Integral Terminal Sliding Mode Cooperative Control of Multi-robot Networks

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Abstract—This paper studies the integral terminal sliding mode cooperative control of multi-robot networks. Here, we first propose an integral terminal sliding mode surface for a class of first order systems. Then, we prove that finite time consensus tracking of multi-robot networks can be achieved on this integral terminal sliding mode surface. Simulation results are presented to validate the analysis.

I. INTRODUCTION

Sliding mode control (SMC) is well known for their robustness to system parameter variations and external disturbances [1-7]. In SMC design, a switching manifold or sliding mode surface is first defined, and a sliding mode controller is then designed to drive the system state variables to the sliding mode surface. On the sliding mode surface, the desired convergence property can be obtained, which is not affected by any parameter variations and external disturbances.

For switching manifold design, the most commonly used switching manifolds are linear hyper planes. Such hyper planes guarantee the asymptotic stability of the SMC systems. It is natural to ask if nonlinear hyper planes can be used to design the switching manifolds. Such switching manifold might improve the performance of the SMC systems. Terminal sliding mode (TSM) surface, which is also a nonlinear hyper plane, has been proposed to improve the transient performance and the robustness of the SMC systems. TSM control has been proven success in many control designs [8-12]. However, all the TSM controls found in the literature are dealing with second or higher order system only.

Cooperative consensus control of multi-robot networks poses significant theoretical and practical challenges. First, the research objective is to develop a system of subsystems rather than a single system. Second, the communication bandwidth and connectivity of the team are often limited, and the information exchange among agents may be unreliable. Third, arbitration between team goals and individual goals needs to be negotiated. Fourth, the computational resources of each individual agent will always be limited [13]. In recent years, there has been an increasing research interest in the consensus control design of multi-agent networks [14-17].

In this paper, we will first propose a new integral terminal sliding mode (ITSM) surface that is able to provide finite-time convergence for a class of first order nonlinear systems with input disturbances. This proposed ITSM control is then applied to the cooperative consensus control of 2-degree-of-freedom first order AmigoBot mobile robots. We proved that finite-time consensus tracking of multi-robot networks can be achieved on this ITSM surface. Our proposed ITSM control scheme is robust to input disturbances. Since not all follower robots in the network with directed communication topology have directed communication with the leader robot, our results assume that, the robots in the network only need to communicate with their neighbours and not the entire community.

The remainder of this paper is organized as follows. Section 2 reviews some concepts of graph theory and Lyapunov theory for fast finite-time stability. The new integral terminal sliding mode control is proposed in Section 3. In Section 4, the Integral terminal sliding mode cooperative control (ITSMCC) scheme for multi-robot system is proposed and the design of this ITSMCC algorithm to guarantee finite-time consensus tracking of multi-robot networks is discussed in detail. Section 5 gives numerical examples to illustrate our results. Concluding remarks are given in Section 6.

II. BACKGROUND AND PRELIMINARIES

A. Concepts in Graph Theory and Multi-agent Networks

Consider a multi-robot system consisting of one leader robot and n follower robots. To solve the coordination problems and to model the information exchange between robots, graph theory is introduced here. Let \( G = \{V, E\} \) be a directed graph, where \( V = \{1, 2, ..., n\} \) is the set of nodes, node \( i \) represents the \( i \)th agent, \( E \) is the set of edges, and an edge in \( G \) is denoted by an ordered pair \((i, j)\) if and only if the \( i \)th agent can send information to the \( j \)th agent directly, but not necessarily vice versa. In contrast to a directed graph, the pairs of nodes in an undirected graph are unordered, where the edge \((i, j)\) denotes that agent \( i \) and \( j \) can obtain information from each other. Therefore, an undirected graph can be viewed as a special case of a directed graph. A directed tree is a directed graph, where every node has exactly one parent except for the root, and the root has a directed path to every other node. A directed
spanning tree of $G$ is a directed tree that contains all nodes of $G$. [13]

$$A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$$

is called the weighted adjacency matrix of $G$ with nonnegative elements where $a_{ii} = 0$ and $a_{ij} = 1$ if there is an edge between the $i$ th agent and the $j$ th agent. Let $D = \text{diag}\{ d_1, \ldots, d_n \} \in \mathbb{R}^{n \times n}$ be a diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 1, \ldots, n$. Then the Laplacian of the weighted graph can be defined as

$$L = D - A. \quad (1)$$

The connection weight between the $i$ th robot and the leader robot is denoted by $b_i$ with $b_i = 1$ if there is an edge between the $i$ th robot and the leader robot. The following theorems present the existing results on Laplacian matrix and graph theory.

**Theorem 1:** [15] The Laplacian matrix $L$ of a directed graph $G$ has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right half plane. Further, $L$ has exactly one zero eigenvalue if and only if $G$ has a directed spanning tree.

**Theorem 2:** [13] The directed graph $G$ has a directed spanning tree if and only if $\{V, E \}$ has at least one node with a directed path to all other nodes.

In this paper, the network considered here consists of $n+1$ robots where an agent indexed by 0 acts as the leader robot and the other robots indexed by 1, $\ldots$, $n$, are referred to as the follower robots. The topology relationships among the leader and followers is described by a directed graph $G = \{V, E \}$ with $V = \{0, 1, \ldots, n \}$ and the adjacency matrix

$$A = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{n1} & \cdots & a_{nn}
\end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)} \quad (2)
$$

Denote $\mathcal{G} = \{\mathcal{V}, \mathcal{E} \}$ as the subgraph of $G$, which is formed by the $n$ followers, where

$$\mathcal{A} = \begin{bmatrix}
a_{12} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{bmatrix} \in \mathbb{R}^{n \times n}. \quad (3)$$

Let $\mathcal{D} = \text{diag}\{ \mathcal{d}_1, \mathcal{d}_2, \ldots, \mathcal{d}_n \} \in \mathbb{R}^{n \times n}$ be the diagonal matrix with $\mathcal{d}_i = \sum_{j=1}^{n} a_{ij}$ for $i = 1, 2, \ldots, n$. Then, it is clear that the Laplacian of the graph, $\mathcal{L}$, can be defined as

$$\mathcal{L} = \mathcal{D} - \mathcal{A}. \quad (4)$$

The connection weight between robot $i$ and the leader robot is denoted by

$$\mathcal{B} = \text{diag}\{ b_1, b_2, \ldots, b_n \}$$

such that $b_i = a_{i0} \quad (5)$

We have the following theorem that state under what condition, consensus can be reached.

**Theorem 3:** [18] Let $x_i$, $i = 0, 1, \ldots, n$ denotes the state variable vector of the robot, and defining the error function as:

$$e_i \triangleq \sum_{j=1}^{n} a_{ij} (x_i - x_j) + b_i (x_i - x_0). \quad (6)$$

If the graph $G$ has a directed spanning tree and if $e_0 = e_1 = \cdots = e_n = 0$, then

$$[x_1 \cdots x_n]^T = 1_{n \times 1}. \quad (7)$$

**B. Lyapunov Theory for Fast Finite Time Stability**

It is know that finite time stability guarantees that every system state reaches the system origin in a finite time, finite time stability has a much stronger requirement than asymptotic stability. The following theorem presents sufficient conditions for fast finite time stability.

**Theorem 4:** Consider the non-Lipschitz continuous nonlinear system $\dot{x} = f(x)$ with $f(0) = 0$. Suppose there is a Lyapunov function $V(x)$ defined on a neighborhood of the origin, and real numbers $\alpha > 0$, $\beta > 0$, and $0 < \gamma < 1$, such that

1. $V(x)$ is positive definite
2. $\dot{V} + \alpha V + \beta V^\gamma \leq 0.$

Then, the origin is locally fast finite-time stable, and the settling time, depending on the initial state $x(0) = x_0$, satisfies

$$T(x_0) \leq \frac{1}{\alpha(1-\gamma)} \ln \left( \frac{\alpha V^\gamma + \beta \gamma}{\beta} \right). \quad (8)$$

**Proof:** The proof is straightforward and therefore is omitted.

**III. INTEGRAL TERMINAL SLIDING MODE CONTROL**

Consider the following SISO first order nonlinear system:

$$\dot{x} = f(x) + u + \delta, \quad (9)$$

where $x$ is system state, $f(x)$ is known nonlinear function, $u$ is the control input, and $\delta$ is the bounded disturbances satisfying the following inequality:

$$|\delta| \leq \mathcal{D}, \text{ where } \mathcal{D} > 0. \quad (10)$$

A new integral terminal sliding variable for the above SISO first order nonlinear system is defined as follows:

$$s = x(t) - x(t_0) + \int_{t_0}^{t} \frac{\alpha}{2} x + \frac{\beta}{2} x^2 dt. \quad (11)$$

The integral terminal sliding mode surface is then defined as $s = \dot{s} = 0$ or

$$\dot{s} + \frac{\alpha}{2} x + \frac{\beta}{2} x^2 = 0. \quad (12)$$

To obtain the sufficient condition of the existence of the integral terminal sliding mode surface (12) for SISO first order system (9), we have the following theorem.
Theorem 5: Consider the SISO first order nonlinear system (9). If the control input is designed as:
\[ u = -f(x) - \frac{\alpha}{2} x - \frac{\beta}{2} x^2 - \frac{\beta}{\sqrt{2}} s - \delta \sign(s) - \Delta \sign(s), \] (13)
then the integral terminal sliding variable \( s \) will reach the integral terminal sliding mode surface \( s = \dot{s} = 0 \) in finite time.

Proof: Defining a Lyapunov function
\[ V = \frac{1}{2} s^2 \] (14)
and differentiating \( V \) with respect to time, we have
\[ \dot{V} = s \dot{s} \]
\[ = s \left[ \dot{x} + \frac{1}{2} \alpha x + \frac{1}{2} \beta x^2 - \frac{\beta}{\sqrt{2}} s \right] \]
\[ = s \left[ f(x) + \dot{s} + \frac{\alpha}{2} x + \frac{\beta}{\sqrt{2}} s \right] \]
\[ = -D|s| + \dot{s} - \frac{\alpha}{2} s^2 - \frac{\beta}{\sqrt{2}} s \]
\[ \leq -\frac{\alpha}{2} s^2 - \frac{\beta}{\sqrt{2}} s \]
\[ = -\alpha \dot{V} - \beta V^{1/2}. \] (15)
By Theorem 4, expression (15) is the sufficient condition for the integral terminal sliding variable \( s \) to converge to zero in finite time. On the ITSM surface, \( s = \dot{s} = 0 \).

The convergence property on the SISO first order nonlinear system in (9) on the ITSM surface is stated in the following theorem.

Theorem 6: Consider the SISO first order nonlinear system (9). On the ITSM surface, the system state \( x \) will converge to zero in a finite time.

Proof: The proof is straightforward and similar to the proof in (15), hence it is omitted here.

On the ITSM surface, \( s = \dot{s} = 0 \), the equivalent control can be described as follows:
\[ u_{eq} = -f(x) - \frac{\alpha}{2} x - \frac{\beta}{2} x^2 - \delta. \] (16)

Remark 1: It should be noted that the ITSM control law in (13) and the equivalent control law in (16) are always nonsingular. Hence, unlike the conventional TSM control, the variable, \( \gamma \) not necessarily need to be odd.

Remark 2: In order to eliminate the chattering, a saturation function \( \sign(s) \) can be used to replace the \( \sign(s) \) function.

IV. INTEGRAL TERMINAL SLIDING MODE COOPERATIVE CONTROL FOR MULTI-ROBOT NETWORKS

Consider a multi-robot network formed by \( n \) AmigoBots. We assume that the robots can communicate with their neighbors but not the entire community. Define the “hand” position of the robot as point \( h = [h_x, h_y]^T \) that lies a distance \( L \) along the line that is normal to the wheel axis and intersects the wheel axis at the center point \( r = [r_x, r_y]^T \), as shown in Figure 1. In this paper, similar to the work in [13], we consider the problem of coordinating the hand position \( (h_x, h_y) \) instead of the center position \( (r_x, r_y) \).

For further analysis, we let \( (r_{u_i}, r_{v_i}), \theta_i \), and \( (v_i, \omega_i) \) represent the center position, orientation, and linear and angular speeds of the \( i \)th AmigoBot, respectively. It is easy to see that the hand position of the \( i \)th AmigoBot is:
\[ \begin{bmatrix} h_u \\ h_v \end{bmatrix} = \begin{bmatrix} r_u \\ r_v \end{bmatrix} + L_i \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix}. \] (17)
Differentiating (17) with respect to time and letting
\[ \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} \begin{bmatrix} u_{u_i} \\ u_{v_i} \end{bmatrix}, \] (18)
this gives
\[ \begin{bmatrix} h_{u_i} \\ h_{v_i} \end{bmatrix} = \begin{bmatrix} u_{u_i} \\ u_{v_i} \end{bmatrix}, \] (19)
which is in the same form as equation (9). For convenience, equation (19) is rewritten as:
\[ \dot{X}_i = U_i. \] (20)
Here, we shall consider the case where the position of the leader robot is available to its neighbors only. Let
\[ \varepsilon = [e_1, e_2, \ldots, e_n]^T, \quad U = [U_1, U_2, \ldots, U_n]^T \] (21)
and by defining the error function as (6), the error dynamics of the interconnection graph can be expressed as:
\[ \varepsilon = [(I + B) \otimes I] U - [B \otimes I_2] U_h. \] (22)

![Fig. 1. Nonholonomic differentially driven wheeled mobile robot](image-url)
fractional power of the vector is defined as:

$$M^\gamma = \begin{bmatrix} m_1^\gamma, & m_2^\gamma, & \ldots, & m_n^\gamma \end{bmatrix}.$$  (23)

To obtain the sufficient condition of the existence of the ITSM controller for the leader-follower multi-robot networks, we have the following theorem.

**Theorem 7:** Consider the network formed by the multi-robot systems (20). If the directed graph of this network, $\mathcal{G}$, has a directed spanning tree, then there exist an integral terminal sliding variable vector and an ITSM control law for the multi-robot systems such that on the ITSM surface, consensus can be reached in finite time.

**Proof:** By defining the integral terminal sliding variable for the $i$th robot as:

$$s_i = e_i(t) - e_i(t_0) + \int_{t_0}^t \frac{\alpha}{2} e_i(t) + \frac{\beta}{2} e_i(t) dt,$$  (24)

the integral terminal sliding variable vector can be written as:

$$S = \mathbf{e}(t) - \mathbf{e}(t_0) + \int_{t_0}^t \frac{\alpha}{2} \mathbf{e}(t) + \frac{\beta}{2} \mathbf{e}(t) dt.$$  (25)

One can simply choose the ITSM control input as:

$$U = \begin{bmatrix} U_1^T, U_2^T, \ldots, U_n^T \end{bmatrix}^T,$$  (26)

with

$$U_i = \left( \sum_{j=1}^n a_{ij} U_j + b_i U_0 - \frac{\alpha}{2} s_i \right. \left. - \frac{\beta}{2} \mathbf{e}_i \right) \times \frac{\mathbf{e}_i}{2} \text{sign}(s_i),$$  (27)

and this results in

$$U = \left[ L + B \right] \otimes I_2 \left\{ \mathcal{B} \otimes I_2 U_0 - \frac{\alpha}{2} \mathbf{e}_i - \frac{\beta}{2} \mathbf{e}_i \mathbf{e}_i \right\},$$  (28)

Consider the Lyapunov function $V = \frac{1}{2} S^T S$. Using (20), (22), (25) and (28), a simple computation gives

$$\dot{V} = S^T \left\{ \dot{s}(t) + \dot{e}(t) + \frac{\beta}{2} e_i \mathbf{e}_i(t) \right\}$$

$$= S^T \left\{ \left[ L + B \right] \otimes I_2 \mathcal{B} \otimes I_2 U_0 + \frac{\alpha}{2} s_i - \frac{\beta}{2} \mathbf{e}_i \mathbf{e}_i \mathbf{e}_i \right\}$$

$$= S^T \left\{ -\frac{\alpha}{2} s_i - \frac{\beta}{2} \mathbf{e}_i \mathbf{e}_i \right\}$$

$$= -\alpha V - \beta V^\frac{\gamma}{2}. \quad \text{(29)}$$

By Theorem 4, expression (29) is the sufficient condition for the integral terminal sliding variable vector $S$ to converge to zero in fast finite time.

V. NUMERICAL EXAMPLE

This section presents the simulation result to illustrate the performance of the proposed ITSMCC scheme. Here, we
consider one leader robot indexed by 0 and four follower robots indexed by 1, 2, 3, and 4, respectively. Suppose that the robot dynamics are given in (19) and the directed graph in Figure 2 is used to model the information exchange among robots, where the information of the leader robot is available only to followers 3 and 4. Note that follower robot 4 has no directed path to all other follower robots, but there exist a directed path from the leader robot to all follower robots.

The Laplacian of the follower network can be written as:

$$L = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

and the diagonal matrices for the interconnection relationship between the leader and the followers is:

$$\bar{B} = \text{diag}(0011)$$

Figure 3 shows that the follower robots can track the leader robot in fast finite time.

VI. CONCLUSION

This paper has presented a robust fast finite time consensus tracking control scheme for leader-follower multi-robot network. The control design is based on the newly proposed integral terminal sliding mode control. It is proven that fast finite time consensus can be reached on the integral terminal sliding mode surface.

REFERENCES


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