In Chapter 2 we analysed the velocity sample of the 6-degree Field Galaxy Survey. Using 2-point statistics of the velocity field, as traced by 6dFGSv, we measured the scale-dependent growth rate of structure at $z = 0$ by fitting the velocity distribution using a maximum-likelihood method. Specifically, we measured

$$f\sigma_8(k_i) = [0.79^{+0.21}_{-0.29}, 0.30^{+0.14}_{-0.19}, 0.32^{+0.19}_{-0.15}, 0.64^{+0.17}_{-0.16}, 0.48^{+0.22}_{-0.21}],$$

for the wavenumber ranges $k_1 \equiv [0.005, 0.02]$, $k_2 \equiv [0.02, 0.05]$, $k_3 \equiv [0.05, 0.08]$, $k_4 \equiv [0.08, 0.12]$ and $k_5 \equiv [0.12, 0.150]$. We emphasize that this method allows us to constrain the growth rate of structure on the largest-scale currently accessible by galaxy surveys. This is due to the sensitivity of peculiar velocities to large-scale modes. In particular, for length scales greater than $\sim 300h^{-1}\text{Mpc}$ ($k < 0.02h\text{Mpc}^{-1}$) we constrain the growth rate to $\sim 30\%$. From these measurements we find no evidence for scale-dependence in the growth rate of structure.

Combining the above growth rate measurements, we determine the scale-independent growth rate to $\sim 15\%$, finding $f\sigma_8(z = 0) = 0.418 \pm 0.065$. This result is consistent with the Planck prediction of $f\sigma_8(z = 0) = 0.443$. Furthermore, we emphasize that this measurement is independent of galaxy bias. And that our result is con-
consistent with the redshift-space distortion analysis of 6dFGS presented by (Beutler et al., 2012), which is dependent on galaxy bias. The consistency between these measurements gives us confidence in the current modeling of galaxy bias.

We improve the treatment of systematics in peculiar velocity analysis by illustrating that for velocities derived using the Fundamental Plane or the Tully-Fisher relation the intrinsic scatter is not Gaussian for the PV. Moreover, we show that assuming the error is Gaussian can significantly bias cosmological constraints. We suggest a combination of the data to use that has Gaussian errors, and hence avoids this source of systematic error. In addition, we improve upon the existing fitting methodology by marginalizing over the unknown magnitude zero-point of the fundamental plane relation, and through non-linear modelling of the velocity power spectrum.

In Chapter 3 we measured time- and scale-dependent deviations to the gravitational field equations of General Relativity. We model these deviations using the time and scale-dependent parameters \( \{G_{\text{matter}}, G_{\text{light}}\} \): each parameter is defined using 2 bins in time and 2 bins in scale.

To constrain these parameters, plus the standard model parameters, we utilize a range of cosmological probes. Two probes of particular importance in measuring \( G_{\text{matter}} \) are the power spectrum multipoles from the WiggleZ and CMASS galaxy redshift samples, and the velocity power spectrum from 6dFGSv.

In terms of 68% CLs, for \( G_{\text{matter}} \) we measure \( G_{\text{matter}}(z < 1; k > 0.01) = 0.65 \pm 0.43, G_{\text{matter}}(z < 1; k < 0.01) = 1.22^{+0.39}_{-0.34}, G_{\text{matter}}(z > 1; k > 0.01) = 0.53 \pm 0.32, G_{\text{matter}}(z > 1; k < 0.01) = 0.87 \pm 0.30 \). These constraints are consistent with GR, where \( G_{\text{matter}} = 1 \), at the 95% confidence level. We observe a small tension (> 1σ) for the small-scale and high-redshift bin. For \( G_{\text{light}} \) we measure \( G_{\text{light}}(z > 1; k > 0.01) = 1.057^{+0.053}_{-0.045}, G_{\text{light}}(z > 1; k < 0.01) = 1.048 \pm 0.048, G_{\text{light}}(z < 1; k > 0.01) = 1.153^{+0.089}_{-0.068}, G_{\text{light}}(z > 1; k < 0.01) = 1.016 \pm 0.026 \). Similarly to above, these measurements are consistent with General Relativity at the 95% confidence level.

We preform Bayesian model comparison to further understand whether the data prefers the introduction of the parameters \( \{G_{\text{matter}}, G_{\text{light}}\} \). Following the scale suggested by Kass & Raftery (1995), the evidence calculations suggests “no support”
for modifying general relativity. This is consistent with the results from the posterior distributions.

We consider a second phenomenological model where deviations to the growth rate are modeled by the growth index $\gamma$, and deviations to the expansion history are modeled by two parameters describing a redshift-dependent equation of state $(w_0, w_a)$. Fixing a $\Lambda$CDM expansion history (in terms of 68% CL) we measure $\gamma = 0.665 \pm 0.067$, while allowing the expansion history to deviate from $\Lambda$CDM we measure $\gamma = 0.69^{+0.09}_{-0.11}$. Both these results are consistent with the standard model. Furthermore, they improve on recent estimates by including extra datasets, and by correctly accounting for the Alcock-Paczynski effect.

To summarise, in Chapters 2 and 3 we have testing gravitational physics using new cosmological probes that are sensitive to a range of length scales, as a result we have improved current constraints on GR.

In Chapter 4, expanding upon the methodology introduced by McQuinn & White (2013), we derive a quadratic estimator that measures the redshift probability distribution of a photometric galaxy sample using the cross-correlation with an overlapping spectroscopic survey.

We test the quadratic estimator using 20 mock galaxy catalogues, which we construct from the Scinet LiLight Cone Simulations (SLICS). From each simulation we generate mock spectroscopic and photometric samples by Monte-Carlo sampling sources from the density field and halo catalogues. For the redshift distributions of the samples we consider the simple case where the spectroscopic sample is uniformly distributed over the range $0.1 < z < 0.9$, and the photometric sample distributed as a Gaussian with mean $z = 0.5$ and standard deviation 0.1.

From the mock catalogues we find that, at the level of statistical errors present in individual mocks, the recovered redshift probability distributions times the bias agrees with the model $P(z) \times b(z)$ distribution, at a level of 2-$\sigma$. We then reduce the statistical error by averaging over all the measurements from the mocks. In this case, we find that above a redshift of 0.5 the recovered distribution deviates from the underlying Gaussian distribution at a level greater than 2-$\sigma$. We expect this discrepancy is caused by the step-wise approximation of the redshift probability distribution.
To summarise, in Chapter 4 we have developed and tested a new technique that can be applied to existing lensing tomographic datasets to test a key systematic error present in weak lensing.

\section*{5.1 Future Applications}

Progress in science is never immediate, nor is its direction transparent. Accordingly, the research we describe in this thesis has been influenced by considering not only the immediate impact, but also the potential for future applications. In this section we outline the latter. In particular, we focus on three surveys—the low-redshift peculiar velocity and galaxy redshift survey TAIPAN, the wide-field spectroscopic survey 2dFLenS, and the weak lensing survey KiDS—and describe the applicability of our work.

In Chapters 2 and 3 we improved upon the methodology to analyse peculiar velocity surveys. The subsequent analysis of the 6dFGSv survey allowed us to infer information about the local growth rate of structure, and to test for deviations from General Relativity. These methods can be readily applied to upcoming peculiar velocity surveys. Furthermore, as statistical errors reduce with larger sample sizes, the systematic error corrections we have outlined will become increasingly important.

The largest upcoming velocity survey is the Taipan Fundamental Plane survey \cite{Colless2013}. Beginning in 2016, using the upgraded UK Schmidt Telescope, Taipan aims to measure spectroscopic redshifts for \( \sim 5 \times 10^5 \) galaxies over most of the southern hemisphere. Furthermore, peculiar velocities will be derived for a subset (\( \sim 5 \times 10^4 \)) of these galaxies using the Fundamental Plane relation. Following a factor of \( \sim 5 \) increase in sample size (relative to 6dFGSv), we expect a significant improvements in constraints, relative to those we have presented; moreover, we expect much more complicated analysis to be possible, which build upon the work we have presented—e.g., a self-consistent analyses of the 3D density and velocity fields.

On a more speculative note, the Maunakea Spectroscopic Explorer (a 10-meter
class telescope) has been proposed to replace the CFHT telescope\(^1\). A number of science proposals have accompanied this proposal, of particular interest is the peculiar velocity survey, MSEv. This survey, while hypothetical, would represent a significant step forward from the TAIPAN survey. In particular, the survey aims to obtain galaxy redshifts out to \(z \sim 0.25\) for 75% of the sky. Forecasts suggest this survey would result in peculiar velocity measurements for \(\sim 100,000\) galaxies. Following this significant reduction in statistical errors, much more subtle features will be observables in the velocity power spectrum (such as oscillations and the turnover) and, when combined with the density field, scale-dependent velocity bias will be measurable.

In Chapter 4 we outline a method to use cross-correlation measurements to infer redshift probability distributions. An application of this method would require a gravitational lensing survey, and an overlapping spectroscopic redshift survey. Fortunately, two such surveys exist: the 2-degree Field Lensing Survey (Blake et al., 2016) and the The Kilo-Degree Survey (KiDS) (de Jong et al., 2015). We outline each survey below.

The 2dFLenS survey was conducted on the Anglo-Australian Telescope over three semesters from 2014 to 2015. After 50 nights of observations the redshifts of \(\sim 50,000\) LRGs and \(\sim 30,000\) bright low-redshift galaxies were measured, where the LRG targets span the redshift range \(z = 0.1\) to \(z = 0.9\). The principal aim of 2dFLenS is to expand on the area of overlap between spectroscopic galaxy surveys and photometric lensing surveys. This facilitates two key science goals. Firstly, it allows a joint analysis of lensing and galaxy redshift samples including all cross-correlation statistics (e.g., Gaztañaaga et al., 2012; Cai & Bernstein, 2012). Secondly, it allows the calibration of photometric redshift distributions using cross-correlation techniques.

The Kilo-Degree Survey (KiDS) is a multi-band imaging survey designed for weak gravitational lensing analysis (de Jong et al., 2015). This survey is being carried out on the 2.6 meter VLT Survey Telescope, where, using the 300-mega-pixel wide-field camera OmegaCAM, images are taken with the four filters \(ugri\). KiDS aims to image \(\sim 1500\) square degrees of the sky down to a limiting \(r\)-band

\(^1\)http://mse.cfht.hawaii.edu/project/
The magnitude of $\sim 25^2$. The first and second data release of KiDS is presented by de Jong et al. (2015). The subsequent lensing analysis of this data release was undertaken by Kuijken et al. (2015).

The 2dFLenS and KiDS surveys are ideal for applying the methodology we have developed for the following reasons. By construction, the overlap between 2dFLenS and KiDS is $\sim 700$ square degrees. 2dFLenS is currently the largest spectroscopic sample that overlaps with a lensing survey. Finally, KiDS is a state-of-the-art weak lensing survey.

The combination of 2dFLenS and KiDS currently represents the most optimal (in terms of overlap) application of our quadratic estimator; however, within the next 10 year there will be a significant increase in depth and sky coverage of gravitational lensing surveys, of particular relevance are the following: the HyperSuprime cam Survey\(^3\), the Dark Energy Survey\(^4\), and Euclid\(^5\). Spectroscopic surveys will also significantly increase in size and overlapping area with photo-z surveys, e.g., the eBOSS survey will have a considerable overlap with the Dark Energy Survey\(^6\). To conclude, the data available for the application of cross-correlation techniques is still in its infancy and in the years to come there will be a significant increase in the constraining power of such methods.

\(^2\)The \textit{r}-band images are used for galaxy shape measurements and hence are slightly deeper than the remaining bands: For the remaining bands the limiting magnitudes are $\sim 24, 25, 23$, for \textit{u}, \textit{g}, \textit{i} respectively (Kuijken et al., 2015)

\(^3\)http://www.naoj.org/Projects/HSC/HSCProject.html

\(^4\)http://www.darkenergysurvey.org

\(^5\)http://sci.esa.int/euclid/42267-science

\(^6\)https://www.sdss3.org/future/eboss.php

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