Bifurcation, amplitude death and oscillation patterns in a system of three coupled van der Pol oscillators with diffusively delayed velocity coupling

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In this paper, we study a system of three coupled van der Pol oscillators that are coupled through the damping terms. Hopf bifurcations and amplitude death induced by the coupling time delay are first investigated by analyzing the related characteristic equation. Then the oscillation patterns of these bifurcating periodic oscillations are determined and we find that there are two kinds of critical values of the coupling time delay: one is related to the synchronous periodic oscillations, the other is related to eight branches of asynchronous periodic solutions bifurcating simultaneously from the zero solution. The stability of these bifurcating periodic solutions are also explicitly determined by calculating the normal forms on center manifolds, and the stable synchronous and stable phase-locked periodic solutions are found. Finally, some numerical simulations are employed to illustrate and extend our obtained theoretical results and numerical studies also describe the switches of stable synchronous and phase-locked periodic oscillations. © 2011 American Institute of Physics.

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Coupled dynamical system arises in a variety of contexts in the physical, biological, and social sciences and has broad relevance to many areas of research. Synchronization phenomena in the coupled system is ubiquitous and of interest to researchers in different research fields. Generally speaking, time delay is inevitable in the coupled systems since the information flows between the individual units are never conveyed instantaneously, but only after some time delay. A nature question is how delay coupling affects the synchronization patterns. Here, we consider a delay-coupled system consisting of three van der Pol oscillators and attempt to analytically investigate how the coupling time delay and the coupling strength affect the stability and synchronization patterns. Hopf bifurcations and amplitude death induced by the coupling time delay are investigated and the corresponding relation between the synchronization patterns of Hopf bifurcating periodic oscillations and the critical values of the coupling time delay is explicitly found.

I. INTRODUCTION

Since synchronization phenomena is frequently encountered in nature and can help to explain many phenomena in biology, chemistry, physics and has potential applications in engineering and communication, it has been considered as one of the fundamental characteristics of collective dynamics of coupled systems and has permanently remained an object of intensive research in nonlinear science. System of coupled self-oscillators is often regarded as a basic model in the theory of oscillators and nonlinear dynamics dealing with synchronization phenomenon. A system of coupled nonlinear oscillators is capable of displaying a rich dynamics and has applications in various areas of science and technology, such as physical, chemical, biological, and other systems.1–4 In particular, the effect of synchronization in systems of coupled oscillators nowadays provides a unifying framework for different phenomena observed in nature (for more reviews, see Refs. 5–7).

The van der Pol equation has long been studied as a quintessential example of a self-excited oscillator and has been used to represent oscillations in a wide variety of applications.8,9 It evolves in time according to the following second order differential equation:

\[ \dot{y}(t) + \varepsilon (y^2(t) - 1)\dot{y}(t) + x(t) = 0, \tag{1} \]

where \( y \) is position coordinate and \( \varepsilon \) is a positive real number indicating the damping strength. It is well known10 that Eq. (1) has a unique periodic solution which attracts all other orbits except the origin, which is the unique equilibrium point of Eq. (1) and unstable. It has been shown in Refs. 11 and 12 that linear feedback can annihilate limit cycles and stabilize the origin.

Recently, there are increasing interests on the study of the synchronization behavior in the coupled van der Pol oscillators (see, e.g., Refs. 13–17 and references therein). In Ref. 13, the synchronization of four coupled van der Pol oscillators has been carried out. It has been shown that, for the coupling constant below a critical value, there exists a region in which a diversity of phase-shift attractors is present, whereas for values above the critical value an in-phase attractor is predominant. It is also observed that the presence

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of an antiphase attractor in the subcritical region is associated with sudden changes in the period of the coupled oscillators. In Ref. 14, the synchronization in the system of coupled nonidentical nonisochronous van der Pol–Duffing oscillators with inertial and dissipative coupling has been studied. In Ref. 15, Ookawara and Endo have investigated the effect of element value deviation on the degenerate modes in a ring of three and four coupled van der Pol oscillators. By using the averaging method, they proved that, for a ring of three coupled oscillators, two frequencies bifurcate from the degenerate mode, synchronize if they are close enough, but lose synchronization when they are separated to some extent; while for a ring of four coupled oscillators, the two frequencies generally cannot be synchronized, even if they are close enough. In Ref. 16, Woafo and Enjieu Kadji have investigated different states of synchronization in a ring of mutually van der Pol oscillators. Using the properties of the variational equations of stability, they calculated the good coupling parameters leading to complete and partial synchronization or disordered states and obtained a stability map showing domains of synchronization to an external excitation locally injected in the ring. In Ref. 17, the stability of the synchronization manifold in a ring and in an open-ended chain of nearest neighbor coupled self-sustained systems, each self-sustained system consisting of multilimit cycle van der Pol oscillators, has been investigated and it has been found that synchronization occurs in a system of many coupled modified van der Pol oscillators, and it is stable even in the presence of a spread of parameters.

However, the coupling delay is not considered in the literatures mentioned above. In fact, the coupling is certainly not instantaneous and might exhibit propagation delays which are comparable to or larger than the characteristic oscillatory time scale of the individual oscillators. It has been shown that a coupling delay can induce complex behavior in a network and the nodes organize in different synchronization patterns. For example, the coupling delay can induce the amplitude death even for zero frequency mismatch between the limit cycle oscillators, which does not occur in the case without coupling time delay, can lead to chaos when two lasers are coupled face to face and cause the oscillators to switch from being in-phase to being out-of-phase in various types of delay-coupled oscillators.

In the following, we consider a system of three delay-coupled van der Pol oscillators where three such oscillators are diffusively coupled through the damping terms

\[ \dot{y}_i(t) + \varepsilon (y_i^2(t) - 1)y_i(t) + y_i(t) = k(y_{i+1}(t - \tau) + y_{i+2}(t - \tau) - y_i(t)), \quad i = 1, 2, 3 \pmod{3}, \]

(2)

where \( k \geq 0 \) represents the coupling strength and \( \tau \geq 0 \) is the coupling time delay. It is well known that the dynamics of the coupled system depends on the properties of each of the units and on their interactions. Clearly, in system (2), the parameters \( k \) and \( \tau \) reflect the interactions of three van der Pol oscillators. Here, we aim at addressing the influence of the coupling strength and time delay on the stability and synchronized patterns of system (2). We will show that the coupling time delay will induce to stable synchronous periodic solutions, stable phase-locked periodic solutions, and unstable mirror-reflecting and standing waves.

We would like to mention that there are several works on delayed coupling van der Pol oscillators similar to system (2). In Ref. 26, Wirkus and Rand have studied the dynamics of a pair of van der Pol oscillators where the coupling is chosen to be through the damping terms but not of diffusive type. They used the method of averaging to reduce the problem to the study of a slow-flow in three dimensions and investigated the steady state solutions of this slow-flow, with special attention given to the bifurcations accompanying their change in number and stability. Sen and Rand entirely by numerical methods, investigated the dynamics of a pair of relaxation oscillators of the van der Pol type with delay position coupling and have found the various phase-locked motions including the in-phase and out-of-phase modes. Li et al. investigated the dynamics of a pair of van der Pol oscillators with delayed position and velocity coupling. Using the method of averaging together with truncation of Taylor expansions, they have found that the dynamics of 1:1 internal resonance is more complex than that of non-1:1 internal resonance. More recently, the system of a pair of coupled van der Pol oscillators has been studied by Atay and Song. In Ref. 29, Atay has investigated the synchronization and amplitude death using averaging theory. The parameter ranges for the coupling strength and delay are calculated such that the oscillators exhibit amplitude death or stable in-phase or antiphase synchronized oscillations with identical amplitudes. In Ref. 24, Song has investigated the stability and oscillation patterns induced by the coupling time delay. It was shown that for appropriate damping and coupling strengths there are stability switches and the synchronized dynamics changes from in-phase to out-of-phase oscillations (or vice versa) as the coupling time delay is increased and the critical values of coupling time delay leading to different oscillation patterns were explicitly calculated.

This paper is organized as follows. Local stability and delay-induced Hopf bifurcations are presented in Sec. II. Spatio-temporal patterns of bifurcating periodic solutions are investigated in Sec. III. In Sec. IV, we calculate the normal forms on center manifolds and then determine stability of bifurcating periodic orbits. Numerical simulations are employed to support and extend the theoretical analysis in Sec. V. A summary of the results is presented in Sec. VI.
The linearization of (3) at the zero solution leads to

\[ \dot{Y}_i(t) = MY_i(t) + N(Y_{i+1}(t - \tau) + Y_{i+r}(t - \tau)) \]

\[ i = 1, 2, 3 \pmod{3}. \]  

(5)

The characteristic matrix of linearized system (5) is given by

\[ \mathcal{M}_e(0, \lambda) = \begin{pmatrix} \lambda I_2 - M - \epsilon N e^{-i\tau} & -Ne^{-i\tau} -Ne^{-i\tau} \\ -Ne^{-i\tau} -Ne^{-i\tau} & \lambda I_2 - M - \epsilon N e^{-i\tau} \end{pmatrix}, \]

where \( I_2 \) is a 2 \times 2 identity matrix. Assuming that \( \eta_j \in \mathbb{R}^2 \) are eigenvectors, respectively, of \( \lambda I_2 - M - \epsilon N e^{-i\tau} - \epsilon Ne^{-i\tau}, j = 0, 1, 2 \), then it is easy to verify that

\[ \mathcal{M}_e(0, \lambda) \eta_j = \text{diag}\{T_2, T_2, T_2\} \eta_j, \quad j = 0, 1, 2, \]  

(6)

where \( T_2 = \lambda I_2 - M - \epsilon N e^{-i\tau} - \epsilon Ne^{-i\tau}, \quad \eta_j = (\eta_{j1}, \eta_{j2})^T, \quad \eta_j \in \mathbb{R}^2 \). So, we have

\[ \det \mathcal{M}_e(0, \lambda) = \prod_{j=0}^2 \det(\lambda I_2 - M - \epsilon N e^{-i\tau} - \epsilon Ne^{-i\tau}), \]

and then the characteristic equation of the linearization of (3) at the zero solution is

\[ \Delta(\tau, \lambda) = \det \mathcal{M}_e(0, \lambda) = \Delta_1 \Delta_2 = 0, \]  

(7)

where

\[ \Delta_1 = \left( \lambda^2 + (k - \epsilon)\lambda + 1 - 2k\epsilon e^{-i\tau} \right), \]

\[ \Delta_2 = \left( \lambda^2 + (k - \epsilon)\lambda + 1 + 2k\epsilon e^{-i\tau} \right). \]

Note the fact that the root \( \lambda \) of the characteristic equation 7 corresponds to the zero eigenvalue of the matrix \( \lambda I_2 - M - \epsilon N e^{-i\tau} - \epsilon Ne^{-i\tau}, j = 0, 1, 2 \), which will be an important information to calculate the eigenvector of system (5) for the purely imaginary roots of the characteristic equation 7 in the following two sections.

In the remaining of this section, we investigate the distribution of roots of Eq. (7), which determines the local stability of the zero solution of (2). It is convenient to start with a more general second order transcendental polynomial equation

\[ \lambda^2 + p\lambda + r + s\lambda e^{-i\tau} = 0, \quad p, r, s \in \mathbb{R} \text{ and } s \neq 0, \]  

(8)

which has been extensively studied (see, e.g., Refs. 30 and 31). From Refs. 30 and 31, we first have the following lemma.

Lemma 2.1. Assuming that \( r > 0 \), then we have the following.

(i) If \( |s| < |p| \), then Eq. (8) has no purely imaginary root.

(ii) If \( |s| > |p| \), then Eq. (8) has a pair of purely imaginary roots \( \pm \eta \lambda \pm \lambda \), respectively at \( \tau = \tau^+_j \) (or \( \tau = \tau^-_j \)), respectively such that \( \tau^+_j \neq \tau^-_j \) for any non-negative integer numbers \( j, k \), while Eq. (8) has two pairs of purely simple imaginary roots \( \pm \iota \lambda \) and \( \pm \iota \lambda \) at \( \tau = \tau_+^j \) (or \( \tau = \tau^-_j \)) for some non-negative integer numbers \( j \) and \( k \) such that \( \tau^+_j = \tau^-_k \).

(iii) If \( |s| = |p| \), then Eq. (8) has a pair of purely imaginary roots \( \pm \iota \sqrt{r} \) at \( \tau = \tau^+_j \).

(iv) Letting

\[ \lambda(\tau) = \eta(\tau) + i\alpha(\tau) \]

is a solution of Eq. (8) satisfying

\[ \eta(\tau) = 0, \quad \omega(\tau) = \omega \pm, \]

then we have

\[ \frac{d\Re(\lambda_1^\pm(\tau))}{d\tau} > 0, \quad \frac{d\Re(\lambda_2^\pm(\tau))}{d\tau} < 0, \quad \text{for } |s| > |p| \]  

(9)

and

\[ \frac{d\Re(\lambda_1^\pm(\tau))}{d\tau} = 0, \quad \text{for } |s| = |p|. \]

Here

\[ \omega = \frac{\sqrt{2}}{2} \left[ s^2 + 2r - p^2 \pm \sqrt{(s^2 + 2r - p^2)^2 - 4r^2} \right]^{1/2} \]

and

\[ \begin{cases} \tau^+_j = \frac{1}{\omega_j} (2j\pi + \pi - \arccos(\xi_j)), & \text{for } s > |p|, \\ \tau^-_j = \frac{1}{\omega_j} (2j\pi + \arccos(\xi_j)), & \text{for } s < |p|, \\ \tau^+_j = \frac{1}{\omega_j} (2j\pi + \pi), & \text{for } s = p, \\ \tau^-_j = \frac{1}{\omega_j} (2j\pi + \pi), & \text{for } s = -p, \end{cases} \]

with \( j \in \{0, 1, 2, \ldots\} \).

From Lemma 2.1, we have the following two lemmas on the distribution of roots of the equations \( \Delta_1 = 0 \) and \( \Delta_2 = 0 \).

Lemma 2.2. For the equation \( \Delta_1 = 0 \), we have the following results:

(i) When \( \tau = 0 \), the equation \( \Delta_1 = 0 \) has only two roots and they both have negative real parts for \( k < -\epsilon \), positive real parts for \( k > -\epsilon \), and zero real parts for \( k = -\epsilon \).

(ii) If \( -\epsilon < k < \epsilon/3 \), then the equation \( \Delta_1 = 0 \) has no purely imaginary root.

(iii) If either \( k < -\epsilon \) or \( k > \epsilon/3 \), then the equation \( \Delta_1 = 0 \) has a pair of purely simple imaginary roots \( \pm \iota \lambda \) (\( \pm \iota \lambda \), respectively) at \( \tau = \tau^+_j \) (\( \tau = \tau^-_j \), respectively) such that \( \tau^+_j \neq \tau^-_j \) for any non-negative integer numbers \( j, k \), while the equation \( \Delta_1 = 0 \) has two pairs of purely simple imaginary roots \( \pm \iota \lambda \) and \( \pm \iota \lambda \) at \( \tau = \tau^+_j \) (or \( \tau = \tau^-_j \)) for some non-negative integer numbers such that \( \tau^+_j = \tau^-_k \).

(iv) If either \( k = -\epsilon \) or \( k = \epsilon/3 \), then Eq. (8) has a pair of purely imaginary roots \( \pm \iota \lambda \) at \( \tau = \tau^+_j \).

Here,

\[ \omega^+_j = \frac{\sqrt{2}}{2} \left[ (3k - \epsilon)(k + \epsilon) + 2 \pm \sqrt{(3k - \epsilon)(k + \epsilon)(3k - \epsilon)(k + \epsilon) + 4} \right]^{1/2} \]

(10)
and

\[
\begin{align*}
\tau_{ij}^+ &= \frac{1}{\omega_i} (2\pi + \pi + \arccos(t_{\pi i} - 1)), \\
\tau_{ij}^- &= \frac{1}{\omega_i} (2\pi + \pi - \arccos(t_{\pi i} - 1)), & \text{for } k > \varepsilon/3, \\
\tau_{ij}^+ &= \frac{1}{\omega_i} (2\pi + \pi + \arccos(t_{\pi i} - 1)), & \text{for } -k < \varepsilon, \\
\tau_{ij}^- &= \frac{1}{\omega_i} (2\pi + \pi + \arccos(t_{\pi i} - 1)), & \text{for } k = \varepsilon/3, \\
\tau_{ij}^+ &= 2\pi + \pi, & \text{for } k = -\varepsilon, \\
\tau_{ij}^- &= 2(j + 1)\pi, & \text{for } k = -\varepsilon.
\end{align*}
\]

Lemma 2.3. For the equation \( \Delta_2 = 0 \), we have the following results:

(i) When \( \tau = 0 \), the equation \( \Delta_2 = 0 \) has only two roots and they both have negative real parts for \( k > \varepsilon/2 \), positive real parts for \( k < \varepsilon/2 \), and zero real parts for \( k = \varepsilon/2 \).

(ii) If \( k < \varepsilon/2 \), then the equation \( \Delta_2 = 0 \) has no purely imaginary root.

(iii) If \( k > \varepsilon/2 \), then the equation \( \Delta_2 = 0 \) has a pair of purely imaginary roots \( \pm i \omega^2_1 \) and \( \pm i \omega^2_2 \), respectively, \( \omega^2_2 > \omega^2_1 \), and all roots of Eq. (7) have negative real parts for \( k < \varepsilon \) and the zero equilibrium of system (2) is always unstable for all \( \tau > 0 \) and \( k > \varepsilon/2 \).

Theorem 2.1: Assuming that \( \tau_{ij}^+ \), \( \tau_{ij}^- \), and \( \tau_{ij}^0 \) are defined, respectively, by (11), (13), and (14), we have the following:

(i) If either \( k < \varepsilon \), then the zero equilibrium of system (2) is stable for all \( \tau > 0 \).

(ii) If \( k > \varepsilon \) and the assumptions (A1) and (A2) hold, then there exists a positive integer \( n \) such that \( \tau_{n-1} < \tau_{n-1}^+ < \tau_{n}^+ < \tau_n \) and all roots of Eq. (7) have negative real parts for \( \tau \in (\tau_n^+, \tau_{n+1}^-) \cup (\tau_{n+1}^+, \infty) \).

(iii) Near the zero equilibrium, system (2) undergoes Hopf bifurcations at the critical values \( \tau_{ij}^+ \) and undergoes equivalent Hopf bifurcations at the critical values \( \tau_{ij}^- \).

According to Theorem 2.1, the stability switches region in the plane of damping and coupling strengths is plotted in Fig. 1. In the shaded region of Fig. 1, there exist stability switches for the zero equilibrium of system (2) with increasing the coupling time delay \( \tau \), and in other regions, the zero equilibrium of system (2) is always unstable for all \( \tau > 0 \). For fixed damping strength, say \( \varepsilon = 0.03 \), the islands of amplitude death in the plane of the coupling time delay and strength, where all oscillations are quenched, is qualitatively same as Fig. 2. Throughout the remainder of the present paper, the damping and coupling strengths are always restricted to the shaded region of Fig. 1.
III. SPATIO-TEMPORAL PATTERNS OF BIFURCATING PERIODIC SOLUTIONS

It follows from Sec. II that due to the symmetry of system (2), the purely imaginary eigenvalues $io\tau_2$ are always multiple at the critical values $\tau_{2j}$. Hence, the standard Hopf bifurcation theorem of functional differential equations (see, e.g., Hale\textsuperscript{32} and Hassard \textit{et al.}\textsuperscript{33}) cannot be applied. In this section, we first explore the symmetry of system (3) and then apply the symmetric Hopf bifurcation theorem for delay differential equations established in Ref. 34 to describe the spatio-temporal patterns of bifurcating periodic solutions as the coupling time delay crosses the critical values $\tau_{2j}$. We would also like to mention that this theory has been successfully employed to study the spatio-temporal patterns in the delay-coupled neural networks (see Refs. 25, 35–38 and references therein).

Let $G : C \rightarrow \mathbb{R}^n$ and $\Gamma$ be a compact Lie group. By a compact Lie group, we mean a closed subgroup of $GL(\mathbb{R}^n)$, the group of all invertible linear transformations of the vector space $\mathbb{R}^n$ into itself. Note that the space of $n \times n$ matrices may be identified with $\mathbb{R}^n$, which contains $GL(\mathbb{R}^n)$ as an open subset. It follows from Ref. 39 that the system $\dot{u}(t) = G(u)$ is said to be $\Gamma$-equivariant if $G(\gamma u) = \gamma G(u)$ for all $\gamma \in \Gamma$. Let $\Gamma = D_3$ be the dihedral group of order 6, which is generated by the cyclic group $Z_3$ order 3 together with the flip of order 2 (see Ref. 39, for more details). Denote by $\rho$, the generator of the cyclic group $Z_3$ and $\kappa$ the flip. Define the action of $D_3$ on $\mathbb{R}^6$ by

$$
\rho \left( \begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ Y_1 \\ Y_3 \\ Y_3 \end{array} \right) = \left( \begin{array}{c} Y_2 \\ Y_3 \\ Y_1 \\ Y_3 \\ Y_3 \end{array} \right), \quad \kappa \left( \begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ Y_1 \\ Y_3 \\ Y_3 \end{array} \right) = \left( \begin{array}{c} Y_1 \\ Y_2 \end{array} \right)
$$

for all $Y_i \in \mathbb{R}^2$, $i \ (\text{mod} \ 3)$,

and then it is easy to verify the following lemma.

Lemma 3.1. System (3) is $D_3$ equivariant.

Assume that the infinitesimal generator $A(\tau)$ of the $C_0$-semigroup generated by the linear system (5) has a pair of purely imaginary eigenvalues $\pm io\tau_2$ at the critical value $\tau_{2j}$. From Sec. II, we also get that for the critical value $\tau_{2j}$, there are two independent eigenvectors $v_1$ and $v_2$, where $v_j = (\eta_j, \chi_j, \eta_j, \chi_j, \eta_j, \chi_j)^T$, $j = 1, 2$, with $\chi_j = e^{i\pi j/3}$ and

$$
\eta_1 = \eta_2 = \left( 1, io\tau_2, 1 \right) \quad \text{for } \tau^*_2, \quad \text{and} \quad \eta_1 = \eta_2 = \left( 1, io\tau_2 \right) \quad \text{for } \tau_2^*.
$$

Denoting the action of $\Gamma = D_3$ on $\mathbb{R}^2$ by $\rho(u_1) = \frac{1}{2} \left( -\frac{1}{2} - \frac{\sqrt{3}}{2} \right) (u_1)$, $\kappa(u_2) = \frac{1}{2} \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} \right) (u_2)$, we have the following lemma.

Lemma 3.2. $\mathbb{R}^2$ is an absolutely irreducible representation of $\Gamma$, and Ker$\Delta(\tau_{2j}^*, io\tau_2^*)$ is isomorphic to $\mathbb{R}^2 \oplus \mathbb{R}^2$. Here a representation $\mathbb{R}^2$ of $\Gamma$ is absolutely irreducible if the only linear mapping that commutes with the action of $\Gamma$ is a scalar multiple of the identity.

Proof: The proof for $\mathbb{R}^2$ to be two-dimensional absolutely irreducible representation of $\Gamma$ is straightforward and can be found in, for example, Ref. 39. Clearly,

$$
\text{Ker} \Delta(\tau_{2j}^*, io\tau_2^*) = \{ (a_1 + b_1i)v_1 + (a_2 + b_2i)v_2, a_1, a_2, b_1, b_2 \in \mathbb{R} \}.
$$

Define a mapping $J$ from Ker$\Delta(\tau_{2j}^*, io\tau_2^*)$ to $\mathbb{R}^4$ by

$$
J((a_1 + b_1i)v_1 + (a_2 + b_2i)v_2) = (a_1 + a_2, b_1 - b_2, b_1 + b_2, a_3 - a_1)^T
$$

Clearly, $J : \text{Ker} \Delta(\tau_{2j}^*, io\tau_2^*) \cong \mathbb{R}^4$ is a linear isomorphism. Note that

$$
\rho((a_1 + b_1i)v_1 + (a_2 + b_2i)v_2) = (a_1 + b_1i)\rho(v_1) + (a_2 + b_2i)\rho(v_2) = (a_1 + b_1i)e^{i2\pi/3}(v_1 + (a_2 + b_2i)e^{-i2\pi/3}v_2 = \left( \frac{-1}{2}a_1 - \frac{\sqrt{3}}{2}b_1 \right) + i \left( \frac{-1}{2}b_1 + \frac{\sqrt{3}}{2}a_1 \right)v_1
$$

Thus, $J(\rho(u)) = J(\kappa(u))$ for all $u \in \mathbb{R}^2$. Hence, $J$ is a representation of $\Delta(\tau_{2j}^*, io\tau_2^*)$. 

FIG. 2. (Color online) Islands of amplitude death in the plane of the coupling time delay $\tau$ and the coupling strength $k$ for fixed $\varepsilon = 0.03$. 

FIG. 1. (Color online) Possible region of amplitude death for the coupling time delay in the $\varepsilon - k$ plane. There exist stability switches for the zero equilibrium of system (2) by increasing the coupling time delay $\tau$ in the shaded region and the zero equilibrium of system (2) is always unstable for any $\tau \geq 0$ otherwise. $l_1$ and $l_2$ are curves determined by the equations are $\cos(\varepsilon - k/2k) - (o_1^2 - o_\varepsilon^2)/(o_1^2 + o_\varepsilon^2)x = 0$ and $o_\varepsilon^2\arccos(\varepsilon/k - 2k) - o_\varepsilon^2\arccos(k/\varepsilon) - \frac{2}{2}(o_1^2 - o_\varepsilon^2) = 0$, respectively.
and
\[
\kappa((a_1 + b_2)i)v_1 + (a_2 + b_2)i)v_2 = (a_1 + b_2i)e^{i(2\pi/3)}v_2 + (a_2 + b_2i)e^{-i(2\pi/3)}v_1 = -\left(\frac{1}{2}a_1 - \sqrt{3}a_2, i\left(-\frac{1}{2}b_1 + \sqrt{3}a_1\right)\right)v_2 + \left(-\frac{1}{2}a_2 + \sqrt{3}b_2\right) + i\left(-\frac{1}{2}b_2 - \sqrt{3}a_2\right)v_1.
\]

Therefore,
\[
J(\kappa((a_1 + b_2)i)v_1 + (a_2 + b_2i)v_2) = \left(-\frac{1}{2}(a_1 + a_2) - \sqrt{3}(b_1 - b_2), \frac{1}{2}(b_1 - b_2) - \frac{1}{2}(b_1 + b_2), -\frac{1}{2}(a_1 + a_2) - \sqrt{3}(a_2 - a_1), \frac{1}{2}(a_2 - a_1) - \sqrt{3}(b_1 + b_2)\right) = \rho(J((a_1 + b_2)i)v_1 + (a_2 + b_2i)v_2)).
\]

and
\[
J(\kappa((a_1 + b_2)i)v_1 + (a_2 + b_2i)v_2)) = \left(-\frac{1}{2}(a_1 + a_2) - \sqrt{3}(b_1 - b_2), \frac{1}{2}(b_1 - b_2) - \frac{1}{2}(b_1 + b_2), -\frac{1}{2}(a_1 + a_2) - \sqrt{3}(a_2 - a_1), \frac{1}{2}(a_2 - a_1) - \sqrt{3}(b_1 + b_2)\right) = \rho(J((a_1 + b_2)i)v_1 + (a_2 + b_2i)v_2)).
\]

This completes the proof.

Let \(\omega = 2\pi/\omega, i = 1,2\), and denote by \(P_{\omega}\) the Banach space of all continuous \(\omega\)-periodic mappings \(x : \mathbb{R} \to \mathbb{R}\). Then \(D_3 \times S_1\) acts on \(P_{\omega}\), where \(S^1\) is a circle group, by
\[
(\gamma, \theta)x(t) = \gamma x(t + \theta), (\gamma, \theta) \in D_3 \times S_1.
\]

Denote by \(SP_{\omega}\) the set of all \(\omega\)-periodic solutions of (5) with \(\tau = \tau_{21}\). Then, for \(\tau_{21}\),
\[
SP_{\omega} = \{z_1\tilde{e}_{21} + z_2\tilde{e}_{22} + z_3\tilde{e}_{23} + z_4\tilde{e}_{24}, z_1, z_2, z_3, z_4 \in \mathbb{R}\},
\]

where
\[
\tilde{e}_{21} = \text{Re}(e^{i2\theta}v_1) = \cos(\omega^2\theta)\text{Re}(v_1) - \sin(\omega^2\theta)\text{Im}(v_1),
\]
\[
\tilde{e}_{22} = \text{Im}(e^{i2\theta}v_1) = \sin(\omega^2\theta)\text{Re}(v_1) + \cos(\omega^2\theta)\text{Im}(v_1),
\]
\[
\tilde{e}_{23} = \text{Re}(e^{i2\theta}v_2) = \cos(\omega^2\theta)\text{Re}(v_2) - \sin(\omega^2\theta)\text{Im}(v_2),
\]
\[
\tilde{e}_{24} = \text{Im}(e^{i2\theta}v_2) = \sin(\omega^2\theta)\text{Re}(v_2) + \cos(\omega^2\theta)\text{Im}(v_2).
\]

By these formulae and (16), the lemma follows from a straightforward calculation.

Proof: Note that \(\eta_1 = \eta_2 = (1, i\omega \tilde{e}_{23})^T\). It is easy to verify that
\[
\kappa(\text{Re}(v_1)) = -\frac{1}{2}\text{Re}(v_2) - \frac{\sqrt{3}}{2}\text{Im}(v_2),
\]
\[
\kappa(\text{Im}(v_1)) = \frac{\sqrt{3}}{2}\text{Re}(v_2) - \frac{1}{2}\text{Im}(v_1),
\]
\[
\kappa(\text{Re}(v_2)) = -\frac{1}{2}\text{Re}(v_1) + \frac{\sqrt{3}}{2}\text{Re}(v_1),
\]
\[
\kappa(\text{Im}(v_1)) = -\frac{\sqrt{3}}{2}\text{Re}(v_1) - \frac{1}{2}\text{Im}(v_1),
\]

and
\[
\rho(\text{Re}(v_1)) = -\frac{1}{2}\text{Re}(v_1) - \frac{\sqrt{3}}{2}\text{Re}(v_1),
\]
\[
\rho(\text{Im}(v_1)) = \frac{\sqrt{3}}{2}\text{Re}(v_1) - \frac{1}{2}\text{Im}(v_1),
\]
\[
\rho(\text{Re}(v_2)) = -\frac{1}{2}\text{Re}(v_2) + \frac{\sqrt{3}}{2}\text{Re}(v_2),
\]
\[
\rho(\text{Im}(v_1)) = -\frac{\sqrt{3}}{2}\text{Re}(v_2) - \frac{1}{2}\text{Im}(v_2).
\]

Setting
\[
(\kappa, \pm 1, 1), (\rho, e^{\pm i(2\pi/3)}),
\]

we can prove the following lemma.

Lemma 3.4. Fix(\(\Sigma_{+2}, SP_{\omega}\)) and Fix(\(\rho, SP_{\omega}\)) are all subspaces of \(SP_{\omega}\) of dimension 2.

Proof: (1) First, \(x \in \text{Fix}(\Sigma_{+2}, SP_{\omega})\) if and only if \(\kappa(x) = x\). For \(x = \sum_{i=1}^4 z_i\tilde{e}_{2i}\), we have
\[
\kappa x = \sum_{i=1}^4 z_i \kappa(\tilde{e}_{2i}) = \left(- \frac{1}{2}z_3 + \frac{\sqrt{3}}{2}z_4\right) \tilde{e}_{21} + \left(\frac{\sqrt{3}}{2}z_3 - \frac{1}{2}z_4\right) \tilde{e}_{22} + \left(-\frac{1}{2}z_1 + \frac{\sqrt{3}}{2}z_2\right) \tilde{e}_{23} - \left(\frac{\sqrt{3}}{2}z_2 + \frac{1}{2}z_1\right) \tilde{e}_{24}
\]

It follows that \(x \in \text{Fix}(\Sigma_{+2}, SP_{\omega})\) if and only if
\[
\left(- \frac{1}{2}z_1 - \frac{\sqrt{3}}{2}z_2\right) \tilde{e}_{23} = \left(\tilde{e}_{21}\right),
\]

This implies that \(\text{Fix}(\Sigma_{+2}, SP_{\omega})\) is spanned by \(\tilde{e}_{1}\) and \(\tilde{e}_{2}\), where
\[
\tilde{e}_1 = -\frac{1}{2}z_1 + \frac{\sqrt{3}}{2}z_2 + \frac{\sqrt{3}}{2}z_2, \quad \tilde{e}_2 = -\frac{\sqrt{3}}{2}z_1 - \frac{1}{2}z_2 + \frac{\sqrt{3}}{2}z_2.
\]

(2) Second, if and only if \(x \in \text{Fix}(\Sigma_{+2}, SP_{\omega})\) if and only if
\[
\left(- \frac{1}{2}z_1 - \frac{\sqrt{3}}{2}z_2\right) \tilde{e}_{23} = \left(\tilde{e}_{21}\right),
\]

This implies that \(x \in \text{Fix}(\Sigma_{+2}, SP_{\omega})\) if and only if
Therefore, Fix \( (\Sigma_{2,3}, SP_{eo}) \) is spanned by \( \epsilon_3^* \) and \( \epsilon_4^* \), where
\[
\epsilon_3^* = \frac{1}{2} \epsilon_2^* - \sqrt{3} \epsilon_4^* + \frac{i}{2} \epsilon_2^*, \quad \epsilon_4^* = \sqrt{\frac{3}{2}} \epsilon_3^* + \frac{1}{2} \epsilon_2^* + \epsilon_4^*.
\]
(3) Third, for \( x = \sum_{i=1}^{4} \epsilon_i \phi_{ij} \), we have
\[
\rho(x) = \left( -\frac{1}{i} \bar{\epsilon}_1 + \frac{\sqrt{3}}{2} \bar{\epsilon}_2 \right) \bar{\epsilon}_3^* + \left( \sqrt{\frac{3}{2}} \bar{\epsilon}_1 - \frac{1}{2} \bar{\epsilon}_2 \right) \bar{\epsilon}_4^* - \left( \frac{1}{2} \bar{\epsilon}_3 + \sqrt{\frac{3}{2}} \bar{\epsilon}_4 \right) \bar{\epsilon}_3^* - \left( \frac{1}{2} \bar{\epsilon}_3 - \sqrt{\frac{3}{2}} \bar{\epsilon}_4 \right) \bar{\epsilon}_4^*.
\]
In addition,
\[
\cos(\omega_t (t - \frac{\pi}{4})) = \cos(\omega_t t - \frac{\pi}{8}) = -\frac{1}{2} \cos(\omega_t t) + \frac{\sqrt{3}}{2} \sin(\omega_t t),
\]
\[
\sin(\omega_t (t - \frac{\pi}{4})) = \sin(\omega_t t - \frac{\pi}{8}) = -\frac{\sqrt{3}}{2} \cos(\omega_t t) - \frac{1}{2} \sin(\omega_t t).
\]
This, together with the expression of each \( \phi_{ij} \) and \( x = \sum_{i=1}^{4} \epsilon_i \phi_{ij} \), leads to
\[
x(t - \frac{\pi}{4}) = -\left( \frac{1}{2} \epsilon_1 + \sqrt{\frac{3}{2}} \epsilon_2 \right) \epsilon_3^* + \left( \sqrt{\frac{3}{2}} \epsilon_1 - \frac{1}{2} \epsilon_2 \right) \epsilon_4^* - \left( \frac{1}{2} \epsilon_3 + \sqrt{\frac{3}{2}} \epsilon_4 \right) \epsilon_3^* - \left( \frac{1}{2} \epsilon_3 - \sqrt{\frac{3}{2}} \epsilon_4 \right) \epsilon_4^*.
\]
So, \( x \in \text{Fix}(\Sigma_{p}^+, SP_{eo}) \), i.e., \( \rho(x(t)) = (x(t - \frac{\pi}{4})) \), if and only if
\[
\begin{align*}
-\frac{1}{2} \bar{\epsilon}_1 + \sqrt{3} \bar{\epsilon}_2 &= -\frac{1}{2} \bar{\epsilon}_1 - \sqrt{3} \bar{\epsilon}_2, \\
-\sqrt{3} \bar{\epsilon}_1 - \frac{1}{2} \bar{\epsilon}_2 &= \sqrt{3} \bar{\epsilon}_1 + \frac{1}{2} \bar{\epsilon}_2, \\
-\frac{1}{2} \bar{\epsilon}_3 + \sqrt{3} \bar{\epsilon}_4 &= -\frac{1}{2} \bar{\epsilon}_3 - \sqrt{3} \bar{\epsilon}_4, \\
\sqrt{3} \bar{\epsilon}_3 + \frac{1}{2} \bar{\epsilon}_4 &= -\sqrt{3} \bar{\epsilon}_3 - \frac{1}{2} \bar{\epsilon}_4.
\end{align*}
\]
That is, \( x \in \text{Fix}(\Sigma_{p}^+, SP_{eo}) \) if and only if \( \bar{\epsilon}_1 = \bar{\epsilon}_2 = 0 \), and \( x \in \text{Fix}(\Sigma_{p}^+, SP_{eo}) \) if and only if \( \bar{\epsilon}_3 = \bar{\epsilon}_4 = 0 \). Therefore, \( \text{Fix}(\Sigma_{p}^+, SP_{eo}) \) is spanned by \( \bar{\epsilon}_3^* \) and \( \bar{\epsilon}_4^* \), and \( \text{Fix}(\Sigma_{p}^+, SP_{eo}) \) is spanned by \( \bar{\epsilon}_3^* \) and \( \bar{\epsilon}_4^* \).

Note that, if \( Y(t) \) is a periodic solution of (3), then so is \( (y, e^{i(2\pi/\omega_0)t})Y(t) \) for every \( (y, e^{i(2\pi/\omega_0)t}) \in \mathbb{D}_n \times \mathbb{S}_1 \). It follows from the symmetric bifurcation theory of delay differential equations due to Wu 34 that, under usual nonresonance and transversality conditions, for every subgroup \( \Sigma \leq \mathbb{D}_n \times \mathbb{S}_1 \) such that the \( \Sigma \)-fixed-point subspace of \( SP_{eo} \) is of dimension 2, symmetric delay differential equations has a bifurcation of periodic solutions whose spatial–temporal symmetries can be completely characterized by \( \Sigma \). By Lemma 3.4, the subgroups \( \Sigma_{2,3} \) and \( \Sigma_p \) can be employed to describe the symmetry of periodic solutions of (3) which exhibit certain spatial–temporal patterns (see Ref. 39 and 34, for more details). Thus, Lemmas 3.1–3.4 allow us to apply the general symmetric Hopf bifurcation theorem due to Wu 34 to describe the spatio-temporal patterns of local Hopf bifurcating periodic oscillations.

**Theorem 3.1:** System (3) has eight branches of asynchronous periodic solutions of period \( p \) near \( 2\pi/\omega_0^* \), bifurcated simultaneously from the zero solution at the critical values \( \tau = \tau_1 \), and these are

(i) two phase-locked oscillations: \( Y_i(t) = Y_{i+1}(t \pm p \tau/3) \), for \( i \) (mod 3) and \( t \in \mathbb{R} \);
(ii) three mirror-reflected waves: \( Y_i(t) = Y_j(t) \neq Y_k(t) \), for \( t \in \mathbb{R} \) and for some distinct \( i, j, k \in \{1, 2, 3\} \); and
(iii) three standing waves: \( Y_i(t) = Y_j(t + (p \tau/2)) \), for \( t \in \mathbb{R} \) and some pair of distinct elements \( i, j \in \{1, 2, 3\} \).

**IV. THE CALCULATION OF NORMAL FORMS ON CENTER MANIFOLDS AND STABILITY OF BIFURCATING PERIODIC ORBITS**

In this section, we employ the algorithm and notations of Faria and Magalhães 40,41 to derive the normal forms of system (3) on center manifolds and then we can determine the direction and stability of bifurcating periodic orbits.

Rescaling the time by \( t \rightarrow t/\tau \) to normalize the delay so that system (3) can be written as a FDE in the phase space \( C = C([1,0], \mathbb{R}^6) \), and then separating the linear terms from the nonlinear terms, (3) becomes
\[
\dot{u}(t) = L_\tau(u(t)) + F(u(t), \tau)
\]
where \( u_i \in C, \ u(0) = u(t + \theta), -1 \leq \theta \leq 0 \), and for \( \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)^T \in C \), we have
\[
L_\tau(\varphi) = \tau M_1 \varphi 2 \times 2(0) + \tau M_2 \varphi(-1)
\]
and
\[
F(\varphi, \tau) = \tau(0, -e^{-\varphi_3 T}(0) \varphi_4(0), 0, -e^{-\varphi_5 T}(0) \varphi_4(0), 0, -e^{-\varphi_5 T}(0) \varphi_6(0), 0),
\]
where \( \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)^T \in C \) and \( M_1, M_2 \) are as follows:
\[
M_1 = \begin{pmatrix} M & O_2 & O_2 \\ O_2 & M & O_2 \\ O_2 & O_2 & M \end{pmatrix}, \quad M_2 = \begin{pmatrix} O_2 & N & N \\ N & O_2 & N \\ N & N & O_2 \end{pmatrix}
\]
where \( M, N \) are defined by (4) and \( O_2 \) is a \( 2 \times 2 \) zero matrix.

There exists a function of bounded variation such that the linear map \( L_\tau \) may be expressed in the following integral form:
\[
L_\tau(\varphi) = \int_{-1}^{0} [d\eta_i(\theta)] \varphi(\theta)
\]
Letting \( C^* = C([1,0], \mathbb{R}^6) \) with \( \mathbb{R}^6 \) being the six-dimensional space of row vectors, define the adjoint bilinear form on \( C^* \times C \) as follows:
\[
\langle \psi(s), \phi(\theta) \rangle = \psi(0)\phi(0) - \int_{-1}^{0} \int_{0}^{\theta} \psi(\xi) d\eta_i(\theta) \phi(\xi) d\xi, \quad \psi \in C^*, \phi \in C.
\]
It follows from Sec. II that the characteristic equation of (18) has purely imaginary roots \( \pm i\omega_c \tau \), at the critical values.
\[ \tau = \tau^* \] of the coupling time delay, where \( \omega_s = \omega^* \) for \( \tau_s = \tau^*_{ij} \) and \( \omega_s = \omega^0 \) for \( \tau_s = \tau^0_{ij} \). Setting \( \Lambda_0 = \{-i\omega_s, \tau_s, i\omega_s, \tau_s \} \) and using the formal adjoint theory for FDEs in Ref. 32, the phase space \( C \) can be decomposed by \( \Lambda_0 \) as \( C = \mathbb{P} \oplus Q \), where \( \mathbb{P} \) is the center space for \( \dot{u}(t) = L_x u_t, \) i.e., \( \mathbb{P} \) is the generalized eigenspace associated with \( \Lambda_0 \). Let \( \Phi \) and \( \Psi \) be bases for \( \mathbb{P} \) and for the phase space \( P \) associated with the eigenvalues \( \pm i\omega_s, \tau_s \) of the formal adjoint equation, respectively, and normalized so that, where \( L_p \) is the \( p \times p \) identity matrix and \( p = \dim P \). Assume that \( B \) is a \( p \times p \) real matrix with the point spectrum \( \sigma(B) = \Lambda \) and satisfies simultaneously satisfy \( \Phi = \Phi B, -\Psi = B^T \Psi \).

Introducing the new parameter \( \mu = \tau - \tau^* \) such that \( \mu = 0 \) is a Hopf bifurcation value of (17), Eq. (17) becomes

\[ \dot{u}(t) = L_x u_t + \tilde{F}(u_t, \mu), \]

where

\[ \tilde{F}(u_t, \mu) = L_\mu u_t + F(\varphi, \tau_s + \mu). \]

Enlarging the phase space \( C \) by considering the Banach space \( BC \) of functions from \([-1, 0]\) into \( \mathbb{R}^3 \) which are uniformly continuous on \([-1, 0]\) and with a jump discontinuity at 0. Using the decomposition \( u_t = \Phi x(t) + y \), with \( x(t) \in C^0 \) and \( y \in C^1 \) (\( C^1 \) is the subset of \( C \) consisting of continuously differentiable functions). Then we can decompose (17) as

\[ \dot{x} = Bx + \Psi(0)\tilde{F}(\Phi x + y, \mu), \]

\[ \dot{y} = A_0 y + (I - \pi)x_0 \tilde{F}(\Phi x + y, \mu), \]

where \( A_0 : C^1 \rightarrow \ker \pi \) is such that \( A_0 \Phi = \hat{\Phi} + X_0 \{ L_x, (\phi) - (\Phi(\mu)) \} \). We write the Taylor formula

\[ \Psi(0)\tilde{F}(\Phi x + y, \mu) = \frac{1}{2} f^1_j(x, x, y, \mu) + \frac{1}{6} f^1_j(x, x, y, \mu) + \cdots, \]

\[ (I - \pi)x_0 \tilde{F}(\Phi x + y, \mu) = \frac{1}{2} f^2_j(x, x, y, \mu) + \frac{1}{6} f^2_j(x, x, y, \mu) + \cdots, \]

(21)

where \( f^1_j(x, x, y, \mu) \) and \( f^2_j(x, x, y, \mu) \) are homogeneous polynomials in \( (x, x, y, \mu) \) of degree \( j \) with coefficients in \( C^p \), \( \ker \pi \), respectively, and \( \text{hot} \) stands for higher order terms. Therefore, differential equation EDE can be written as

\[ \dot{x} = Bx + \sum_{j=2}^s \frac{1}{j!} f^1_j(x, x, y, \mu), \]

\[ \dot{y} = A_0 y + \sum_{j=2}^s \frac{1}{j!} f^2_j(x, x, y, \mu). \]

(22)

From the normal form theory of FDEs due to Faria and Magalhães,40,41 the normal form of (17) on the center manifold of the origin at \( \mu = 0 \) is given by

\[ \dot{x} = Bx + \frac{1}{2!} g^1_j(x, 0, \mu) + \frac{1}{3!} g^1_j(x, 0, \mu) + \text{h.o.t.}, \]

(23)

where \( g^1_j(x, 0, \mu) \) are the second and third order terms in \( (x, \mu) \), respectively, and \( \text{hot} \) stands for higher order terms. In the following part of this section, we will calculate the normal form (23) at the critical values \( \tau^*_{ij} \) and \( \tau^0_{ij} \).

### A. For the critical values \( \tau^*_{ij} \)

From Sec. II, at \( \tau = \tau^*_{ij} \) the characteristic equation of (18) has purely imaginary roots \( \pm i\omega^*_{ij} \), which are simple. So, in this subsection, \( p = 4 \), \( \omega_s = \omega^*_{ij} \) for \( \tau_s = \tau^*_{ij} \) and then we have

\[ B = \text{diag}(i\omega_s, -i\omega_s, \tau^*_{ij}). \]

It follows from (6) that the base of the center space \( P \) at \( \tau = \tau^*_{ij} \) can be taken as \( \Phi = (\phi_1, \phi_2) \), where

\[ \phi_1(\theta) = e^{i\omega_s \tau^*_{ij} \theta}v_0, \quad \phi_2(\theta) = e^{-i\omega_s \tau^*_{ij} \theta}v_0, \quad -1 \leq \theta \leq 0, \]

where

\[ v_0 = (\eta_0, \eta_0, \eta_0)^T, \quad \text{and} \quad \eta_0 = (1, i\omega^*_{ij}) \quad \text{for} \quad \tau^*_{ij}, \]

(24)

One can easily show that the adjoint basis satisfying \( \langle \Psi(s), \Phi(\theta) \rangle \) is \( I_2 \) is

\[ \Psi(s) = \left( \begin{array}{c} \psi_1(s) \\ \psi_2(s) \end{array} \right) = \frac{1}{3} \left( \begin{array}{c} d_v^0 e^{-i\omega_s \tau^*_{ij} s} \\ d_v^0 e^{i\omega_s \tau^*_{ij} s} \end{array} \right), \quad 0 \leq s \leq 1, \]

where

\[ d = \frac{1}{1 + \omega^2_s + \tau_s^2(k - 1) - \tau_s \omega_s (1 - \omega^2_s)/i}. \]

(25)

Let \( V^j_2(C^2) \) be the homogeneous polynomials of degree \( j \) in three variables, \( x_1, x_2, \mu \), with coefficients in \( C^2 \), and let \( M^j_1 \) denote the operator from \( V^j_2(C^2) \) into itself defined by

\[ (M^j_1h)(x, \mu) = Dh(x, \mu)Bx - Bh(x, \mu), \]

(26)

where \( h \in V^j_2(C^2) \). Then, since \( B \) is the \( 2 \times 2 \) diagonal matrix, it follows from Ref. 41 that

\[ V^j_2(C^2) = \text{Im}(M^1_1) \oplus \text{Ker}(M^1_1), \quad g^1_j(x, 0, \mu) \in \text{Ker}(M^1_1) \]

and

\[ \text{Ker}(M^j_1) = \text{Ker}\{x_1^2 x_2^2 \mu^j_1 e_k : q_1 \lambda_1 + q_2 \lambda_2 = \lambda_k, \}

\[ k = 1, 2, q_1, q_2, l \in N_0, q_1 + q_2 + l = j \}, \]

where \( \lambda_1 = i\omega_s \tau_s, \lambda_2 = -i\omega_s \tau_s \), and \( \{e_1, e_2\} \) is canonical basis of \( \mathbb{R}^2 \). Hence,

\[ \text{Ker}(M^1_1) = \text{span}\left\{ \left( \begin{array}{c} x_1 \mu \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ x_2 \mu \end{array} \right) \right\} \}

\[ \text{Ker}(M^j_1) = \text{span}\left\{ \left( \begin{array}{c} x_1^2 x_2 \mu^j_1 \end{array} \right), \left( \begin{array}{c} x_1 \mu^2 \end{array} \right), \left( \begin{array}{c} 0 \\ x_1 x_2^2 \mu^2 \end{array} \right) \right\} \}

(26)

Since \( \Phi''(0) = 0 \), by (21), we have

\[ \frac{1}{2!} f^1_j(x, 0, \mu) = \Psi(0)L_\mu(\Phi x) \]

\[ = \left( \begin{array}{c} \frac{1}{3} d_v^0 \frac{d^2}{d\theta^2} \\ \frac{1}{3} d_v^0 \frac{d^2}{d\theta^2} \end{array} \right) \left( L_\mu(\phi_1)x_1 + L_\mu(\phi_2)x_2 \right). \]
determining the generic Hopf bifurcation. Since 
the center manifold are given by
\[ g(x, y, z) = (x_1, x_2, x_3). \]
We first note that
\[ L_\mu(x_1) = A_1 x_1 + L_\mu(x_2), \]
with
\[ A_1 = \frac{1}{d} \omega_1 \nu_1. \]
Next we compute the cubic terms \( g_3(x, y, z) \) as described in
(23). We first note that
\[ g_3(x, y, z) = \frac{1}{3!} \mathcal{F}_3(x, y, z), \]
for \( S = \text{span} \left\{ (x_1 x_2, 0, x_1 x_2) \right\} \),
where \( \mathcal{F}_3(x, y, z) \) denotes the third order terms after the com-
putation of the normal form up to the second order terms. It is
sufficient to compute only \( \mathcal{F}_3(x, y, z) \) for the purpose of
determining the generic Hopf bifurcation. Since \( f_0(x) = 0 \),
implies \( f_1(x, y) = 0 \), we can deduce that, after the change
of variables that transformed the quadratic terms \( f_2(x, y, z) \) of
the first equation in (22) into \( g_2(x, y, z) \), the coefficients of
third order at \( y = 0, \mu = 0 \) are still given by \( \mathcal{F}_3(x, y, z) \), i.e.,
\[ g_3(x, y, z) = \frac{1}{3!} \mathcal{F}_3(x, y, z). \]
Thus, from (19) and (21), we have
\[ g_3(x, y, z) = \frac{1}{3!} \mathcal{F}_3(x, y, z). \]
where \( \nu_0 = (v_{01}, v_{02}, v_{03}, v_{04}, v_{05}, v_{06})^T \). It follows that
\[ g_3(x, y, z) = \frac{1}{3!} \mathcal{F}_3(x, y, z). \]
with
\[ \nu_0 = (v_{01}, v_{02}, v_{03}, v_{04}, v_{05}, v_{06})^T \]. It follows that
\[ A_2 = -\frac{\epsilon t_1^3}{3} (v_{01}^2 v_{02} + v_{03}^2 v_{04} + v_{05}^2 v_{06}) + 2|v_{01}|^2 |v_{02}|^2 + 2|v_{03}|^2 |v_{04}|^2. \]
This, together with (24), yields that
\[ A_2 = -\frac{\epsilon t_1^3}{3} \frac{d^2}{dt^2} \]
So, the normal form of (17) on the center manifold reads to
\[ \dot{x} = \frac{A_1 x_1 \mu}{A_1 x_2 \mu} + \left( \frac{A_2 x_1^2 x_2}{A_1 x_2^2} \right) + O(|x|^2 + |x|^4), \]
where \( A_1 \) and \( A_2 \) are defined by (27) and (28), respectively.
Changing (29) to real coordinates by the change of variables
\( x_1 = w_1 - i w_2, x_2 = w_1 + i w_2 \) and then using polar coordinates
\( (\rho, \xi), w_1 = \rho \cos \xi, w_2 = \rho \sin \xi \), we obtain
\[ \dot{\rho} = K_1 \rho + K_2 \rho^3 + O(\rho^4 + |(\rho, \xi)|^4), \]
with \( K_1 = \text{Re}(A_1), K_2 = \text{Re}(A_2). \)
It is well known \(^{22}\) that the sign of \( K_1, K_2 \) determines the
direction of the bifurcation (supercritical if \( K_1, K_2 > 0 \), sub-
critical if \( K_1, K_2 < 0 \) and the sign of \( K_2 \) determines the sta-
bility of the nontrivial periodic orbits (stable if \( K_2 < 0 \), unstable
if \( K_2 > 0 \)). From (27) and (28), we have
\[ K_1 = \text{Re}(A_1) = -\omega_1 (1 + \omega_2^2) \text{Re}(d), \]
and
\[ K_2 = \text{Re}(A_2) = -\omega_1 \omega_2 \text{Re}(d). \]
From (10) and (25), it is easy to verify that for \( k > e, \)
\[ \text{Re}(d) = \frac{1 + \omega_1^2 + \tau_3 \omega_2^2 (k - e)}{\Gamma} > 0, \]
\[ \text{Im}(d) = \frac{\tau_3 \omega_2 (1 - \omega_1^2)}{\Gamma} \left\{ \begin{array}{ll} < 0, & \text{for } \tau_3 = \tau_3^+, \\ > 0, & \text{for } \tau_3 = \tau_3^-, \end{array} \right. \]
where
\[ \Gamma = \left( 1 + \omega_1^2 + \tau_3 \omega_2 (k - e) \right)^2 + \tau_3^2 \omega_2^2 (1 - \omega_1^2)^2. \]
By (30), (31), and (32), we have
\[ K_1 = \left\{ \begin{array}{ll} > 0, & \text{for } \tau_3 = \tau_3^+, K_2 < 0, \\ < 0, & \text{for } \tau_3 = \tau_3^-, K_2 > 0, \end{array} \right. \]
which immediately leads to the following theorem.

**Theorem 4.1:** Assuming that \( k > e \) and the assumptions
\( (A1) \) and \( (A2) \) hold, then near \( \tau_3^+ \) (respectively, \( \tau_3^- \)), there exists a supercritical (respectively, subcritical) bifurcation of stable synchronous periodic solutions of period \( p \), near \( 2\pi/\omega_1^+ \) (respectively, \( 2\pi/\omega_1^- \)), bifurcated from the zero
solution of system (2), and those nontrivial periodic orbits are all stable on the center manifold.

B. For the critical values $\tau_{2j}$

From Sec. II, at $\tau = \tau_{2j}$ the characteristic equation of (18) has purely imaginary roots $\pm i\omega_{2j}$, which are double. So, in this subsection, $p = 4$, $\omega_{s} = \omega_{2j}$ for $\tau_{s} = \tau_{2j}$ and $\omega_{s} = \omega_{2j}$ for $\tau_{s} = \tau_{2j}$ and then we have

$$B = \text{diag}(i\omega_{s}, \tau_{s}, -i\omega_{s}, \tau_{s}, i\omega_{s}, \tau_{s}, -i\omega_{s}, \tau_{s})$$

By (6), the base of the center space $P$ at $\tau = \tau_{s}$ can be taken as $\Phi = (\phi_{1}, \phi_{2}, \phi_{3}, \phi_{4})$, where

$$\phi_{1}(\theta) = e^{i\omega_{s}\tau_{s}\theta}v_{1}, \quad \phi_{2}(\theta) = e^{-i\omega_{s}\tau_{s}\theta}v_{1},$$
$$\phi_{3}(\theta) = e^{i\omega_{s}\tau_{s}\theta}v_{2}, \quad \phi_{4}(\theta) = e^{-i\omega_{s}\tau_{s}\theta}v_{2}, \quad -1 \leq \theta \leq 0.$$  \hspace{1cm} (33)

It is easy to check that a basis for the adjoint space $P^*$ is

$$\Psi(s) = \begin{pmatrix} \psi_{1}(s) \\ \psi_{2}(s) \\ \psi_{3}(s) \\ \psi_{4}(s) \end{pmatrix} = \frac{1}{3} \begin{pmatrix} d e^{-i\omega_{s}\tau_{s}v_{1}} \\ -d e^{i\omega_{s}\tau_{s}v_{1}} \\ d e^{-i\omega_{s}\tau_{s}v_{2}} \\ -d e^{i\omega_{s}\tau_{s}v_{2}} \end{pmatrix}, \quad 0 \leq s \leq 1.$$  \hspace{1cm} (34)

with $\langle \Psi(s), \Phi(\theta) \rangle = I_{4}$ for the adjoint bilinear form on $C^{*} \times C$ defined by (20), where $d$ is defined by (25).

Let $V_{2j}^{j}(\mathbb{C}^{3})$ be the homogeneous polynomials of degree $j$ in 5 variables, $x$, with coefficients in $\mathbb{C}^{4}$, and let $M_{j}^{3}$ denote the operator from $V_{2j}^{j}(\mathbb{C}^{3})$ into itself defined by (26) with $h \in V_{2j}^{j}(\mathbb{C}^{3})$. Thus, we have

$$M_{j}^{3}(\mu^{\theta}e_{k}) = i\omega_{s}\tau_{s}\mu(q_{1} - q_{2} + q_{3} - q_{4} + (-1)^{j})\mu^{\theta}e_{k},$$

$$|\mu| = j - 1,$$

where $j \geq 2, 1 \leq k \leq 4$, and $\{e_{1}, e_{2}, e_{3}, e_{4}\}$ is the canonical basis of $\mathbb{R}^{4}$. In particular, if $|\mu| = 1$, then

$$\text{Ker}(M_{j}^{3}) \cap \text{span}\{\mu^{\theta}e_{k} : |\mu| = 1, k = 1, 2, 3, 4\} = \text{span}\{\mu x_{1}e_{1}, \mu x_{3}e_{1}, \mu x_{2}e_{2}, \mu x_{4}e_{2}, \mu x_{1}e_{3}, \mu x_{3}e_{3}, \mu x_{2}e_{4}, \mu x_{4}e_{4}\}.$$  \hspace{1cm} (35)

It follows from (33) and (34) that

$$\Psi(0) = \frac{1}{3} \begin{pmatrix} d \bar{v}_{1} \\ d \bar{v}_{1} \\ d \bar{v}_{2} \\ d \bar{v}_{2} \end{pmatrix},$$

$$\Phi(0)x = (v_{1}, \bar{v}_{1}, v_{2}, \bar{v}_{2})x = x_{1}v_{1} + x_{2}v_{2} + x_{3}v_{2} + x_{4}v_{2},$$
and

$$\Phi(\theta)x = x_{1}e^{i\omega_{s}\tau_{s}\theta}v_{1} + x_{2}e^{i\omega_{s}\tau_{s}\theta}v_{1} + x_{3}e^{-i\omega_{s}\tau_{s}\theta}v_{2} + x_{4}e^{i\omega_{s}\tau_{s}\theta}v_{2},$$

By (21), we have

$$\frac{1}{2\pi_{2}}f_{2}(x, 0, \mu) = \Psi(0)L_{\mu}(\Phi x)$$
$$= i\mu\omega_{s}\Psi(0)(x_{1}v_{1} - x_{2}v_{2} + x_{3}v_{2} - x_{4}v_{2})$$
$$= i\mu\omega_{s}(1 + \omega_{2}^{2}) \begin{pmatrix} d(x_{3} - x_{2}) \\ d(x_{3} - x_{2}) \\ d(x_{1} - x_{4}) \end{pmatrix}. \hspace{1cm} (36)$$

and then by (35) the second order terms in $(x, \mu)$ of the normal form on the center manifold are given by

$$\frac{1}{2}g_{4}^{4}(x, 0, \mu) = \frac{1}{2} \text{Proj}_{\text{Ker}(M_{j}^{3})}f_{4}^{4}(x, 0, \mu)$$
$$= \begin{pmatrix} B_{1}x_{1}\mu \\ B_{2}x_{2}\mu \\ B_{3}x_{3}\mu \end{pmatrix}.$$  \hspace{1cm} (37)

where

$$B_{1} = i\omega_{s}(1 + \omega_{2}^{2})d.$$  \hspace{1cm} (38)

Next we compute the cubic terms $g_{3}^{3}(x, 0, \mu)$ as described in (23). We first note that

$$g_{3}^{3}(x, 0, \mu) = \text{Proj}_{\text{Ker}(M_{j}^{3})}f_{3}^{3}(x, 0, \mu)$$
$$= \text{Proj}_{\text{Ker}(M_{j}^{3})}f_{3}^{3}(x, 0, 0) + O(|x|^{\mu}),$$

where $\frac{1}{3}f_{3}^{3}(x, 0, 0)$ denotes the third order terms after the computation of the normal form up to the second order terms. It is easy to see from (36) and (37) that $g_{3}^{3}(x, 0, 0) = f_{3}^{3}(x, 0, 0) = 0$.

So,

$$1 = \Psi(0)F_{3}(\Phi x, \tau_{s})$$
$$= \begin{pmatrix} 0 \\ -\varepsilon[\Phi(0)x_{1}]^{2}\Phi(0)x_{2} \\ 0 \\ -\varepsilon[\Phi(0)x_{3}]^{2}\Phi(0)x_{4} \\ 0 \\ -\varepsilon[\Phi(0)x_{5}]^{2}\Phi(0)x_{6} \end{pmatrix},$$

where

$$\Phi(0)x_{j} = x_{1}v_{1j} + x_{2}v_{2j} + x_{3}v_{2j} + x_{4}v_{2j}, \quad j = 1, 2, 3, 4, 5, 6.$$  \hspace{1cm} (39)

Also note that for $q = 3,

$$M_{j}^{3}(x^{\theta}e_{j}) = 0,$$  \hspace{1cm} \text{if and only if} \quad q_{1} - q_{2} + q_{3} - q_{4} + (-1)^{j} = 0,$$

$$j = 1, 2, 3, 4.$$  \hspace{1cm} (40)

So,

$$\text{Ker}(M_{j}^{3}) = \text{span}\{x_{1}^{2}x_{2}e_{1}, x_{2}^{2}x_{4}e_{1}, x_{3}^{2}x_{4}e_{1}, x_{1}x_{2}x_{3}e_{1}, x_{1}x_{3}x_{4}e_{1}, x_{2}x_{3}x_{4}e_{1}, x_{2}x_{1}x_{2}e_{1}, x_{2}x_{1}x_{3}e_{1}, x_{2}x_{1}x_{4}e_{1}, x_{2}x_{1}x_{3}e_{2}, x_{2}x_{1}x_{4}e_{2}, x_{2}x_{1}x_{3}e_{3}, x_{2}x_{1}x_{4}e_{3}, x_{2}x_{1}x_{3}e_{4}, x_{2}x_{1}x_{4}e_{4}, x_{2}x_{1}x_{3}e_{5}, x_{2}x_{1}x_{4}e_{5}, x_{2}x_{1}x_{3}e_{6}, x_{2}x_{1}x_{4}e_{6} \}$$
and then
\[ \frac{1}{3!} g_3^1(x, 0, 0) = \frac{1}{4} \text{Proj}_{\text{ker}[\mathcal{M}]} f_3^1(x, 0, 0) \]
\[ = \begin{pmatrix} B_2(x_1^2 x_2 + 2x_1 x_3 x_4) \\ B_2(x_1^2 x_1 + 2x_2 x_3 x_4) \\ B_2(x_3^2 x_4 + 2x_1 x_2 x_3) \\ B_2(x_3^2 x_3 + 2x_1 x_2 x_4) \end{pmatrix}, \]
where
\[ B_2 = -\varepsilon \tau (\omega x)^2 d. \] (39)

So, the normal form of (17) on the center manifold reads to
\[ \dot{x} = \omega \tau x + \begin{pmatrix} B_1 x_1 \mu \\ B_1 x_2 \mu \\ B_1 x_3 \mu \\ B_1 x_4 \mu \end{pmatrix} + \begin{pmatrix} B_2(x_1^2 x_2 + 2x_1 x_3 x_4) \\ B_2(x_1^2 x_1 + 2x_2 x_3 x_4) \\ B_2(x_3^2 x_4 + 2x_1 x_2 x_3) \\ B_2(x_3^2 x_3 + 2x_1 x_2 x_4) \end{pmatrix} + O(|x|^2 + |x|^4), \] (40)

where \( B_1 \) and \( B_2 \) are defined by (38) and (39), respectively. Changing (40) to real coordinates by the change of variables
\[ x = \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \end{pmatrix} w \]
and letting
\[ \rho_1^2 = x_1 x_2 = w_1^2 + w_2^2, \quad \rho_2^2 = x_3 x_4 = w_3^2 + w_4^2, \]
we obtain
\[ \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \\ \dot{w}_4 \end{pmatrix} = \omega \tau \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix} + \mu \begin{pmatrix} \text{Re}(B_1) w_1 + \text{Re}(B_1) w_2 \\ -\text{Im}(B_1) w_1 + \text{Re}(B_1) w_2 \end{pmatrix} + \begin{pmatrix} \rho_1^2 + 2\rho_2^2 \\ 2\rho_1^2 \end{pmatrix} \begin{pmatrix} \text{Re}(B_2) w_1 + \text{Re}(B_2) w_2 \\ -\text{Im}(B_2) w_1 + \text{Re}(B_2) w_2 \end{pmatrix} + O(|w|^2 + |w|^4). \]

Introducing the periodic-scaling parameter \( \omega \) and letting
\[ z_1(t) = w_1(s) + iw_2(s), \]
\[ z_2(t) = w_3(s) + iw_4(s) \]
with
\[ s = \left[(1 + \alpha) \omega \tau \right]^{-1} t, \]
we obtain
\[ (1 + \sigma) \dot{z}_1 = -iz_1(t) + \frac{\mu}{\omega \tau} (\text{Re}(B_1) - i\text{Re}(B_1)) z_1(t) + \frac{1}{\omega \tau} (|z_1|^2 + 2|z_2|^2)(\text{Re}(B_2) - i\text{Re}(B_2)) z_1(t) + O(|z|^2 + |z|^4), \]
\[ = -iz_1(t) + \frac{\mu}{\omega \tau} B_1 z_1(t) + \frac{1}{\omega \tau} B_2 z_1(t) \times (|z_1|^2 + 2|z_2|^2) + O(|z|^2 + |z|^4). \]

Similarly, we get an equation for \( z_2(t) \). Thus, ignoring the terms \( O(|z|^2) \) and \( O(|z|^4) \), we get the normal form
\[ (1 + \sigma) \dot{z}_1 = -iz_1(t) + \frac{\mu}{\omega \tau} B_1 z_1(t) + \frac{1}{\omega \tau} B_2 z_1(t)(|z_1|^2 + 2|z_2|^2), \]
\[ (1 + \sigma) \dot{z}_2 = -iz_2(t) + \frac{\mu}{\omega \tau} B_1 z_2(t) + \frac{1}{\omega \tau} B_2 z_2(t)(|z_1|^2 + 2|z_2|^2). \]
(41)

Let \( g : C \oplus C \oplus R \to C \oplus C \) be given so that \( -g(z_1, z_2, \mu) \) is the right-hand side of (41). Then (41) can be written as
\[ (1 + \alpha) \dot{z} + g(z, \mu) = 0. \]

According to Ref. 39 (pp. 296–297, Theorems 6.3 and 6.5), the bifurcations of small-amplitude periodic solutions of (41) are completely determined by the zeros of equation,
\[ -i(1 + \sigma) z + g(z, \mu) = 0, \]
and their (orbital) stability is determined by the signs of three eigenvalues of the following matrix:
\[ D g(z, 0) - i(1 + \sigma) I d \]

By (41), (42) is equivalent to
\[ i \sigma z_1 + \frac{\mu}{\omega \tau} B_1 z_1 + \frac{1}{\omega \tau} B_2 (|z_1|^2 + 2|z_2|^2) z_1 = 0, \]
\[ i \sigma z_2 + \frac{\mu}{\omega \tau} B_1 z_2 + \frac{1}{\omega \tau} B_2 (|z_2|^2 + 2|z_1|^2) z_2 = 0. \]
(43)
To apply the results of Ref. 39, we write (43) as

\[ A \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) + B \left( \begin{array}{c} z_1^2 \\ z_2^2 \end{array} \right) = 0 \]

with

\[ A = A_0 + A_N(|z_1|^2 + |z_2|^2), \quad B = B_0 \]

for some complex numbers \( A_0, A_N, \) and \( B_0 \) given by

\[ A_0 = -\frac{\mu}{\omega_s \tau_s}B_1 - i\omega_s, \]

\[ A_N = -\frac{2}{\omega_s \tau_s}B_2, \]

\[ B_0 = \frac{1}{\omega_s \tau_s}B_2. \]

(44)

It follows from (25), (32), (39), and (44) that for \( k > \varepsilon, \)

\[ \text{Re}(A_N + B_0) = -\frac{1}{\omega_s \tau_s} \text{Re}(B_2) = \varepsilon \omega_s \text{Re}(d) > 0, \]

\[ \text{Re}(2A_N + B_0) = -\frac{3}{\omega_s \tau_s} \text{Re}(B_2) 3 \varepsilon \omega_s \text{Re}(d) > 0, \]

and

\[ \text{Re}B_0 = \frac{1}{\omega_s \tau_s} \text{Re}(B_2) = -\varepsilon \omega_s \text{Re}(d) < 0. \]

In addition, from (12), (25), and (38), we obtain that

\[ \text{Re} \left( \frac{B_1}{\omega_s \tau_s} \right) = \left( 1 + \frac{\omega_s^2}{\tau_s^2} \right) \text{Re}(d) \]

\[ = \left( 1 + \frac{\omega_s^2}{\tau_s^2} \right) \left( \frac{\omega_s^2 - 1}{\Gamma} \right) \begin{cases} > 0, & \text{for } \tau_s = \tau_N^1, \\ < 0, & \text{for } \tau_s = \tau_N^2. \end{cases} \]

By the results of Ref. 39 (pp. 383, Theorem 3.1), we have the following theorem on the properties of bifurcation at the critical values \( \tau_N^1. \)

**Theorem 4.2:** Assuming that \( k > \varepsilon \) and the assumptions (A1) and (A2) hold, then the bifurcations are supercritical (respectively, subcritical) at the critical values \( \tau_N^1 \) (respectively, \( \tau_N^2 \)), and then on the center manifold the phase-locked oscillations are orbitally asymptotically stable and the mirror-reflecting waves and standing waves are unstable.

**V. NUMERICAL CONTINUATION**

In this section, we perform some numerical simulations to illustrate and expand the results obtained above. In the following numerical simulations, we take \( \varepsilon = 0.03, k = 0.1 \) such that \( (\varepsilon, k) \) belongs to the shaded region of Fig. 1, where stability switches occur with increasing the coupling time delay. The following numerical simulations were performed by Simulink of MATLAB and the initial functions are chosen as default of Simulink for delay differential equations with \( y_1(0) = 0.2, y_1(0) = 0.1, y_2(0) = -0.3, y_2(0) = -0.1, y_3(0) = 0.5, y_3(0) = -0.4. \)

From (11) and (13), we have

\[ \tau_{10} = 1.3322, \quad \tau_{11} = 4.6172, \quad \tau_{12} = 8.2315, \quad \tau_{13} = 10.3393, \]

\[ \tau_{14} = 15.1307, \quad \tau_{15} = 16.0615, \quad \tau_{16} = 21.7836, \quad \tau_{17} = 22.0300, \]

and

\[ \tau_{20} = 2.2639, \quad \tau_{21} = 4.0801, \quad \tau_{22} = 8.3267, \quad \tau_{23} = 10.5916, \]

\[ \tau_{24} = 14.3896, \quad \tau_{25} = 17.1032, \quad \tau_{26} = 20.4524, \]

\[ \tau_{27} = 23.6147, \quad \tau_{28} = 265153. \]

So, the critical values of the coupling time delay can be arranged as

\[ \tau_{10} < \tau_{20} < \tau_{10} < \tau_{11} < \tau_{12} < \tau_{13} < \tau_{22} < \tau_{23} < \tau_{24}. \]

According to Theorem 2.1, the zero equilibrium of system (2) is asymptotically stable for \( \tau \in (\tau_{10}, \tau_{20}) \cup (\tau_{20}, \tau_{10}) \cup (\tau_{11}, \tau_{21}) \). Figure 3 illustrate this result for \( \tau = 2 \in (\tau_{10}, \tau_{20}). \)

According to Theorems 3.1, 4.1, and 4.2, when \( \tau \) is less than and close to \( \tau_{10} \) stable synchronous periodic solutions bifurcate from the zero solution. When \( \tau \) is greater than and close to \( \tau_{20} \) or \( \tau \) is less than and close to \( \tau_{20} \), stable phase-locked periodic solutions bifurcate from the zero solution. The existence of these stable phase-locked periodic solutions are illustrated in Figs. 4(a) and 4(b) for \( \tau = 2.3 \) greater than and close to \( \tau_{20} \) and in Figs. 4(e) and 4(f) for \( \tau = 2.4 \) less than and close to \( \tau_{20} \). When \( \tau \) is greater than and close to \( \tau_{10} \) or \( \tau \) is less than and close to \( \tau_{11} \), stable synchronous periodic solutions again bifurcate from the zero solution. The existence of these stable synchronous periodic solutions is illustrated in Figs. 5(a) and 5(b) for \( \tau = 4.7 \) greater than and close to \( \tau_{10} \) and in Figs. 5(e) and 5(f) for \( \tau = 8.1 \) less than and close to \( \tau_{11} \). These numerical simulations are exactly consistent with the results obtained in Secs. II–IV. When \( \tau \) is far away from these critical values, our theoretical analysis above is not available. However, numerical simulations show that these stable phase-locked periodic solutions always exist for all \( \tau \in (\tau_{20}, \tau_{26}) \) and stable synchronous periodic solutions always exist for all \( \tau \in (\tau_{10}, \tau_{11}) \) and that further the delay is far away from the critical values, the bigger the amplitude of the periodic solution is. Figures 5(c) and 5(d) illustrate the trajectories of these periodic oscillations.

![FIG. 3. (Color online) Trajectories of the oscillators \( y_1 \) (solid line), \( y_2 \) (dashed line), and \( y_3 \) (dotted line): the zero equilibrium is asymptotically stable for \( \tau \in (\tau_{10}, \tau_{20}) \cup (\tau_{20}, \tau_{10}) \cup (\tau_{11}, \tau_{21}) \). Here \( \tau = 2 \in (\tau_{10}, \tau_{20}). \)](image-url)
FIG. 4. (Color online) Trajectories of the oscillators $y_1$ (solid line), $y_2$ (dashed line), and $y_3$ (dotted line) with stable phase-locked periodic motion at $\tau = 2.3$ (a and b), $\tau = 3.1$ (c and d), and $\tau = 4$ (e and f), where $\tau \in (\tau_0, \tau_N)$.

FIG. 5. (Color online) Trajectories of the oscillators $y_1$ (solid line), $y_2$ (dashed line), and $y_3$ (dotted line) with stable synchronous periodic motion at $\tau = 4.7$ (a and b), $\tau = 6.5$ (c and d), and $\tau = 8.1$ (e and f), where $\tau \in (\tau_0, \tau_N)$. 

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oscillations for \( \tau = 3.1 \) far away from the critical values \( \tau_{20}^+ \) and \( \tau_{20}^- \), and Figs. 6(c) and 6(d) illustrate the trajectories of these periodic oscillations for \( \tau = 6.5 \) far away from the critical values \( \tau_{10}^+ \) and \( \tau_{11}^- \). The results of these numerical continuation are described as shown in Fig. 7.

When \( \tau > \tau_{21}^+ \), the zero equilibrium of system (2) loses its stability permanently. When \( \tau \) is greater than and close to \( \tau_{21}^- \), Theorems 3.1, 4.1, and 4.2 show that there are stable phase-locked periodic orbits bifurcating from the zero equilibrium. Numerical simulations show that system (2) always possesses for all \( \tau > \tau_{21}^- \). With increasing the coupling time delay, oscillation patterns of these stable periodic orbits switch from being phase-locked to synchronous and back to phase-locked several times and finally becoming synchronous. Numerical simulations also show that these stable periodic orbit changes its oscillation patterns only when \( \tau \) is less than and very close to the critical values \( \tau_{10}^- \) and \( \tau_{11}^- \). More precisely, there exist sufficiently small positive number \( \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \) such that stable phase-locked periodic orbits exist for

\[
(\tau_{21}^- \tau_{21}^+ - \mu_1) \cup (\tau_{12}^- \tau_{22}^+ - \mu_2) \cup (\tau_{13}^- \tau_{23}^+ - \mu_3)
\]

and stable synchronous periodic orbits exist for

\[
(\tau_{21}^- \tau_{21}^+ - \mu_1) \cup (\tau_{12}^- \tau_{22}^+ - \mu_2) \cup (\tau_{13}^- \tau_{23}^+ - \mu_3) \cup (\tau_{23}^+ - \mu_5, +\infty).
\]

Figure 7 shows the switch from being phase-locked to synchronous when \( \tau \) is less than and close to \( \tau_{21}^- \). Figures 6(a) and 6(b) show the existence of stable phase-locked periodic oscillation for \( \tau = 10.4 < \tau_{21}^- \mu_1 \) and Figs. 7(c) and 7(d) show the existence of stable synchronous periodic oscillation for \( \tau = 10.5 > \tau_{21}^- \mu_1 \).

VI. CONCLUSIONS

We have studied the bifurcation, amplitude death, and oscillation patterns induced by the coupling time delay in a system of three coupled van der Pol, which are coupled through the damping terms. We have shown that the coupling time delay can lead to rich dynamics. The stability of zero solution is determined in the parameter space. The existence of

![Graphs showing oscillations and trajectories](image-url)
stability switches induced by the coupling time delay is found in the region of the parameter plane of the damping strength and the coupling strength. The islands of amplitude death are also described in the parameter plane of the coupling time delay and strength for appropriate fixed damping strength.

In the region where stability switches occur, there are two kinds of critical values $\tau_{1}^c$ and $\tau_{2}^c$ of the coupling time delay leading to Hopf bifurcating periodic solutions. By using the symmetric bifurcation theory of delay differential equations due to Wu, we have shown that at the critical values $\tau_{1}^c$, there exist a synchronous periodic solution bifurcation and at the critical values $\tau_{2}^c$, there are eight branches of asynchrony periodic solutions bifurcating simultaneously from the zero solution: two phase-locked oscillations, mirror-reflecting waves and three standing waves. The direction and stability of these bifurcations is based on the normal form calculations. We have shown that the bifurcation is subcritical at the critical values $\tau_{2}^c$ and supercritical at the critical values $\tau_{1}^c$, $k = 1, 2, j = 0, 1, 2, \ldots$. We have also shown that on the center manifold the synchronous periodic and phase-locked periodic solutions are stable and other mirror-reflecting and standing waves are unstable.

Numerical studies have been employed to support and extend the obtained theoretical results. For fixed damping strength and coupling strength, the numerical simulations illustrate the existence of stable synchronous and phase-locked periodic solutions near the critical values of the coupling time delay which is in good agreement with our theoretical analysis results. For the coupling time delay far away from the critical values, our theoretical analysis is not available. However, numerical simulations show that with increasing the coupling time delay from the zero, the dynamics of the system switches from being synchronous to stability, to being phase-locked to stability, and so on. When the zero solution ultimately loses its stability, the oscillation patterns of these stable periodic solutions switch from being phase-locked to being synchronous (or vice versa) and finally becoming synchronous with increasing the coupling time delay.

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