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# Uniform approximation by polynomial splines of the highest defect: necessary and sufficient optimality conditions and their generalisations

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## Abstract

In this paper necessary and sufficient optimality conditions for uniform approximation of continuous and discrete functions by continuous piece-wise polynomial functions (polynomial splines) are obtained. In general, such polynomial splines are nonsmooth functions. This type of polynomial splines appears in applications, for example, in taxation problems. The obtained results are generalisations of the results, obtained for polynomial approximation (Chebyshev theorem) and polynomial spline approximation (Tarashnin theorem). The main result, obtained in this study, is a generalisation of Tarashnin's results to the case when the degree of the polynomials, which compose polynomial splines, can vary from one interval to another.

## 1 Introduction

Polynomial splines is an efficient tool for data and function approximation. In this paper polynomial splines are thought to be continuous piece-wise polynomial functions. Their derivatives may be discontinuous at the points where the polynomials are joined together. A simple representative of such splines is a continuous piece-wise linear function.

The problem of polynomial approximation has been studied extensively for several centuries. In 1715 B. Taylor published the power series formulas, named later the Taylor series. The series (namely, their truncated parts) can be used for approximate computations of function values.

Later polynomial approximations were studied by K. Weierstrass. The main results is Weierstrass approximation theorem. Also, a significant contribution to this area was made by Pafnuty Chebyshev (alternance theorem, see [19] for more details).

The main advantage of polynomials is their simplicity. One of the main disadvantages of polynomial approximation is that high degree polynomials are needed in order to reach a satisfactory accuracy level for polynomial approximations.

Another powerful approximation tool is polynomial splines (continuous piece-wise polynomial functions). This technique has been extensively studied, see [1], [6], [7], [8],

[10], [20], [21], [22] and others. Polynomial splines combine simplicity of polynomials and flexibility, which allows significant decrease of the degree of the corresponding polynomials and therefore polynomial splines are thought to be a powerful tool for function and data approximation. In this paper necessary and sufficient optimality conditions to a specific class of polynomial spline approximation are obtained.

The paper is constructed as follows. Section 2 introduces the problem, namely, it contains necessary definitions and presents results obtained earlier (Chebyshev, Rice and Tarashnin). Section 3 presents a problem of polynomial spline approximation as an optimisation problem, formulate and prove a new theorem, which is a generalisation of Tarashnin theorem. Several important corollaries from the generalisation of Tarashnin theorem are also presented in section 3. Section 4 presents conclusions and new research directions.

## 2 Definitions and preliminary results

### 2.1 Polynomial splines

There are several definitions for polynomial splines. In this paper polynomial splines are constructed in the following way: suppose that a polynomial spline  $Sp(t)$  is determined on an interval  $[a, b]$  (approximation interval), suppose also that this interval has been divided into  $n$  segments, such that

$$a = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_n = b \quad (2.1)$$

and the points  $\theta_i, i = 1, \dots, n - 1$  are fixed.

**Definition 2.1** *Polynomial spline (or spline) is a continuous piece-wise polynomial function, such that in each segment  $T_i = [\theta_{i-1}, \theta_i], i = 1, \dots, n$  it is represented by a polynomial of a degree less than or equal to  $m_i, i = 1, \dots, n$ . If  $m_1 = \dots = m_n = m$ , then  $m$  is the degree of the polynomial spline.*

**Definition 2.2** *The points  $\theta_i, i = 0, \dots, n$  are called the polynomial spline knots (or knots). The points  $\theta_i, i = 1, \dots, n - 1$  are called the internal knots.*

Most researchers work with smooth polynomial splines. In this paper splines are defined as continuous functions, which can be nonsmooth at their knots.

**Definition 2.3** *The difference between the degree of the spline and the order of the highest continuous derivative is called the defect of the spline.*

In this paper the research is concentrated on the highest defect splines, the splines with the defect equals to the degree of the spline.

Consider an example of polynomial spline constructing (see [18]).

$$S_m(A, t) = a_0 + \sum_{i=1}^n \sum_{j=m-d+1}^m a_{ij} (t - \theta_{i-1})_+^j, \quad (2.2)$$

where  $A = (a_0, a_{11}, \dots, a_{nm}) \in R^{mn+1}$  is the vector of spline parameters,  $m$  is the spline degree,  $d$  is the spline defect,  $\theta_i$  are the spline knots, such that  $i = 0, \dots, n$ ,

$$(\xi(x))_+ = \begin{cases} \xi(x), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

**Remark 2.1** Notice also that according to Haar theorem (see [5] and references within) in the case of polynomial approximation the best polynomial approximation is unique, since  $1, x, x^2, \dots, x^n$  is a Chebyshev system. In the case of polynomial spline approximation the best polynomial spline approximation is not necessarily unique, since  $(t - \theta_{i-1})_+^j, i = 1, \dots, n, j = 1, \dots, m$  is not a Chebyshev system.

In some cases it is necessary to specify the degree of the polynomials, which represent the spline in each segment more precisely.

**Definition 2.4** If in each segment  $[\theta_{i-1}, \theta_i]$  the spline is considered as a polynomial of the degree less than or equal to  $m_i$  then the generalised (vector) degree of the polynomial spline is the vector  $M = (m_1, m_2, \dots, m_n)$ .

A polynomial spline of the generalised degree  $M = (m_1, \dots, m_n)$  can be constructed as follows: in each segment  $T_i = [\theta_{i-1}, \theta_i]$  it is presented by the polynomial  $P_i(t)$ , such that

$$P_1(t) = \sum_{j=1}^{m_1} a_{1j} (t - \theta_0)^j + a_0, \quad P_i(t) = \sum_{j=1}^{m_i} a_{ij} (t - \theta_{i-1})^j + P_{i-1}(\theta_{i-1}), \quad i = 2, \dots, n. \quad (2.3)$$

**Definition 2.5** The vector

$$A = (a_0, a_{11}, a_{12}, \dots, a_{1m_1}, a_{21}, \dots, a_{2m_2}, \dots, a_{nm_n})$$

is called the vector of the parameters of the polynomial spline of the generalised degree  $M = (m_1, \dots, m_n)$

**Remark 2.2** If the degrees of the corresponding polynomials are different in different intervals it is quite difficult to give a strict definition to the spline defect. However, the splines constructed in (2.3) can be called the highest defect splines (continuous piece-wise polynomial functions, not necessarily smooth at any of the knots).

## 2.2 Necessary and sufficient optimality conditions (Chebyshev, Rice, Tarashnin)

In this paper necessary and sufficient optimality conditions for continuous function approximation and discrete function (dataset) approximation by polynomial splines with fixed knots under the uniform (Chebyshev) optimisation criterion are obtained.

Apart from their theoretical importance, the optimality conditions can be used for developing an algorithm of optimal spline constructing. Remez algorithm [10] plays an important role in the area of polynomial approximation. At each iteration of this algorithm necessary and sufficient optimality conditions for the case of polynomial approximation are verified. This method can be generalised to the case of polynomial splines (see [17]). In this generalisation the necessary and sufficient optimality conditions for polynomial spline approximation, obtained in [18], are also subject to verification at each iteration.

Necessary and sufficient optimality conditions for polynomial and polynomial spline approximation are based on the notion of alternating and alternance points.

**Definition 2.6** *A function  $g(t)$  alternates  $p$  times on a set  $[a, b]$  if there exist  $p + 1$  points  $t_i < t_{i+1} \in [a, b]$ , such that*

$$g(t_i) = -g(t_{i+1}) = \pm \max_{t \in [a, b]} |g(t)|.$$

**Definition 2.7** *Alternance points are the points where the absolute value of the deviation is maximal and the sign of the deviation at any two consequent point is opposite.*

Necessary and sufficient optimality conditions in the case of polynomials have been obtained by Chebyshev (see [19]). Later they were generalised to some particular types of polynomial splines:

- polynomial splines of the degree  $m$  and the defect 1 (Rice, see [22] for details);
- polynomial splines of the degree  $m$  and the highest possible defect  $m$  (Tarashnin, see [18]).

**Theorem 2.1** *(Chebyshev) Necessary and sufficient optimality conditions for a polynomial of degree  $m$  (Chebyshev approximation problem of a function  $f(t)$  on the interval  $[a, b]$ ) are there exist  $m + 2$  alternance points.*

In 1967 this theorem was generalised to the case of polynomial splines of the defect equals 1 (Rice, see [22], the characterization theorem).

**Theorem 2.2** *(Rice) Let  $f(t)$  be continuous on  $[a, b]$  and let  $S(A, t)$  be a polynomial spline of the defect equals 1 of degree  $m$ . Then necessary and sufficient conditions that  $S(A^*, t)$  be a Chebyshev approximation to  $f(t)$  is that  $f(t) - S(A^*, t)$  alternates at least  $m + p + 1$  times in some interval  $[\theta_{i-1}, \theta_{i+p}]$ , which means that there exist  $m + p + 2$  alternance points.*

In 1996 M. Tarashnin proved a corresponding theorem to the case of the highest defect splines, namely, to the splines of the degree equals  $m$  and the defect equals  $m$  (see [18] for details). In his research Tarashnin approximated continuous functions  $f(t)$  and discrete functions (datasets, presented as collections of pairs  $\{(t_i, f(t_i))\}_{i=1}^N$ , such that  $t_i \in [a, b], i = 1, \dots, N$ ) by polynomial splines. The theorem is as follows.

**Theorem 2.3** (Tarashnin) *Necessary and sufficient optimality conditions for the spline  $S(A^*, t)$  of the degree  $m$  and the defect  $m$  are as follows:*

1. *in one of the sub-intervals  $[\theta_{i-1}, \theta_i]$  there exist at least  $m + 2$  alternance points  $t_1, \dots, t_{m+2}$ , which are points of maximal deviation of the spline  $S(A^*, t)$  from  $f(t)$ , such that*

$$\begin{aligned} F(A^*, t_i) = f(t_i) - S(A^*, t_i) = -F(A^*, t_{i+1}) = -(f(t_{i+1}) - S(A^*, t_{i+1})) = \\ = \pm \max_{t \in [a, b]} |f(t) - S(A^*, t)| \end{aligned}$$

or

2. *there exist sub-intervals  $[\theta_{i-1}, \theta_i]$  and  $(\theta_{j-1}, \theta_j]$ , where  $1 \leq i < j \leq n$ , such that the maximal deviation points are distributed as follows:*
  - *there exist at least  $m + 1$  alternance points in the  $i$ -th interval,*
  - *there exist at least  $m + 1$  alternance points in the  $j$ -th and*
  - *there exist at least  $m$  alternance points in the  $k$ -th interval ( $i < k < j$ )*

**Definition 2.8** *A minimal length chain of sub-intervals, where the conditions of Tarashnin theorem are satisfied is called a minimal chain.*

The length of the minimal chain is 1 if the first condition of Tarashnin theorem is satisfied or  $i - j$  if the second condition is satisfied.

Recall that polynomial splines of the highest defect are generally nonsmooth functions. It can be argued that it is better to approximate functions by smooth functions. However, in some applications it is necessary to create a nonsmooth approximation to some modelling functions, for example, taxation problems (see [4]). In these problems a continuous function (modelling function) has to be approximated by a continuous piece-wise linear function (which is a polynomial spline of the highest defect) or a piece-wise fraction-linear function. The nonsmooth approximations represent corresponding taxation tables. The goal is to find an approximation which “fits best” to the corresponding modelling function. Therefore, there are some practical needs for constructing nonsmooth approximations to some modelling functions.

The goal of this paper is to generalise necessary and sufficient optimality conditions for polynomial spline approximation, obtained by Tarashnin, to generalised (vector) degree polynomial splines. This generalisation is presented in the next section.

### 3 A generalisation of Tarashnin theorem

This section contains a generalisation of Tarashnin theorem and some important corollaries to this generalisation.

Suppose that a continuous function  $f(t)$  is approximated in the interval  $[a, b]$ , which has been divided into  $n$  sub-intervals, by a fixed knots polynomial spline of the generalised (vector) degree  $M = (m_1, \dots, m_n)$ . The polynomial spline  $S(A, t)$ , used in the approximation, is a polynomial spline with the knots  $\Theta = (\theta_0, \dots, \theta_n) \in R^{n+1}$ , such that  $a = \theta_0$ , and  $b = \theta_n$  and the vector of spline parameters  $A = (a_0, a_{11}, \dots, a_{nm_n}) \in R^{\gamma+1}$ , where  $\gamma = \sum_{k=1}^n m_k$ . Then the optimisation problem for finding an optimal polynomial spline is as follows

$$\text{Minimise } \max_{t \in [a, b]} |f(t) - S(A, t)| \quad \text{subject to } A \in R^{\gamma+1}. \quad (3.1)$$

Problem (3.1) is a Chebyshev approximation problem in the case of polynomial spline approximation. If  $\Theta = (a, b)$  in (3.1) then the polynomial spline is simply a polynomial and problem (3.1) is a Chebyshev approximation problem in the case of polynomials.

A polynomial spline of the generalised (vector) degree  $M = (m_1, \dots, m_n)$  can be presented as follows

$$S_M(A, t) = a_0 + \sum_{i=1}^n \sum_{j=1}^{m_i} a_{ij} (\min\{t, \theta_i\} - \theta_{i-1})_+^j, \quad (3.2)$$

where  $A = (a_0, a_{11}, \dots, a_{nm_n}) \in R^{\gamma+1}$  is the vector of spline parameters,  $\gamma = \sum_{i=1}^n m_i$ ,  $\theta_i$  are the spline knots ( $i = 0, \dots, n$ ,  $a = \theta_0, b = \theta_n$ ),

$$(\xi(x))_+ = \begin{cases} \xi(x), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

This presentation for polynomial splines is similar to the one used in [18]. However, there are two main advantages for the presentation (3.2). First of all, it allows the construction of polynomial splines with different degree polynomials in different intervals, which is not strait forward in the case of (2.2) if, for example,  $m_i > m_{i+1}$ . Second, the components of the vector of the spline parameters are the coefficients for the corresponding polynomials which represent the spline, namely

$a_0, a_{11}, \dots, a_{1m_1}$  are the parameters of the polynomial  $P_1(t)$  from (2.3);

$P_i(\theta_{i-1}), a_{i1}, \dots, a_{2i_2}$  are the parameters of the polynomial  $P_i(t)$  from (2.3),  $i = 2, \dots, n$ .

The following theorem holds.

**Theorem 3.1** *Necessary and sufficient optimality conditions for the spline  $S(A^*, t)$  of the generalised (vector) degree  $M = (m_1, \dots, m_n)$  and the highest defect are as follows:*

1. in one of the sub-intervals  $[\theta_{i-1}, \theta_i]$  there exist at least  $m_i + 2$  alternance points  $t_1, \dots, t_{m_i+2}$ , which are points of maximal deviation of the spline  $S(A^*, t)$  from  $f(t)$ , such that

$$\begin{aligned} F(A^*, t_k) &= f(t_k) - S(A^*, t_k) = -F(A^*, t_{k+1}) = -(f(t_{k+1}) - S(A^*, t_{k+1})) = \\ &= \pm \max_{t \in [a, b]} |f(t) - S(A^*, t)| \end{aligned}$$

or

2. there exist sub-intervals  $[\theta_{i-1}, \theta_i]$  and  $(\theta_{j-1}, \theta_j]$ , such that  $1 \leq i < j \leq n$  and on the chain

$$[\theta_{i-1}, \theta_i], (\theta_i, \theta_{i+1}], \dots, (\theta_{j-1}, \theta_j]$$

the maximal deviation points are distributed as follows:

- there exist at least  $m_i + 1$  alternance points in the  $i$ -th interval,
- there exist at least  $m_j + 1$  alternance points in the  $j$ -th and
- there exist at least  $m_k$  alternance points in the  $k$ -th intervals ( $i < k < j$ )

**Proof.** The structure of the proof is similar to the one, proposed in [18]. However some more accurate techniques are needed in order to vary the degree of the corresponding polynomials in different intervals. Similar to Tarashnin theorem, the minimal chain is the chain of subintervals of the minimal length, where the first or the second condition the current theorem is satisfied.

Assume that the minimal chain is known. Suppose for simplicity that it consists of  $n$  intervals. Further in the proof, the first interval is the first interval of the minimal chain, the second interval is the second interval of the minimal chain, the  $n$ -th interval is the  $n$ -th interval of the minimal chain (which is also the last interval of the minimal chain).

Necessity.

Consider the following problem

$$\max_{t \in [a, b]} |F(A, t)| \rightarrow \min_{A \in R^{\gamma+1}}, \quad (3.3)$$

where  $F(A, t) = f(t) - S_M(A, t)$ ,

$f(t)$  is the function to approximate,

$S_M(A, t)$  is the spline of the generalised (vector) degree  $M = (m_1, \dots, m_n)$ ,

$$\gamma = \sum_{i=1}^n m_i, \quad A \in R^{\gamma+1}.$$

Consider an equivalent problem

$$\max_{t \in [a, b]} \frac{1}{2} (F(A, t))^2 \rightarrow \min_{A \in R^{\gamma+1}}. \quad (3.4)$$



Use the following notation

$$\varphi(A) = \max_{t \in [a,b]} \frac{1}{2} (F(A, t))^2, \quad (3.5)$$

The function  $\varphi(A)$  is convex on  $R^{\gamma+1}$ . Fix the vector  $A^* = (a_0^*, a_{11}^*, \dots, a_{nm_n}^*)$ . Suppose that a maximum with respect to  $t$  of the function  $\frac{1}{2}(F(A, t))^2$  is reached at the points  $t_{l\tau(l)}, l = 1, \dots, n$ . The index  $l$  corresponds to the segment number. For maximal deviation points the following notation is used  $I_l$  is the set of indexes of all maximal deviation points of the  $l$ -th interval, therefore  $\tau(l) \in I_l$ .

The following notation is used

$$F(A^*, t_{l\tau(l)}) = F(t_{l\tau(l)}), \quad |F(t_{l\tau(l)})| = F, \quad \tau(l) \in I_l.$$

The subdifferential of the function  $\varphi(A)$  can be constructed as follows

$$\underline{\partial}\varphi(A) = -\text{co} \left\{ \begin{array}{l} F \text{sign}(F(t_{1\tau(1)})) \\ \vdots \\ F \text{sign}(F(t_{2\tau(2)})) \end{array} \right. , \quad \left. \begin{array}{l} \begin{pmatrix} 1 \\ (t_{1\tau(1)} - \theta_0) \\ (t_{1\tau(1)} - \theta_0)^2 \\ \vdots \\ (t_{1\tau(1)} - \theta_0)^{m_1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \tau(1) \in I_1 \end{array} \right. , \quad \left. \begin{array}{l} \begin{pmatrix} 1 \\ (\theta_1 - \theta_0) \\ (\theta_1 - \theta_0)^2 \\ \vdots \\ (\theta_1 - \theta_0)^{m_1} \\ (t_{2\tau(2)} - \theta_1) \\ (t_{2\tau(2)} - \theta_1)^2 \\ \vdots \\ (t_{2\tau(2)} - \theta_1)^{m_2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \tau(2) \in I_2 \end{array} \right.$$

$$\dots, \left. \begin{array}{c} F \text{sign}(F(t_{n\tau(n)})) \\ \left( \begin{array}{c} 1 \\ (\theta_1 - \theta_0) \\ (\theta_1 - \theta_0)^2 \\ \vdots \\ (\theta_1 - \theta_0)^{m_1} \\ (\theta_2 - \theta_1) \\ (\theta_2 - \theta_1)^2 \\ \vdots \\ (\theta_2 - \theta_1)^{m_2} \\ \vdots \\ (t_{n\tau(n)} - \theta_{n-1}) \\ (t_{n\tau(n)} - \theta_{n-1})^2 \\ \vdots \\ (t_{n\tau(n)} - \theta_{n-1})^{m_2} \end{array} \right) \end{array} \right\} \\ \tau(n) \in I_n$$

The condition

$$0_{\gamma+1} \in \underline{\partial}\varphi(A^*) \quad (3.6)$$

is a necessary and sufficient condition for the convex function  $\varphi(A)$  to reach a minimal value at the point  $A^*$  (see [2]). According to Caratheodory theorem there exists a set of no more than  $\gamma + 2$  extreme points of the subdifferential, such that their convex hull composes  $0_{\gamma+1}$  at a point of minimum. Suppose that in the first interval there exist  $k_1$  points from this set of  $\gamma + 2$  extreme points, in the second interval —  $k_2$  points, ..., in the  $n$ -th interval —  $k_n$  points. Then the condition (3.6) can be presented as follows: there exist nonnegative coefficients  $\alpha'_{ij}$ , such that their sum gives 1 and such that the following condition holds:

$$0_{\gamma+1} = \sum_{i=1}^n \sum_{j=1}^{k_i} T_{ij} \alpha'_{ij}, \quad (3.7)$$

where  $T_{ij}$  are the vectors from the subdifferential (the extreme points which form the convex hull and correspond to the points  $t_{ij}, i = 1, \dots, n, j = 1, \dots, k_i$ ).

Note that if the minimal chain consists of only one interval then the conditions of Chebyshev theorem are satisfied.

The expression (3.7) can be presented as a linear homogeneous system

$$P\Lambda = 0_{\gamma+1}, \quad (3.8)$$

where  $\Lambda = (\alpha'_{11}, \dots, \alpha'_{1k_1}, \dots, \alpha'_{n1}, \dots, \alpha'_{nk_n})^T$ . The system matrix  $P$  can be constructed as follows

$$\begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 & \dots & 1 & \dots & 1 \\ t_{11}-\theta_0 & \dots & t_{1k_1}-\theta_0 & \theta_1-\theta_0 & \dots & \theta_1-\theta_0 & \dots & \theta_1-\theta_0 & \dots & \theta_1-\theta_0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (t_{11}-\theta_0)^{m_1} & \dots & (t_{1k_1}-\theta_0)^{m_1} & (\theta_1-\theta_0)^{m_1} & \dots & (\theta_1-\theta_0)^{m_1} & \dots & (\theta_1-\theta_0)^{m_1} & \dots & (\theta_1-\theta_0)^{m_1} \\ 0 & \dots & 0 & t_{21}-\theta_1 & \dots & t_{2k_2}-\theta_1 & \dots & \theta_2-\theta_1 & \dots & \theta_2-\theta_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & (t_{21}-\theta_1)^{m_2} & \dots & (t_{2k_2}-\theta_1)^{m_2} & \dots & (\theta_2-\theta_1)^{m_2} & \dots & (\theta_2-\theta_1)^{m_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & t_{n1}-\theta_{n-1} & \dots & t_{nk_n}-\theta_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & (t_{n1}-\theta_{n-1})^{m_n} & \dots & (t_{nk_n}-\theta_{n-1})^{m_n} \end{pmatrix}$$

The first step is to show that in the last interval there exist  $m_n + 1$  alternance points. Consider the last  $m_n$  rows of the matrix  $P$ , excluding the zero-columns:

$$\begin{pmatrix} t_{n1} - \theta_{n-1} & t_{n2} - \theta_{n-1} & \dots & t_{nk_n} - \theta_{n-1} \\ (t_{n1} - \theta_{n-1})^2 & (t_{n2} - \theta_{n-1})^2 & \dots & (t_{nk_n} - \theta_{n-1})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (t_{n1} - \theta_{n-1})^{m_n} & (t_{n2} - \theta_{n-1})^{m_n} & \dots & (t_{nk_n} - \theta_{n-1})^{m_n} \end{pmatrix} \begin{pmatrix} \alpha'_{n1} \\ \alpha'_{n2} \\ \vdots \\ \alpha'_{nk_n} \end{pmatrix} = 0_{m_n} \quad (3.9)$$

The system (3.9) can be rewritten as follows

$$(T_{n1}, \dots, T_{nk_n-1}) \begin{pmatrix} \alpha'_{n1} \\ \alpha'_{n2} \\ \vdots \\ \alpha'_{nk_n-1} \end{pmatrix} = -\alpha'_{nk_n} T_{nk_n}, \quad (3.10)$$

where  $T_1, \dots, T_n$  are the columns of the matrix of the system (3.9). In the current theorem  $t_{nj} > \theta_{n-1}, j = 1, \dots, k_n$ , then the system can be presented as follows

$$\begin{pmatrix} 1 & \dots & 1 \\ t_{n1}-\theta_{n-1} & \dots & t_{nk_n-1}-\theta_{n-1} \\ (t_{n1}-\theta_{n-1})^2 & \dots & (t_{nk_n-1}-\theta_{n-1})^2 \\ \vdots & \ddots & \vdots \\ (t_{n1}-\theta_{n-1})^{m_n-1} & \dots & (t_{nk_n-1}-\theta_{n-1})^{m_n-1} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{k_n-1} \end{pmatrix} = -\beta_{k_n} \begin{pmatrix} 1 \\ t_{nk_n}-\theta_{n-1} \\ (t_{nk_n}-\theta_{n-1})^2 \\ \vdots \\ (t_{nk_n}-\theta_{n-1})^{m_n-1} \end{pmatrix}, \quad (3.11)$$

where  $\beta_l = \alpha'_{nl}(t_{nl} - \theta_{n-1}), l = 1, \dots, k_n$ . Notice that  $\beta_l \neq 0$  for  $n > 1$ . If  $n = 1$  then the problem is a polynomial approximation problem and the conditions of Chebyshev theorem are satisfied, which coincide with the first condition of the current theorem. Notice also that the transposition of the system matrix in (3.11) is a Vandermonde matrix. It is

clear, that the system (3.11) has only one solution which satisfies the condition  $\beta_l \neq 0$ ,  $l = 1, \dots, k_n$ , also notice that  $k_m > m_n$ .

If  $k_n \geq m_n + 2$  then the conditions of Chebyshev theorem are satisfied (the first condition of the current theorem), therefore the minimal chain contains only one interval, namely the  $n$ -th interval.

Notice that

$$\text{sign}(\beta_l) = -\text{sign}(\beta_{l+1}), \quad l = 1, \dots, k_n - 1,$$

then  $\text{sign}(\alpha'_{nl}) = -\text{sign}(\alpha'_{n,l+1}), l = 1, \dots, k_n - 1$ . It means that the deviation function consequently changes the sign at the points of maximal deviations. Therefore, the last interval contains at least  $m_n + 1$  alternance points.

The second step is to show that the first interval contains  $m_1 + 1$  alternance points. Consider the first  $m_1 + 1$  rows of the matrix  $P$ . Multiply the  $m_1$ -th row by  $(\theta_1 - \theta_0)$  and subtract from the  $(m_1 + 1)$ -th row. Then multiply the  $(m_1 - 1)$ -th row by  $(\theta_1 - \theta_0)$  and subtract from the  $m_1$ -th row. Continue the process and obtain the first  $m_1 + 1$  rows of an equivalent matrix as follows

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ t_{11} - \theta_1 & t_{12} - \theta_1 & \dots & t_{1k_1} - \theta_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (t_{11} - \theta_0)^{m_1 - 1}(t_{11} - \theta_1) & (t_{12} - \theta_0)^{m_1 - 1}(t_{11} - \theta_1) & \dots & (t_{1k_1} - \theta_0)^{m_1 - 1}(t_{1k_1} - \theta_1) & 0 & \dots & 0 \end{pmatrix} \quad (3.12)$$

Consider the rows starting from the second one and finishing by the  $(m_1 + 1)$ -th of the matrix (3.12). Several cases should be studied separately.

1.  $t_{1k_1} \neq \theta_1$ .

Then repeating the same logic steps as in the study of the last interval, obtain that in the first interval there exist at least  $m_1 + 1$  maximal deviation points and the deviation function changes its sign at the maximal deviation points (alternance).

2.  $t_{1k_1} = \theta_1$ .

- If  $k_1 \geq m_1 + 2$  then the conditions of Chebyshev theorem (the first option of the current theorem) are satisfied.
- If  $k_1 < m_1 + 2$  then  $\alpha'_{11} = \alpha'_{12} = \dots = \alpha'_{1m_1} = 0$ . Therefore, including the point  $t_{1k_1}$  in the second interval, the minimal chain can be reduced and  $0_{\gamma+1}$  can be constructed using the intervals from the second till the  $n$ -th. This is a contradiction to the assumption that the chosen chain is minimal. Therefore, on the first interval there exist at least  $(m_1 + 1)$  alternance points.

The third step is to show that internal intervals contain  $m_i$  alternance points,  $1 < i < n$ .

Consider the rows, which correspond to the maximal deviation points in the  $i$ -th interval ( $1 < i < n$ ). The zero columns are removed:

$$\begin{pmatrix} t_{i1}-\theta_{i-1} & t_{i2}-\theta_{i-1} & \dots & t_{ik_i}-\theta_{i-1} & (\theta_i-\theta_{i-1}) & \dots & (\theta_i-\theta_{i-1}) \\ (t_{i1}-\theta_{i-1})^2 & (t_{i2}-\theta_{i-1})^2 & \dots & (t_{ik_i}-\theta_{i-1})^2 & (\theta_i-\theta_{i-1})^2 & \dots & (\theta_i-\theta_{i-1})^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (t_{i1}-\theta_{i-1})^{m_i} & (t_{i2}-\theta_{i-1})^{m_i} & \dots & (t_{ik_i}-\theta_{i-1})^{m_i} & (\theta_i-\theta_{i-1})^{m_i} & \dots & (\theta_i-\theta_{i-1})^{m_i} \end{pmatrix} \quad (3.13)$$

Multiply the  $(m_i - 1)$ -th row of the matrix (3.13) by  $(\theta_{i-1} - \theta_i)$  and subtract from the  $m_i$ -th row. Continue the process and obtain:

$$\begin{pmatrix} t_{i1}-\theta_{i-1} & \dots & t_{ik_i}-\theta_{i-1} & (\theta_i-\theta_{i-1}) & \dots & (\theta_i-\theta_{i-1}) \\ (t_{i1}-\theta_{i-1})(t_{i1}-\theta_i) & \dots & (t_{ik_i}-\theta_{i-1})(t_{ik_i}-\theta_i) & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (t_{i1}-\theta_{i-1})^{m_i-1}(t_{i1}-\theta_i) & \dots & (t_{ik_i}-\theta_{i-1})^{m_i-1}(t_{ik_i}-\theta_i) & 0 & \dots & 0 \end{pmatrix} \quad (3.14)$$

Notice that  $t_{i1} \neq \theta_{i-1}$  for any internal interval.

Consider the following situations

1.  $t_{ik_i} \neq \theta_i$ ;
2.  $t_{ik_i} = \theta_i$ .

In the first case, repeating the same logic steps as in the study of the first and the last intervals, obtain that on the  $i$ -th interval there exist at least  $m_i$  alternance points.

Therefore, for each internal interval at least one of the following conditions holds: the  $i$ -th interval contains  $m_i$  alternance points or  $t_{ik_i} = \theta_i$ . Suppose that for the  $i$ -th interval only the second condition satisfies.

First suppose, that on the  $(i + 1)$ -th interval there exist  $m_{i+1}$  alternance points. In this case  $0_{\gamma+1}$  can be constructed using the vectors, which correspond to the points  $t_{ik_i}, t_{i+11}, \dots, t_{i+1k_{i+1}}$ .

Now suppose, that on the  $(i + 1)$ -th interval there exist a maximal deviation point which coincides to the point  $\theta_{i+1}$ . In this case, moving along the minimal chain towards the last interval till an internal interval (the  $j$ -th interval) which contains  $m_j$  alternance points is reached or till the last interval, which contains  $m_n + 1$  alternance points, is reached. In the first case  $0_{\gamma+1}$  can be constructed using only the intervals, starting from the  $j$ -th interval and ending by the  $n$ -th interval. In the second case the minimal chain contains only one interval (the  $n$ -th interval). Therefore, in both cases a contradiction to the statements of the theorem that the chosen chain is minimal is obtained.

Therefore, any internal  $i$ -th interval contains  $m_i$  alternance points.

Since the deviation signs are alternating inside each interval, the next step is to prove, that the deviation function changes the sign at any two nearest alternance points which belong to different intervals (the signs are alternating).

Consider the  $n$ -th and the  $(n-1)$ -th intervals. Using the rows, which correspond to the  $(n-1)$ -th interval, construct the following linear system

$$\begin{pmatrix} t_{n-11}-\theta_{n-2} & \dots & t_{n-1m_{n-1}}-\theta_{n-2} \\ (t_{n-11}-\theta_{n-2})^2 & \dots & (t_{n-1m_{n-1}}-\theta_{n-2})^2 \\ \vdots & \ddots & \vdots \\ (t_{n-11}-\theta_{n-2})^{m_{n-1}} & \dots & (t_{n-1m_{n-1}}-\theta_{n-2})^{m_{n-1}} \end{pmatrix} \begin{pmatrix} \alpha_{n-11} \\ \alpha_{n-12} \\ \vdots \\ \alpha_{n-1m_{n-1}} \end{pmatrix} = - \sum_{i=1}^{m_{n-1}} \alpha_{ni} \begin{pmatrix} \theta_n-\theta_{n-1} \\ (\theta_n-\theta_{n-1})^2 \\ \vdots \\ (\theta_n-\theta_{n-1})^{m_{n-1}} \end{pmatrix}$$

Analysing the sign of  $\alpha_{n-1m_{n-1}}$  using the Cramer's Rules, obtain

$$\text{sign}(\alpha_{n-1m_{n-1}}) = -\text{sign}\left(\sum_{i=1}^{m_{n-1}} \alpha_{ni}\right). \quad (3.15)$$

Notice, that

$$|\alpha_{nm_{n+1}}| < |\alpha_{nm_n}| < \dots < |\alpha_{n1}|$$

and the signs are alternating, then  $\text{sign}(\alpha_{nk}) = \text{sign}\left(\sum_{i=k}^{m_{n+1}} \alpha_{ni}\right)$ . Therefore,

$$\text{sign} \alpha_{n1} = \text{sign}\left(\sum_{i=1}^{m_{n+1}} \alpha_{ni}\right) = -\text{sign}(\alpha_{n-1m_{n-1}}).$$

The next step is to prove that

$$\text{sign}(\alpha_{1m_1+1}) = -\text{sign}(\alpha_{21}).$$

The first equality of (3.7) gives

$$\sum_{i=1}^{m_1+1} \alpha_{1i} + \sum_{j=2}^n \sum_{i=1}^{m_j} \alpha_{ji} + \alpha_{nm_{n+1}} = 0, \quad (3.16)$$

therefore

$$\sum_{i=1}^{m_1+1} \alpha_{1i} = - \left( \sum_{j=2}^n \sum_{i=1}^{m_j} \alpha_{ji} + \alpha_{nm_{n+1}} \right), \quad (3.17)$$

Consider the first  $m_1 + 1$  equalities of (3.7)

$$\begin{pmatrix} 1 & \dots & 1 \\ t_{11}-\theta_0 & \dots & t_{1m_1+1}-\theta_0 \\ (t_{11}-\theta_0)^2 & \dots & (t_{1m_1+1}-\theta_0)^2 \\ \vdots & \ddots & \vdots \\ (t_{11}-\theta_0)^{m_1} & \dots & (t_{1m_1+1}-\theta_0)^{m_1} \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1m_1+1} \end{pmatrix} = \sum_{i=1}^{m_1+1} \alpha_{1i} \begin{pmatrix} \theta_1-\theta_0 \\ (\theta_1-\theta_0)^2 \\ \vdots \\ (\theta_1-\theta_0)^{m_1} \end{pmatrix}. \quad (3.18)$$

Using Cramer's Rules, obtain

$$\text{sign}(\alpha_{1m_1+1}) = \text{sign} \left( \sum_{i=1}^{m_1+1} \alpha_{1i} \right). \quad (3.19)$$

Consider the equations starting from  $m_1 + 2$  till  $m_1 + m_2 + 1$  (excluding the first  $m_1 + 1$  columns, which are zeros). Together with (3.17) obtain a system, which is similar to (3.18)

$$\begin{pmatrix} t_{21}-\theta_1 & \dots & t_{2m_2}-\theta_1 \\ (t_{21}-\theta_1)^2 & \dots & (t_{2m_2}-\theta_1)^2 \\ \vdots & \ddots & \vdots \\ (t_{21}-\theta_1)^{m_2} & \dots & (t_{2m_2}-\theta_1)^{m_2} \end{pmatrix} \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2m_2} \end{pmatrix} = \left( \sum_{i=1}^{m_1+1} \alpha_{1i} + \sum_{i=1}^{m_2} \alpha_{2i} \right) \begin{pmatrix} \theta_2-\theta_1 \\ (\theta_2-\theta_1)^2 \\ \vdots \\ (\theta_2-\theta_1)^{m_2} \end{pmatrix}. \quad (3.20)$$

The first equation of (3.20) gives

$$\sum_{i=1}^{m_2} \alpha_{2i}(t_{2i} - \theta_1) = \left( \sum_{i=1}^{m_1+1} \alpha_{1i} + \sum_{i=1}^{m_2} \alpha_{2i} \right) (\theta_2 - \theta_1) \quad (3.21)$$

Similar to (3.15) obtain

$$\text{sign}(\alpha_{21}) = \text{sign} \left( \sum_{i=1}^{m_2} \alpha_{2i} \right). \quad (3.22)$$

Therefore, the goal is to prove that

$$\text{sign} \left( \sum_{i=1}^{m_2} \alpha_{2i} \right) = -\text{sign} \left( \sum_{i=1}^{m_1+1} \alpha_{1i} \right) \quad (3.23)$$

Consider now

$$\sum_{i=1}^{m_2} \alpha_{2i}(t_{2i} - \theta_2) = \sum_{i=1}^{m_2} \alpha_{2i}(t_{2i} - \theta_1) - \sum_{i=1}^{m_2} \alpha_{2i}(\theta_2 - \theta_1). \quad (3.24)$$

Notice that

$$\text{sign} \left( \sum_{i=1}^{m_2} \alpha_{2i}(t_{2i} - \theta_2) \right) = -\text{sign} \left( \sum_{i=1}^{m_2} \alpha_{2i} \right).$$

Indeed,

$$\sum_{i=1}^{m_2} \alpha_{2i}(t_{2i} - \theta_2) = \sum_{i=1}^{m_2} \beta_{2i},$$

where  $\beta_{2i} = \alpha_{2i}(t_{2i} - \theta_2)$ ,  $i = 1, \dots, m_2$ , therefore

$$|\beta_{2i}| = |\alpha_{2i}| |t_{2i} - \theta_2|, \quad i = 1, \dots, m_2.$$

Then  $|\beta_{2i}| < |\beta_{2i-1}|$ ,  $i = 2, \dots, m_2$  and

$$\text{sign} \left( \sum_{i=1}^{m_2} \beta_{2i} \right) = \text{sign}(\beta_{2i}) = -\text{sign}(\alpha_{2i}) = -\text{sign} \left( \sum_{i=1}^{m_2} \alpha_{2i} \right).$$

Combine (3.21) and (3.24)

$$\sum_{i=1}^{m_2} \alpha_{2i}(t_{2i} - \theta_1) = \left( \sum_{i=1}^{m_1+1} \alpha_{1i} + \sum_{i=1}^{m_2} \alpha_{2i} \right) (\theta_2 - \theta_1) = \sum_{i=1}^{m_2} \alpha_{2i}(t_{2i} - \theta_2) + \sum_{i=1}^{m_2} \alpha_{2i}(\theta_2 - \theta_1).$$

Therefore,

$$\text{sign} \left( \sum_{i=1}^{m_1+1} \alpha_{1i} \right) = \text{sign} \left( \sum_{i=1}^{m_2} \alpha_{2i}(t_{2i} - \theta_2) \right) = -\text{sign} \left( \sum_{i=1}^{m_2} \alpha_{2i} \right)$$

and (3.23) holds.

Continue the process till the last interval is reached. Therefore, the deviation signs are alternating at all the maximal deviation points, which are used in the constructing the necessary condition. The Necessity is proved.

Sufficiency.

The condition (3.6) is not only necessary, but it is also a sufficient optimality condition. If the minimal chain contains only one interval, then the conditions of Chebyshev theorem are satisfied. Therefore, it is only necessary to prove the sufficiency for the case  $n > 1$ .

If the maximal deviation points are distributed as follows: there exist  $m_1 + 1$  points on the first interval,  $m_i$  points on each internal (the  $i$ -th) interval and  $m_n + 1$  on the last (the  $n$ -th) interval, such that the deviation sign is alternating, then solution of the system (3.8) constructed during the proof of the necessariness of the theorem can be used for constructing the convex hull of the extreme points of the differential which forms  $0_{\gamma+1}$ .

The sufficiency is proved.



The theorem is proved.

△

**Corollary 3.1** *Maximal deviation points from the minimal chain can not coincide with any of the internal knots.*

**Proof.** Suppose, that there exists an internal interval (the  $i$ -th interval), such that  $t_{im_i} = \theta_i$   $i = 2, \dots, n - 1$  (in the case when  $i = 1$   $t_{1m_1+1} = \theta_1$ ). Then the minimal chain can be reduced, namely, it can be constructed without the usage of the intervals from the first till the  $i$ -th.

△

In some practical applications it is necessary to approximate functions by splines with some conditions on the value at the points  $a$  and  $b$ , which are the borders of the approximation interval  $[a, b]$ .

**Definition 3.1** *If the value of the spline  $Sp(t)$  is fixed at the point  $a$  and/or  $b$ , such that  $Sp(a) = y_0$  and/or  $Sp(b) = y_n$ , then the spline  $Sp(t)$  is a spline with a fixed left and/or right tail.*

Necessary and sufficient optimality conditions for the spline  $S(A^*, t)$  of the generalised (vector) degree  $M = (m_1, \dots, m_n)$  and the highest defect with a fixed left or right tail are formulated in the following corollary.

**Corollary 3.2** *A polynomial spline with a fixed left (right) tail is optimal if and only if one of the following statements is satisfied:*

1) *in one of the intervals  $[\theta_{i-1}, \theta_i]$ ,  $i \neq 1$  ( $i \neq n$ ) there exist at least  $m_i + 2$  alternance points  $t_1, \dots, t_{m_i+2}$ , which are points of maximal deviation of the spline  $S(A^*, t)$  from  $f(t)$ , such that*

$$\begin{aligned} F(A^*, t_k) &= f(t_k) - S(A^*, t_k) = -F(A^*, t_{k+1}) = -(f(t_{k+1}) - S(A^*, t_{k+1})) = \\ &= \pm \max_{t \in [a, b]} |f(t) - S(A^*, t)|, \end{aligned}$$

*if  $i = 1$  ( $i = n$ ) then there exist at least  $m_i + 1$  alternance points;*

*or*

2) *there exist intervals  $[\theta_{i-1}, \theta_i], (\theta_i, \theta_{i+1}], \dots, (\theta_{j-1}, \theta_j]$ , the minimal chain, where  $1 < i < j \leq n$  ( $1 \leq i < j < n$ ), such that the maximal deviation points are distributed as follows:*

- *there exist at least  $m_i + 1$  alternance points in the  $i$ -th interval,*
- *there exist at least  $m_j + 1$  alternance points in the  $j$ -th and*

- there exist at least  $m_k$  alternance points in the  $k$ -th intervals ( $i < k < j$ ), if  $i = 1$  ( $j = n$ ) then the number of alternance points on the  $i$ -th ( $j$ -th) interval is reduced by one;

or

3) the point  $a$  ( $b$ ) is one of the maximal deviation points.

**Proof.** Assume for simplicity that the minimal chain is known and it consists of  $n$  intervals (the same approach as it was in the proof of theorem 3.1).

It is clear that if the first (last) interval is not included into the minimal chain then the necessary and sufficient optimality conditions for splines with free tails (theorem 3.1) and with the fixed left (right) tail are coincided. Therefore, it is only necessary to prove the case when the first (last) interval are in the minimal chain.

First consider splines with the fixed left tail.

If one of the maximal deviation points is the point  $\theta_0 = a$ , then it is clear that this approximation can not be improved. If the point  $\theta_0 = a$  is not one of the maximal deviation points then the spline can be constructed as follows:

$$S_M(A, t) = y_0 + \sum_{i=1}^n \sum_{j=1}^{m_i} a_{ij} (\min\{t, \theta_i\} - \theta_{i-1})_+^j, \quad (3.25)$$

where  $y_0$  is the value of the spline at the point  $a$  (this value is fixed). Similar to (3.8) construct a linear homogeneous system with the system matrix  $P_L$ , which can be obtained from the matrix  $P$  of the system (3.8) by eliminating the first row. Repeat the same logic steps as in the proof of theorem 3.1 working with the matrix  $P_L$ . Indeed, in this case the spline behavior in each of the intervals is the same as it is in the case of free tails. The behavior of the spline on the first interval is the same as it is in the case of the internal intervals in theorem 3.1. Therefore, on the first interval there exist  $m_1$  maximal deviation points.

Now consider splines with the fixed right tail. Such splines can be presented as following (an alternative formula to (3.25))

$$S_M(B, t) = b_0 + \sum_{i=1}^n \sum_{j=1}^{m_i} b_{ij} (-\max\{t, \theta_{i-1}\} - \theta_i)_+^j, \quad (3.26)$$

where  $B = (b_0, b_{11}, \dots, b_{nm_n}) \in R^{\gamma+1}$  is the vector of spline parameters,  $\gamma = \sum_{i=1}^n m_i$   $\theta_i$  are the knots, where  $i = 0, \dots, n$ .

This spline is constructed according to the same scheme as (3.25), but in the opposite direction (from the point  $b$  to the point  $a$ ). It is obvious that the matrix  $P_R$  has the same structure as the matrix  $P_L$ , but the intervals are numerated in the opposite order. Repeat the same steps as they are in the case of the fixed left tail. Therefore, the corollary is proved.

△

Now consider a situation where both tails are fixed.

**Corollary 3.3** *Necessary and sufficient optimality conditions for the spline  $S(A^*, t)$  of the generalised (vector) degree  $M = (m_1, \dots, m_n)$  and the highest defect with the fixed left and right tail are as follows:*

1) *on one of the intervals  $[\theta_{i-1}, \theta_i]$ , there exist at least  $m_i + 2$  alternance points  $t_1, \dots, t_{m_i+2}$ , which are points of maximal deviation of the spline  $S(A^*, t)$  from  $f(t)$ , such that*

$$\begin{aligned} F(A^*, t_k) = f(t_k) - S(A^*, t_k) &= -F(A^*, t_{k+1}) = -(f(t_{k+1}) - S(A^*, t_{k+1})) = \\ &= \pm \max_{t \in [a, b]} |f(t) - S(A^*, t)|, \end{aligned}$$

*if  $i = 1$  or  $i = n$  then there exist at least  $m_i + 1$  alternance points;*

*or*

2) *there exist intervals  $[\theta_{i-1}, \theta_i]$  and  $(\theta_{j-1}, \theta_j]$ , the minimal chain, where  $1 \leq i < j \leq n$ , such that the maximal deviation points are distributed as follows:*

- *there exist at least  $m_i + 1$  alternance points in the  $i$ -th interval,*
- *there exist at least  $m_j + 1$  alternance points in the  $j$ -th and*
- *there exist at least  $m_k$  alternance points in the  $k$ -th intervals ( $i < k < j$ ), if  $i = 1$  then the number of alternance points on the  $i$ -th interval can be reduced by one, if  $j = n$  then the number of alternance points on the  $j$ -th interval can be reduced by one.*

*or*

3) *one of the maximal deviation points is the point  $\theta_0 = a$  or the point  $\theta_1 = b$ .*

**Proof.** It is obvious that if the minimal chain contains more than one interval, the current corollary is true (it is a direct consequence of the previous corollary). It is only necessary to prove the case when the minimal chain contains only one interval.

In this case the spline is a polynomial  $S_m(t)$  of the degree  $m$  on the interval  $[\theta_0, \theta_1]$ . Suppose that  $S(\theta_0) = y_0$ ,  $S(\theta_1) = y_1$ , then the polynomial  $S_m(t)$  can be constructed as follows.

$$S_m(t) = \sum_{i=2}^m a^i (t - \theta_0)^i + \left( \frac{y_1}{(\theta_1 - \theta_0)} - \sum_{i=2}^{m-1} a^i (\theta_1 - \theta_0)^{i-1} \right) (t - \theta_0). \quad (3.27)$$

The necessary and sufficient optimality condition is as follows

$$0_{m-1} \in \text{co} \left\{ \left( \begin{array}{c} -(\theta_1 - \theta_0)(t_k - \theta_0) + (t_k - \theta_0)^2 \\ -(\theta_1 - \theta_0)^2(t_k - \theta_0) + (t_k - \theta_0)^3 \\ \vdots \\ -(\theta_1 - \theta_0)^{m-2}(t_k - \theta_0) + (t_k - \theta_0)^m \end{array} \right) \right\}, \quad (3.28)$$

where the points  $t_k, k \in I_1$  are the maximal deviation points. Suppose that there exist  $k_m$  maximal deviation points. According to Caratheodory theorem  $0_{m-1}$  can be obtained as a convex hull of no more than  $m$  extreme points of the subdifferential. Consider the following linear homogeneous system and study its positive solutions.

$$P\Lambda = 0_m, \quad (3.29)$$

where  $\Lambda = (\alpha'_1, \dots, \alpha'_{k_m})^T$ , The matrix of the system  $P$  can be constructed as follows

$$\left( \begin{array}{ccc} -(\theta_1 - \theta_0)(t_1 - \theta_0) + (t_1 - \theta_0)^2 & \dots & -(\theta_1 - \theta_0)(t_{k_m} - \theta_0) + (t_{k_m} - \theta_0)^2 \\ -(\theta_1 - \theta_0)^2(t_1 - \theta_0) + (t_1 - \theta_0)^3 & \dots & -(\theta_1 - \theta_0)^2(t_{k_m} - \theta_0) + (t_{k_m} - \theta_0)^3 \\ \vdots & \ddots & \vdots \\ -(\theta_1 - \theta_0)^{m-2}(t_1 - \theta_0) + (t_1 - \theta_0)^m & \dots & -(\theta_1 - \theta_0)^{m-2}(t_{k_m} - \theta_0) + (t_{k_m} - \theta_0)^m \end{array} \right).$$

Multiply the  $(m-2)$ -th row of the matrix  $P$  by  $(\theta_1 - \theta_0)$  and subtract it the  $(m-1)$ -th row, then multiply the  $(m-3)$ -th row of the matrix  $P$  by  $(\theta_1 - \theta_0)$  and subtract it the  $(m-2)$ -th). Continue the process, and obtain the following equivalent matrix

$$\left( \begin{array}{ccc} (t_1 - \theta_0)(t_1 - \theta_1) & \dots & (t_{k_m} - \theta_0)(t_{k_m} - \theta_1) \\ (t_1 - \theta_0)^2(t_1 - \theta_1) & \dots & (t_{k_m} - \theta_0)^2(t_{k_m} - \theta_1) \\ \vdots & \ddots & \vdots \\ (t_1 - \theta_0)^{m-1}(t_1 - \theta_1) & \dots & (t_{k_m} - \theta_0)^{m-1}(t_{k_m} - \theta_1) \end{array} \right).$$

Note that if one of the maximal deviation points is the point  $\theta_0 = a$  or the point  $\theta_1 = b$ , then the obtained approximation can not be improved and the obtained spline is optimal. If none of the maximal deviation points coincides with  $\theta_0 = a$  or  $\theta_1 = b$ , then there exist at least  $m$  maximal deviation points on the interval  $[\theta_0, \theta_1]$ , such that the sign of the deviation alternates. Therefore, the corollary is proved.

△

## 4 Conclusions and further research

In this paper necessary and sufficient optimality conditions for polynomial spline approximation (uniform approximation criterion, the highest defect, the knots are fixed) have been obtained. These results are generalisations of the results, obtained by M. Tarashnin,

to the case when the degree of the polynomials, which compose the spline, can vary from one interval to another. Then these results have been extended to the cases when the polynomial splines have some border conditions, namely, left or right (or both) tails are fixed.

Necessary and sufficient conditions, obtained in this paper, can be also treated as natural generalisations of the classical results, obtained by Chebyshev.

Apart from their theoretical importance, the optimality conditions can be used for optimal spline constructing, for example, in a generalised version of Remez algorithm, developed for polynomial splines.

The problem of approximating a continuous function by polynomial splines of the highest defect has several applications. One of them appeared in the area of taxation scale constructing (polynomial splines of the degree equals 1 and the defect equals 1 with a fixed left tail).

One of the most important future research directions is the derivation of necessary and sufficient optimality conditions for polynomial splines when polynomial spline knots are free. This problem is much more complicated than the problem studied in this paper (knots are fixed) and it is possible that necessary and sufficient optimality conditions for the case of polynomial splines with free knots are very different from the optimality conditions obtained in this paper. In the case of fixed knots the problem of polynomial spline approximation is convex, however if the knots are not fixed the problem is not convex. Therefore, it is very likely that in the case of free knot polynomial spline approximation necessary and sufficient optimality conditions do not coincide as it is in the case of fixed knots polynomial splines.

In some practical applications, for example, classification problems, the problem of polynomial spline approximation has to be extended to a problem of approximating by more than one polynomial spline. This is another important research direction which will be studied in the future.

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