Rapid method of stress intensity factor calculation for semi-elliptical surface breaking cracks under three-dimensional contact loading

D I Fletcher and A Kapoor
School of Mechanical and Systems Engineering, The University of Newcastle upon Tyne, Barrow Hill, UK

The manuscript was received on 20 June 2005 and was accepted after revision for publication 12 April 2006.

DOI: 10.1243/09544097JRRT27

Abstract: Fast methods of stress intensity factor calculation for inclined surface breaking cracks under contact loading are presented. Cracks are loaded in normal and tangential traction by a three-dimensional Hertzian elliptical contact patch, and friction between the crack faces is considered. Stress intensities are calculated from Green’s functions originally developed for two-dimensional cracks through application of stresses on a plane below the three-dimensional contact patch, in place of those previously considered for a two-dimensional contact. This approach gives the method great speed advantages over fully three-dimensional methods. Both semi-circular and semi-elliptical cracks are examined. The validity of the approximations and the results are judged by validation with results from alternative fully three-dimensional cases. Very good agreement is found between trends in stress intensity factor with changes in crack size and applied tractions. Absolute values of stress intensity factor agree well for semi-circular and shallow semi-elliptical cracks, but values were below those of the reference case for deep, narrow semi-elliptical cracks. Calibration of the model to overcome this under-prediction is discussed.

A case of special value in railway rail–wheel contact modelling is that of a contact offset to the side of a crack, representing the wheel running alongside rather than directly across an existing crack. This configuration results from the common procedure of grinding the rail to change or maintain its cross-sectional profile. The three-dimensional contact patch methods presented here enable this case to be modelled while retaining very fast running times for the calculations.

Keywords: stress intensity factor, contact loading, rail/wheel, crack face friction, grinding

1 INTRODUCTION

This article presents a new method of obtaining stress intensity factors for semi-circular and semi-elliptical surface breaking cracks under three-dimensional contact loading. Both mode I and II stress intensity factors are calculated, with the focus on shear mode crack growth influenced by friction between the crack faces. This friction may be modified by fluid in the crack, but the fluid is assumed to be unpressurized. The method retains the very high speed and simplicity usually associated with two-dimensional solutions through the use of an approximate approach to the three-dimensional calculations. This gives stress intensity factors in good agreement with those from alternative models, where these are available. The new method avoids the very time-consuming process of undertaking full three-dimensional modelling of three-dimensional cracks under elliptical contact loads. It is applicable to a much wider range of contact and crack shapes than are presented here, the range having been restricted to those required for validation of the method.

To illustrate the use of the new stress intensity factor calculation model, its application to railway...
rail–wheel contacts is presented. This application benefits greatly from the three-dimensional nature of the method and includes the case of a contact which is laterally offset from a crack (Fig. 1). The ability to calculate stress intensity factors (and hence crack growth rates) for cracks, which are laterally offset from a contact load, has particular application in the planning of rail grinding and rail re-profiling. These processes can relocate the wheel contact on the rail head, moving it away from existing cracks. The method provides quantification of the reduction of crack growth rate which this relocation can provide.

1.1 Review of existing alternative stress intensity factor methods

Factors, which any stress intensity factor calculation method for rolling contact fatigue (RCF) cracks must consider, include the effect of surface traction, the presence of fluid inside cracks, and the friction which exists between the crack faces when they are pressed together by the compressive contact load. The importance of fluids in the growth of RCF cracks was first identified by Way [1] in experimental work in the 1930s. Fluid may apply hydraulic pressure to the faces of cracks or vary the friction level between crack faces sliding over one another. These mechanisms are illustrated in Fig. 2. The complexity of accounting for the effect of fluids forms a common thread linking all the models reviewed subsequently, which together provide the background to the development of the new model.

1.1.1 Body force method

One of the earliest and most widely respected methods of calculating stress intensity factors for RCF cracks is the ‘Body Force Method’ developed by Kaneta and coworkers [2–4]. This was developed for semi-circular and semi-elliptical cracks and later applied to modelling more realistic crack shapes seen in twin-disc rolling contact tests and in rails [5, 6].

The work by Kaneta and Murakami (K&M) et al. considered fluid in the cracks as a transmitter of contact pressure to the crack faces by a hydraulic mechanism and also as a modifier of friction between the crack faces in regions where they are in contact with one another. Using the body force method, the areas of the crack which are open and closed may be predicted for every position of a contact passing over a crack. The action of internal fluid pressure in the cracks affects primarily the mode I (opening) stress intensity factor. Cases in which fluid was taken to apply no pressure and to act solely as a lubricant were also considered, and
these focus on the mode II (shearing) stress intensity factors and growth.

In the K&M models that include fluid pressure, this pressure was assumed to decrease linearly along the length of the crack, being equal to the contact pressure at the crack mouth (e.g. at a railway rail–wheel contact) and falling to zero at the crack tip. In a static system, such a variation would not exist, but it was chosen to represent the dynamic nature of a fluid briefly pressurized at the crack mouth and then allowed to depressurize after the contact has passed. In cases of contact and sliding between the crack faces, the stress driving crack growth was modified by the friction between the faces. Resistance to sliding was defined by the product of the ‘crack face friction coefficient’ and the force resolved normal to the crack face. Where the shear stress resolved parallel to the crack was insufficient to overcome this friction force, that region of the crack was assumed to be locked, whereas other regions were taken to be sliding. These distributions of pressure and friction were also applied in the later work of Fletcher and Beynon [7, 8], discussed in section 1.2.

A recent application of the body force method is in examining the combined effect of crack face fluid pressure and frictional heating on a crack under rolling–sliding contact [9]. Here it is found that with fluid pressure present, the crack growth is dominated by tensile mode growth (mode I), with limited thermal and frictional effects. Without fluid pressurization, the growth is by a shear mechanism (mode II), and thermal and frictional effects are larger.

1.1.2 Finite element and dislocation-based studies

The study undertaken by Bower [10] is one of the most comprehensive examinations of the mechanisms of RCF. Three possible mechanisms were examined to explain the experimental observations of RCF cracking. The mechanisms were as follows.

1. The cracks may propagate in shear (mode II) driven by the repeated reversing shear stresses due to the contact.
2. Fluid may be forced into the crack by the passing load, allowing the contact stress to be transmitted to the crack walls by hydraulic action, producing tensile (mode I) stresses at the crack tip, therefore, propagating the crack.
3. Fluid may enter the crack, become trapped as the load moves over the crack, and then become pressurized by the load. This ‘entrapment’ would also generate tensile stresses at the crack tip, propagating the crack.

Theoretical analysis showed that all the mechanisms considered by Bower [10] would produce some of the characteristics of RCF crack growth but that each also had deficiencies preventing it from completely explaining the growth observed in practice.

Mechanism A (mode II cyclic shear) was consistent with both the experimental and railway experience of RCF cracking, but it was found that propagation by cyclic shear was unlikely unless crack face friction coefficient for typical cracks was below 0.2. Mechanism B (direct hydraulic transmission of contact pressure, without entrapment) was found unable to account for the effect of load motion or traction direction on crack growth, although the stress intensity factor ranges predicted for this mechanism were sufficient to produce crack growth. Mechanism C (fluid entrapment) was found to predict sufficiently high stress intensities for crack growth, and these were sensitive to the direction of the motion of the load, as they should be to explain observed crack growth. However, the complex non-proportional mixed mode cycle of stress intensities at the crack tip made it difficult to predict the crack growth direction, so this could not be correlated with observed crack growth.

More recently, the work of Bogdanski and coworkers [11–14] has been significant in increasing the understanding of the mixed mode stress intensity factors generated during rolling/sliding contact such as that between a rail and a wheel. Most importantly, Bogdanski and coworkers have moved from two-dimensional modelling to full three-dimensional cases, with the ability to predict the full cycle of stress intensity factor to which the crack will be subjected during the passage of a contact. Much of this work was conducted in association with the EU project ICON [15] and considers the ‘squeeze film’ lubrication mechanism [11] to account for the action of fluid within the crack. This theory was developed for situations in which lubricant is alternately squeezed out and drawn into a gap and, therefore, represents many of the properties of cracks in rolling contact. However, although very comprehensive, this modelling method has the major drawback that its complexity makes model generation very time-consuming, and solving the models is slower than for alternative approaches.

Concentrating on crack initiation and the very early phases of propagation, extensive finite-element models of the rail–wheel contact have also been produced by the group at Chalmers University in Sweden [16–18]. These models use approaches such as cycle by cycle modelling of ratchet strain accumulation, low-cycle fatigue, and a critical plane approach in which a search algorithm is used to find the most damaging crack plane, using the stress and strain tensors determined from FE analyses. The methods are focused on crack initiation modelling and have not been developed for growth rate prediction of longer cracks.
Alternative work on this early stage of crack growth and initiation has been undertaken by Franklin and coworkers [19, 20] on the basis of ratchetting strain accumulation. This work combines the prediction of wear and the early stages of RCF through a computer simulation which avoids the lengthy computational characteristic of finite element analysis.

Moving away from finite-element methods, Nowell and Hills [21, 22] present methods for the calculation of stress intensity factors for both open and closed surface breaking cracks, using a technique of edge dislocation distribution along the line of a crack within a body. The technique had previously been used by Keer and Bryant [23] to model a crack in which both fluid pressurization and crack face friction were considered. More recently, Seed [24] has used this technique to obtain mode I and II stress intensity factors for an inclined crack in a half-plane under loading from an indenter of arbitrary profile. The work showed that contact friction greatly influenced the mode I and II stress intensity factors calculated.

1.2 Influence and Green’s functions

Green’s functions are used to apply results from a stress intensity factor calculation (e.g. boundary element analysis) for a wide range of system input conditions. For example, if a boundary element analysis provides the stress intensity factor produced by individual point forces at positions along a crack face, Green’s functions provide means to treat an arbitrary crack face load as a series of point forces which would be required to close up a crack and to sum up the combined stress intensity factor for the entire arbitrary load.

Treating the stresses present along the line of an inclined surface breaking crack as a series of point loads, the stress can be converted into a stress intensity factor using Green’s functions developed by Rooke et al. [25]. The method of stress intensity factor calculation developed by Fletcher and Beynon (F&B model) [7, 8] was based on this approach and is further developed in this article. Figure 3 illustrates the method, which is also summarized by equation (1) in which \( d \) is the crack length, \( \eta \) is a dummy variable representing the coordinate direction running along the length of the crack, \( \sigma \) is the stress distribution, and \( g \) is the Green’s function. Separate Green’s functions are required for forces normal and parallel to the crack faces and for the contribution of these forces to mode I and II stress intensity factors to be assessed. Full details are given in the references for the F&B model.

\[
K = \frac{1}{\sqrt{\pi d}} \int_0^d \sigma(\eta) g(\eta) \, d\eta
\]  

Fig. 3 Schematic representation of a surface crack showing stress intensity factor calculation by combining the stress distribution and Green’s function along the line of the crack. Separate Green’s functions are needed to calculate the effect of stress normal and parallel to the crack on the modes I and II SIF. Stress distributions are those in the uncracked body.

Green’s functions were originally developed for fretting fatigue situations but are not restricted to this source of surface contact stress. To apply Green’s functions, the stresses present along the line of the crack are calculated for the uncracked body. These are much more rapidly calculated than if the stress distribution for the cracked body itself was required. This approach is in agreement with Bueckner’s principle [26] which states that the forces which would be required to close up a crack must be equivalent to the stress distribution in an uncracked body of the same geometry subject to the same external loading. In the work of Rooke et al. [25], Green’s functions are based on boundary element models of the cracked surface, although similar functions could be produced from alternative stress intensity calculation models.

The F&B model is basically two-dimensional (i.e. a line contact passes over an infinitely wide inclined crack) but it includes a geometry term to produce results for semi-circular cracks. The derivation of this term is presented in the original publication of the models [7, 8] and is outlined in section 2.2 in which the model is extended to both deep and shallow semi-elliptical crack geometries. The stress distribution in the F&B model is that of a Hertzian line contact [27], with input parameters being the Hertzian contact pressure, surface traction coefficient, crack face friction coefficient, and crack size. The model is extremely fast to run, taking only a few seconds on a desktop computer to give results for the entire passage of a wheel contact over a crack. Crack growth is assumed to take place either by pressurization of fluid inside the crack (rainwater, oil, etc.) producing a crack opening stress or by shear of the crack faces, in which case fluid entry into the
crack modifies the friction between the crack faces, but does not apply pressure to these faces. These mechanisms are summarized in Fig. 2.

The F&B model produces results in good agreement with those from the body force method described in section 1.1.1. Both models are based on very similar assumptions about the stresses driving crack growth and the ways in which lubricants affect crack growth. Further development of the F&B model to include elliptic contact patches and additional crack shapes is described subsequently.

2 AN APPROXIMATE APPROACH TO THREE-DIMENSIONAL CONTACTS

The F&B model is advanced here in three ways. First, the contact is changed from a two-dimensional Hertzian line contact to a three-dimensional elliptical contact. Second, further crack shapes are considered allowing stress intensity factor calculation for shallow and deep elliptical cracks as well as semi-circular cracks. Third, the use of a three-dimensional elliptical contact stress distribution allows the contact and crack to be offset from one another as shown in Fig. 1. This configuration is particularly valuable in application of the work in railway rail–wheel contact.

2.1 Elliptical contact patches

Green’s functions developed by Rooke et al. [25], which underlie the stress intensity factor calculation method developed by F&B, are for two-dimensional cracks. However, their use depends only on the stress present along the line of the crack, and this stress can be generated by any arbitrary surface loading on the boundary of the cracked body. Previously, a two-dimensional contact loading was used in the stress intensity factor calculation, but in the newly developed cases, this is replaced by a three-dimensional Hertzian elliptical contact patch. For a contact running centrally over a crack, it is assumed that the stress on the plane below the centre-line of the three-dimensional contact patch controls crack growth. In rail–wheel contact, successive wheels may not follow each other over exactly the same path. Different parts of the crack may therefore be below the contact centre-line and subject to the highest levels of stress. The method presented here is for a central contact position only, but contacts laterally offset from the crack are discussed in section 2.3. This plane is where the highest stresses will lie for a contact under normal pressure and tangential traction in the direction of motion across the crack. Importantly, the stress on this plane is calculated for a three-dimensional elliptical contact rather than a two-dimensional contact and therefore depends on the lateral size of the contact as well as its longitudinal dimension. To reflect the combination of two- and three-dimensional components, the model has become known as the ‘2.5d’ model [28], and this is shown schematically in Fig. 4.

An elliptical contact patch produces stresses which diminish more rapidly with depth into the material than does a line contact [29], and this difference is successfully captured by combining the three-dimensional contact patch with the two-dimensional Green’s functions.

Stress diminishes with increasing depth into the material, but also falls with increasing lateral distance either side of an elliptical contact. With the exception of gross offset of the contact away from the crack (section 2.3), it is not possible to capture this reduction or its effect on crack locking and closure across the crack faces because of the underlying two-dimensional nature of the method. Similarly, the variation of stress intensity factor with position around the crack front of a three-dimensional crack cannot be captured. Stress intensity factors are based on the locking and closure of the crack below the centre-line of the contact. This represents the deepest and, therefore, most critical point of the crack front for determining growth.

2.2 Semi-elliptical cracks

As in previous models [7, 8], Green’s functions underlying the stress intensity factor calculations

![Fig. 4 Schematic representation of the ‘2.5d’ model. Stresses on the centre-line below a three-dimensional elliptical contact are used to predict crack growth using Green’s functions for an infinitely wide ‘slot’ type crack. A geometry factor is used to translate the results to a SIF for the deepest point of a semi-elliptical crack](image)
produce stress intensity factors for an infinitely wide slot crack. This can be converted to values for different crack shapes using a geometry factor. Equation (2) shows how stress intensity factor \( K \) for a given crack shape is calculated from the value for an infinitely wide crack \( K_{\infty} \) using a geometry factor \( Y \).

\[
K = K_{\infty} \times Y \tag{2}
\]

In the F&B model, the only crack shape considered was a semi-circle, with a geometry factor of 0.59 in equation (2). Calculation of geometry factors for shallow and deep semi-elliptical cracks (Fig. 5) was carried out on the basis of standard solutions for beams in tension and pure bending, with the geometry factors found by taking the ratio of the solutions for infinitely wide and semi-elliptical cracks under these loading conditions [30]. This approach allows a wide range of semi-elliptical crack shapes to be considered, but to enable validation of the results from the newly developed model, two crack shapes for which alternative stress intensity factor results are available were chosen.

Kaneta and Murakami [4] considered a wide aspect semi-elliptical crack (Fig. 5(d)) having a length to crack mouth width ratio of 0.5. For this crack, the geometry factor \( Y \) of 0.8 was calculated from standard solutions for cracks normal to the surface of a beam loaded in bending and tension. These calculations assumed the crack to be much smaller than the beam so as to be insensitive to the crack size and to remain similar to the current case, which is modelling a crack in a half-space. Kaneta and Murakami also considered a narrow aspect crack (Fig. 5(d)) giving a geometry factor value of 0.376. The progressive change in geometry factor, with the change from wide aspect crack to semi-circular crack and then the narrow aspect crack, indicates the increasing deviation from the infinitely wide slot crack for which the underlying calculations are performed.

### 2.3 Application to railways: offset contacts

A common case in which inclined surface breaking cracks develop is in railway rail–wheel contact. In railway rails cracks typically develop in the band on which the wheel contact runs, but this ‘running band’ can be relocated to a different position on the rail head by grinding the rail to change its cross-sectional profile [31, 32]. The prediction of stress intensity factors (and hence crack growth rate) for cracks which are offset from a contact is therefore of use in predicting the extension of rail life which grinding may produce and the frequency at which grinding should be conducted.

Using an elliptical contact patch rather than the line contact used in previous models [7, 8], the contact can be moved laterally relative to the crack, as it is no longer of infinite width. This allows the centrelines of the contact and crack to be offset from one another, as they may be following grinding of the rail surface, and is shown in Fig. 1. In the ‘2.5d’
model, this offset is represented by assuming that it is the stress on a plane offset laterally from the centre of the contact which is driving crack growth, rather than the stresses on the plane below the centre-line of the contact, which were used when the crack and contact were inline with one another.

Results for the 'offset contact' model are presented in section 3.2. Although the results of these calculations are of great interest for railway application, they are presented with the caveat that there are currently no alternative solutions against which they can be compared and validated. Such comparisons may be undertaken through development of full three-dimensional finite or boundary element models, but this is a long term and complex task outside the scope of the current project. In particular, such models must consider the friction and sliding of the crack faces under compressive loading, and multiple solutions are needed to understand the range of stress intensities present during the passage of the contact across the crack. The new results have, however, been calculated in the same way as the other results presented for validation of the model, with the exception of using stresses on a plane offset from the centre-line of the contact.

An additional consideration for rails undergoing grinding is the effect of metal removal on the crack growth rate. This must be considered in addition to any offsetting of the contact patch, because both processes vary the crack length and net growth rate. The effects of interaction between these two processes (and other metal removal processes such as wear or corrosion) is the key to assessing the effect of grinding on rail-rolling contact fatigue life [33–35].

3 RESULTS, VALIDATION, AND DISCUSSION

Results are presented first for contacts without offset between the crack and the contact patch ('in-line contacts') and then for cases in which there is a lateral offset between the crack and contact centre-lines.

The stress intensity factors for in-line contacts are presented in the non-dimensional form used by K&M [4] to model a circular contact patch crossing the centre-line of a semi-circular crack. Figure 6 shows the stress intensity factor results for this case, together with results from the '2.5d' model for the same contact size and shape. The crack modelled is a short crack, with a radius of 0.1 times the contact patch radius, lying at 45° below the surface. For a rail–wheel contact, this would give a crack radius of ~0.5–1 mm.

The results shown in Fig. 6 indicate good agreement between the K&M reference data and the output of the '2.5d' model. The agreement at low values of surface traction is extremely good. At higher surface traction levels (the highest examined was 0.3), the '2.5d' model indicates a stress intensity factor range of around 80 per cent that predicted by K&M [4]. This deviation is almost identical to that observed for the previously published line contact versions of the '2.5d' model [7, 8].

Most importantly, the trends predicted by both the K&M data and the '2.5d' model are identical, i.e. both models predict that stress intensity factor values (and hence crack growth rates) fall as the crack face friction coefficient increases and rise with increasing surface traction.

3.1 In-line contacts

3.1.1 Circular contact patch, semi-circular crack

Figure 5(a) shows the crack configuration used by K&M [3] to model a circular contact patch crossing the centre-line of a semi-circular crack. Figure 6 shows the stress intensity factor results for this case, together with results from the '2.5d' model for the same contact size and shape. The crack modelled is a short crack, with a radius of 0.1 times the contact patch radius, lying at 45° below the surface. For a rail–wheel contact, this would give a crack radius of ~0.5–1 mm.

The results shown in Fig. 6 indicate good agreement between the K&M reference data and the output of the '2.5d' model. The agreement at low values of surface traction is extremely good. At higher surface traction levels (the highest examined was 0.3), the '2.5d' model indicates a stress intensity factor range of around 80 per cent that predicted by K&M [3]. This deviation is almost identical to that observed for the previously published line contact versions of the '2.5d' model [7, 8].

Most importantly, the trends predicted by both the K&M data and the '2.5d' model are identical, i.e. both models predict that stress intensity factor values (and hence crack growth rates) fall as the crack face friction coefficient increases and rise with increasing surface traction.

3.1.2 Elliptical contact patch, wide aspect semi-elliptical crack

Figures 5(b) and (c) show the crack configuration used by K&M [4] to model an elliptical contact patch crossing the centre-line of a semi-elliptical crack. Two different orientations of contact patch were modelled, type A with its short axis parallel to the direction of motion and type B with its longer axis parallel to this direction. The ratio of major to minor axis length was 5, the cracks modelled were

\[
K = Fp_0 \sqrt{\pi h} \tag{3}
\]

Stress intensities are presented using the shear mode stress intensity factor \( K_I \) defined by equation (4) [36], which combines mode I (\( K_I \)) and mode II (\( K_{II} \)) stress intensities at an angle \( \theta \) ahead of the crack and is solved to find the largest shear mode stress intensity present

\[
K_I = \frac{1}{2} \cos \left( \frac{\theta}{2} \right) [K_I \sin \theta + K_{II}(3 \cos \theta - 1)] \tag{4}
\]

Graphs of the stress intensity factors show the normalized position of the contact patch relative to an origin at the crack mouth. In each case, the position is normalized by the contact half-width (or radius for circular contacts) lying in the direction of motion across the crack.
at 45° below the surface, and a crack face friction coefficient of 0.5 was used in all cases. Both the wide and narrow aspect semi-elliptical cracks (Fig. 5(d)) were modelled. Although these contact configurations were chosen for validation with existing data, the current model may be applied to a wide range of contact and crack shapes and to cracks from vertical through to 30° below the surface.

Fig. 6 Normalized SIF results for shear growth of a semi-circular crack beneath a circular contact patch. Results from Kaneta et al. (K&M) are shown together with results from the ‘2.5d’ model. The key indicates the surface traction coefficient applied in each case. (a) Crack face friction coefficient of 0.2. (b) Crack face friction coefficient of 0.5.
For the wide aspect semi-elliptical cracks, K&M [4] modelled a crack with a mouth width equal to one-fifth of the major contact ellipse dimension. The crack length into the body was one-quarter of this crack mouth width. Figure 7 shows non-dimensional shear mode stress intensity factor results (see section 3 for details of the non-dimensionalization) for both type A and type B cracks.

Fig. 7 Normalized SIF results for shear growth of a wide aspect semi-elliptical crack beneath an elliptical contact patch. Results from Kaneta and Murakami (K&M, thick lines) are shown together with results from the ‘2.5d’ model (thin lines). Curves are labelled with the surface traction coefficient applied in each case. (a) Type A contact patch, minor axis parallel to the direction of travel and (b) type B crack, major axis parallel to the direction of sliding.
elliptical contact patches crossing this wide aspect semi-elliptical crack.

Comparison of the ‘2.5d’ model results for the wide aspect semi-elliptical crack with those for the same shape crack modelled by K&M [4] showed very good agreement of both the absolute values of the stress intensity factors predicted and of the trends in these values with changes in crack face and surface friction coefficients. As for the semi-circular cracks in section 3.1.1, the best agreement was for the lowest surface friction levels. At the highest surface friction level examined (coefficient of friction 0.3), the stress intensity factor range predicted by the ‘2.5d’ model was approximately 83 per cent of that predicted by Kaneta and Murakami. This indicates slightly better agreement than for semi-circular cracks.

For contact orientations A and B, both the ‘2.5d’ and K&M data predicted increasing stress intensity factors (and hence crack growth rate) with increasing surface friction levels, and that stress intensity factors were higher for elliptical contact patches whose major axis lies parallel to the direction of motion (type B).

3.1.3 Elliptical contact patch, narrow aspect semi-elliptical crack

Reference data from K&M [4] on narrow aspect semi-elliptical cracks used the same contact types A and B used for the wide aspect cracks, but with a crack length into the body equal to the crack mouth width, as shown in Fig. 5(d).

Results for the narrow aspect crack are shown in Fig. 8. In this case, the results from K&M [4] and those from the current model are presented on separate plots for clarity because the curves otherwise cross over one another. The scales of all the graphs are the same to enable comparisons to be made.

From the results for the narrow aspect cracks, it can be seen that the form of the curves and the trend with changes in surface friction levels are almost identical for the ‘2.5d’ and the Kaneta and Murakami model. However, the magnitudes of the stress intensity values predicted by the ‘2.5d’ are lower than those predicted by the K&M model. In the worst case, which as for the other crack shapes is at the highest surface friction levels, the ‘2.5d’ model predicts a stress intensity factor range of around 65 per cent of that predicted by the Kaneta and Murakami model. That of all those examined, the stress intensity factors for the narrow aspect crack have the least good agreement with the K&M data is to be expected because the narrow aspect crack differs most greatly from the infinitely wide slot for which the underlying calculations are performed.

The geometry factor used for the narrow aspect semi-elliptical cracks was derived from the ratio of stress intensity factors for semi-elliptical and infinitely wide crack shapes found using standard solutions for beams in tension and bending. The factor was calculated for cracks normal to the surface. It was initially thought that the application of this factor for an inclined crack was partially responsible for the difference between the current results and the K&M data. This possibility was discounted because much better agreement was found for the shallow semi-elliptical and semi-circular inclined cracks, showing that the difference in inclination is unlikely to be a significant factor, as the geometry factors for these cracks were calculated in the same way. There is, however, nothing to prevent the use of a different geometry factor value which can give much better agreement between the ‘2.5d’ and alternative models for deep narrow cracks, and this is effectively a calibration of the ‘2.5d’ model. Figure 9 shows the results from both ‘2.5d’ and K&M models if a factor Y of 0.564 is used in place of the standard value of 0.376. It can be seen that the difference between the corresponding curves has been reduced, with less than 10 per cent difference in stress intensity factor range in the worst case.

3.2 Offset contacts

The effect of off-setting the contact patch from the centre-line of an existing crack is dependant on the size of the crack modelled relative to the contact patch and also the degree of offset. Results for offset contacts are presented in dimensional form using contacts and units appropriate for the railway application of the method. A maximum Hertzian contact pressure of 1500 MPa was considered, with an elliptical contact patch of 4.1 mm wide by 6.2 mm long in the direction of motion.

Figure 10 shows the shear mode stress intensity factor \( K_s \) experienced by a semi-circular crack of radius 4.7 mm, inclined at 30° below the rail surface. A crack of this size would present a 9.4 mm visible crack mouth on the rail surface. The contact offsets were chosen to include cases in which the contact remained over the majority of the crack, and cases in which it moved completely off the crack to run beside it. (While using the terms ‘over’ and ‘beside’ the crack, it should be noted that the model does not directly recognize these positions. It instead relies on the separation between the centre-lines of the contact and the crack to calculate the reduced stress acting on the crack when the contact is offset.)

The results show that a small offset of the contact, by 1.6 mm, or 40 per cent of the crack radius,
Fig. 8 Normalized SIF results for shear growth of a narrow aspect semi-elliptical crack beneath an elliptical contact patch. Curves are labelled with the surface traction coefficient applied in each case. (a) Type A contact patch, minor axis parallel to the direction of travel (K&M). (b) Type A contact, results from the ‘2.5d’ model. (c) Type B contact patch, major axis parallel to the direction of travel (K&M). (d) Type B contact, results from the ‘2.5d’ model.
produced a 10 per cent reduction in the predicted shear mode stress intensity factor (SIF). Increasing the offset to 3.1 mm (76 per cent of the contact radius) produced a reduction in SIF by almost 40 per cent. However, as would be expected, the largest reduction was with an offset which would move the contact to run beside rather than over the crack.

Interpretation of the significance of the reductions in SIF requires a crack growth law for RCF. Such a law is available from the work of Bold et al. [37]
Fig. 9  Example of calibration using a modified geometry factor to create agreement between normalized shear SIF results from the Kaneta and Murakami (K&M) model and those from the ‘2.5d’ model. The geometry factor $Y$ was changed from 0.376 to 0.564.

Fig. 10  Shear mode SIF results for an elliptical contact patch of 4.1 mm wide by 6.2 mm long in the direction of motion for a semi-circular crack of 4.7 mm radius. Maximum Hertzian contact pressure 1500 MPa, with surface and crack face friction coefficients of 0.18, and the crack inclined at 30° below the surface.
Table 1 Stress intensity and crack growth rate predictions for a contact offset from an existing crack, based on a semi-circular crack of 4.7 mm radius. Without offset, the tensile mode stress intensity factor range ($\Delta K_I$) was 3.75 MPa m$^{1/2}$ and the shear mode range ($\Delta K_{II}$) was 22.0 MPa m$^{1/2}$, giving a predicted growth rate of 14 nm per contact cycle.

<table>
<thead>
<tr>
<th>Contact offset (mm)</th>
<th>Offset as a proportion of crack radius</th>
<th>Percentage $\Delta K_I$</th>
<th>Percentage $\Delta K_{II}$</th>
<th>Percentage crack growth rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>1.6</td>
<td>0.4</td>
<td>86</td>
<td>90</td>
<td>71</td>
</tr>
<tr>
<td>3.1</td>
<td>0.76</td>
<td>52</td>
<td>61</td>
<td>19</td>
</tr>
<tr>
<td>4.7</td>
<td>1.15</td>
<td>31</td>
<td>19</td>
<td>0</td>
</tr>
<tr>
<td>6.2</td>
<td>1.50</td>
<td>21</td>
<td>14</td>
<td>0</td>
</tr>
<tr>
<td>7.8</td>
<td>1.90</td>
<td>15</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

who have conducted extensive mixed mode fatigue tests on rail steel. Results for offset contacts are presented in Table 1 and, on the basis of a threshold SIF range of 4 MPa m$^{1/2}$, the final column of Table 1 indicates the growth rate for each offset, as a percentage of the rate for the contact without an offset. The results show that the crack growth rate is sensitive to even small changes in stress intensity factor, so even small offsets of the contact can dramatically reduce crack growth rate. A 3.1 mm offset, which may be very easily achieved by changing the cross-sectional profile of the rail through grinding, is predicted to cut crack growth rate by over 80 per cent. It should, however, be noted that for larger cracks, this offset would represent a smaller proportion of the crack radius, and its impact in reducing crack growth rate would be reduced to below that shown for the 4.7 mm radius (9.4 mm visible surface crack length) case presented here.

4 CONCLUSIONS

A model has been developed for the rapid calculation of shear mode stress intensity factors for inclined surface breaking semi-elliptical cracks beneath an elliptical contact patch. Such cracks are typically found in railway rail–wheel contact. The method makes a number of assumptions to allow the utilization of existing two-dimensional crack solutions to solve this three-dimensional crack and contact problem, avoiding the extremely time-consuming process of full three-dimensional modelling.

Results from the new model were compared with data available for semi-circular cracks and two forms of semi-elliptical crack beneath circular and elliptical contact patches. In all cases, the trends with movement of the contact patch across the crack and with surface friction were the same for the new model and the reference cases by Kaneta and Murakami. Absolute values of SIF were closest to the reference case for shallow semi-elliptical cracks (within 15 per cent) but were less good for deep narrow cracks, which showed up to 35 per cent reduction in SIF relative to the reference case. That the deep narrow crack showed the least good agreement between the cases was thought to be because this crack geometry differed most greatly from the underlying two-dimensional crack on which the calculations are based. Calibration of the model is possible to improve results for deep narrow cracks.

The method may be applied to a wide range of crack and contact patch shapes, although those presented here are restricted to the cases available for validation of the method. An extension is described to assess crack growth when an elliptical contact runs to the side of an existing crack (i.e. the centre-lines of the crack and contact are offset). This case has application in railway RCF modelling of rails in which surface grinding re-positions the rail–wheel contact away from existing cracks. Using typical values for rail–wheel contact, it was predicted that a contact offset of just 3.1 mm (76 per cent of the crack size) would reduce crack growth rate to less than 20 per cent of its value without an offset between the crack and contact, for a 4.7 mm radius crack (i.e. 9.4 mm visible surface crack length).

5 ACKNOWLEDGEMENTS

The authors would like to thank the Railway Safety and Standards Board, and the Engineering and Physical Sciences Research Council Rail Research UK (RRUK) for funding this work.

REFERENCES

5 Murakami, Y., Sakae, C., and Ichimaru, K. Three-dimensional fracture mechanics analysis of pit formation mechanism under lubricated rolling–sliding

APPENDIX

Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>crack mouth half-width at the surface</td>
</tr>
<tr>
<td>d</td>
<td>maximum crack length inside the body</td>
</tr>
<tr>
<td>F</td>
<td>general non-dimensional stress intensity factor</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$g$</td>
<td>Green’s function</td>
</tr>
<tr>
<td>$h$</td>
<td>crack radius for the case of circular contact with a semi-circular crack, or the major axis of the contact ellipse for cases of elliptical contact</td>
</tr>
<tr>
<td>$K$</td>
<td>general dimensional stress intensity factor</td>
</tr>
<tr>
<td>$K_I$</td>
<td>mode I dimensional stress intensity factor</td>
</tr>
<tr>
<td>$K_{II}$</td>
<td>mode II dimensional stress intensity factor</td>
</tr>
<tr>
<td>$K_Y$</td>
<td>equivalent shear stress intensity factor</td>
</tr>
<tr>
<td>$p_0$</td>
<td>maximum Hertzian contact pressure</td>
</tr>
<tr>
<td>$Y$</td>
<td>geometry factor</td>
</tr>
<tr>
<td>$\eta$</td>
<td>coordinate direction of the length of the crack</td>
</tr>
<tr>
<td>$\theta$</td>
<td>angle ahead of a crack tip at which equivalent stress intensities are calculated</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>general stress distribution in the uncracked body, along the line of the crack</td>
</tr>
</tbody>
</table>