Transactions Briefs

Binary Output of Cellular Neural Networks with Smooth Activation

Lachlan L. H. Andrew

***Abstract*—An important property of cellular neural networks (CNN's) is the binary output property, that, when the self-feedback is greater than one, the final activations are ± 1. This brief considers the generalization of this property to networks with sigmoidal output functions. It is shown that in this case the property cannot be stated without reference to the cross feedback, and conditions are found under which the property remains valid.**

I. Introduction

Cellular neural networks (CNN's) form an important neural network paradigm which has the important property of being realisable in hardware [1]–[3]. An n neuron CNN implements a dynamical system of the form

$$\dot{\mathbf{x}} = -\mathbf{x} + A\mathbf{y} + \mathbf{k} \tag{1}$$

where $\mathbf{x}$ is the state vector and $\mathbf{y}$ is the corresponding output vector, with ith elements x_i and $y_i = f(x_i)$, respectively, $A = (a_{ij})$ is the feedback matrix, and $\mathbf{k}$ represents the input and the bias. Ideally f is the piecewise linear function $f_L(x) = (|x+1| - |x-1|)/2$, but more generally will be a continuous, piecewise differentiable, bounded, monotonic function. Since a real hardware implementation can only approximate f_L it is important to show that the properties of CNN's do not depend heavily on the precise form of f.

Important applications of CNN's and their derivatives include binary image processing [4]–[9], binary associative memories [10]–[12] and winner-take-all networks [13]–[15]. In each of these applications, the outputs represent binary quantities. Hence it should be clear whether a neuron is "on" or "off". This can be guaranteed by the binary output property (BOP) [1], [2], [16], which is possessed by all CNN's with $f = f_L$.

Definition 1: The BOP is the property that a convergent CNN with self feedback $a_{ii} > 1$ for all i will, for a given input $\mathbf{k}$, converge to a state with $|y_i| = 1$ for all i for almost all initial conditions.

This property has also been used in proving stability results [17], [18]. A form of this important property applicable to more general functions f is presented in Section II, and it is then shown that the effect of cross feedback terms, a_{ij} with $i \neq j$, cannot in general be ignored. Sections III and IV then prove that the precise values of the cross feedback terms can indeed be ignored in the important cases of reciprocal networks and transpose diagonally dominant networks, which between them include many useful CNN's.

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The author was with the Department of Electrical and Electronic Engineering at the University of Melbourne, Parkville 3052, Victoria, Australia. He is now with the Department of Communication and Electronic Engineering, Royal Melbourne Institute of Technology, Melbourne 3001, Victoria, Australia.

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II. Binary Output with Smooth Activation

When the output function f is smooth, a neuron will gradually saturate as $|x| \to \infty$ rather than attaining a limit for finite x. Despite this, the notion of all neurons being clearly "on" or "off" can be expressed in this case by the smooth binary output property (SBOP):

Definition 2: Let P be the statement that a) $f'(x) \geq 1$ for all $x \in (b,c)$ and b) $a_{ii} > 1$ for all i. Let Q be the statement that for each stable equilibrium, $\mathbf{x}$, there is no output $y_i \in (f(b), f(c))$. A class of CNN's is said to have the SBOP for the interval (b,c) if, for every CNN in that class, P implies Q. A class of CNN's is simply said to have the SBOP if it has the SBOP for every interval (b,c).

In the context of smooth f, a *binary output state* with respect to some interval (b,c) may be defined to be a state in which, for all i, $y_i \notin (f(b), f(c))$. Having the SBOP ensures that, if a CNN converges, it almost surely converges to a binary output state.

An equilibrium point will be unstable if there is at least one positive eigenvalue of the corresponding linearized system

$$\dot{\mathbf{x}} = (AD - I)\mathbf{x} \tag{2}$$

where $D = \operatorname{diag}(f'(x_1), \ldots, f'(x_n))$, I denotes the identity matrix and $\mathbf{x}$ denotes the deviation of the state from the equilibrium point. This system is homogeneous since the linearization is about an equilibrium point.

Although it has been shown [16] that when $f = f_L$ the BOP holds with no restriction on the off-diagonal elements of A (the cross feedback), this is not the case for general f. The following example, using a smooth function which can approximate f_L arbitrarily closely, has a stable equilibrium with one output in the active region, (-1,+1).

Consider the 2×2 case of (2) with f not f_L. Let $\delta_i = f'(x_i)$. If $\delta_1 a_{11} + \delta_2 a_{22} < 2$ and $(\delta_1 a_{11} - \delta_2 a_{22})^2 + 4\delta_1\delta_2 a_{12} a_{21} < 0$, then the trace $T := \operatorname{Tr}(AD - I) < 0$, and the complex eigenvalues both have real part $T/2 < 0$, so $AD - I$ will have no eigenvalues with positive real part. Let f_ϵ be given by

$$f_\epsilon(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 + \epsilon\left(1 - e^{(1-x)/\epsilon}\right) & \text{if } x > 1 \\ -f_\epsilon(-x) & \text{if } x < 0 \end{cases}. \tag{3}$$

As $\epsilon \to 0$, $f_\epsilon \to f_L$. Now let $1 < a_{11} = a_{22} < 2$, let δ be a constant such that $0 < \delta < (2 - a_{11})/a_{22} < 1$, let $a_{12} = -a_{21} = 1/\delta$ and $\epsilon = 1/\log(1/\delta)$, and let x_1^+ be an arbitrary value in $(-1,+1)$ and $x_2^+ = 2$. Linearizing at the point $\mathbf{x}^+$ gives $\delta_1^+ = 1$ and $\delta_2^+ = \delta$. Now if $\mathbf{k} = \mathbf{x}^+ - Af_\epsilon(\mathbf{x}^+)$ then the system (1) with $f = f_\epsilon$ has a stable equilibrium point (x_1^+, x_2^+) with $f(x_1^+) \in (-1,+1)$, and thus does not have the SBOP on (-1,+1).

For a more concrete example, consider the special case of

$$A = \begin{pmatrix} 1.2 & 100 \\ -100 & 1.2 \end{pmatrix}$$
$$\epsilon = 1/\log(100) \approx 0.217$$
$$\mathbf{k} = (0.5,2)^T - Af_\epsilon\left((0.5,2)^T\right).$$

Fig. 1 shows f_ϵ in this case. Linearizing about the equilibrium point $\mathbf{x}^+ = (0.5,2)$ gives

$$\dot{\mathbf{x}} = \begin{pmatrix} 0.2 & 1 \\ -100 & -0.988 \end{pmatrix}\mathbf{x}. \tag{4}$$

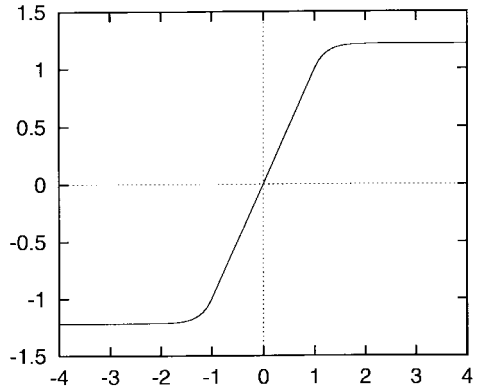

Fig. 1. Activation function $f_{0.22}$.

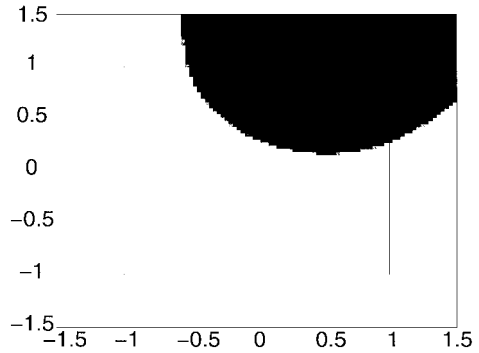

Fig. 2. Basins of attraction of the two stable equilibria of the example CNN without the SBOP. The black area converges to the nonbinary output equilibrium.

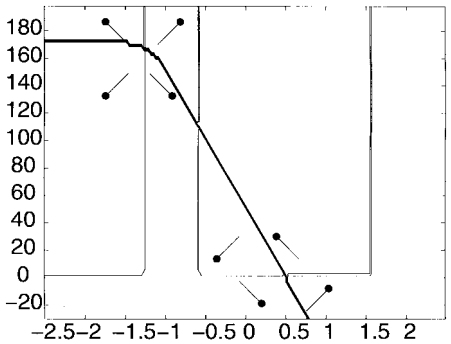

Fig. 3. Pin diagram of the signs of $\dot{x}_1$ and $\dot{x}_2$. Curves indicate points at which $\dot{x}_i = 0$. The three equilibria where $\dot{\mathbf{x}} = 0$ can be clearly seen.

TABLE I
EIGENVALUES λ_m WITH GREATEST REAL PART OF $AD - I$ FOR $n \times n$ TRIDIAGONAL TOEPLITZ A

n	λ_m
2	$-0.175000 + 0.651441i$
3	$-0.247693 + 0.000000i$
4	$-0.295359 + 0.374517i$
5	$-0.322510 + 0.575111i$
6	$-0.340325 + 0.689794i$
7	$-0.353876 + 0.759355i$
8	$-0.365731 + 0.803846i$
9	$-0.375483 + 0.636467i$
10	$-0.380491 + 0.698926i$

The system matrix can easily be shown to have eigenvalues with real part -0.394 and thus this equilibrium point is stable. Indeed, simulations indicate that it is one of two stable equilibria of the system, as shown in Fig. 2. However, for slightly smaller values of $a_{11} = a_{22}$ only one equilibrium exists. The difference in behavior in these two cases can be explained by observing the signs of $\dot{x}_i$ as a function of $\mathbf{x}$ in each case. Fig. 3 shows this for $a_{11} = a_{22} = 1.2$ and $\delta = 0.01$. The state vector will move in the direction of the line away from the circle in each of the regions. The pins have a leftward component if $\dot{x}_1 < 0$ and a downward component if $\dot{x}_2 < 0$. This shows that there is a stable equilibrium at around $(-1.267, 167.38)^{\mathrm{T}}$ and an unstable equilibrium at around $(-0.586, 110.6)^{\mathrm{T}}$ in addition to the stable equilibrium at $(0.5, 2)^{\mathrm{T}}$. As a_{11} decreases, the vertical band separating these two additional equilibria gets narrower, until it vanishes with $a_{11} \approx 1.143$, and the two spurious equilibria vanish with it. On the other hand, as δ increases and $f \to f_L$, the band widens and the nonbinary output equilibrium becomes decreasingly stable.

This example shows that, for activation functions other than f_L, the BOP does not depend only on the self-feedback. It is easy to show that for any bounded monotonic differentiable function, f, such that there is an $x \in (b, c)$ with $f'(x) < 2$, there will be a 2×2 matrix A (i.e., a two neuron CNN) satisfying the requirements of P in the SBOP, and operating point $\mathbf{x}^+$ such that $\mathbf{x}^+$ is stable and $x_1 \in (b, c)$. However the above example also shows that care must be taken when replacing a function by an arbitrarily close approximation as must be done in VLSI implementation. Although $f_\epsilon \to f_L$, the BOP holds for f_L but does not in general for f_ϵ.

Note also that this example relies on the fact that $a_{12}a_{21} < 0$. It can be shown that if $a_{12}a_{21} \geq 0$ then the larger eigenvalue of AD satisfies $\lambda \geq \max(\delta_1 a_{11}, \delta_2 a_{22})$. This quantity is greater than 1 since $a_{ii} > 1$ by hypothesis, and for one of the state variables to be in (b, c), there must be at least one $\delta_i \geq 1$.

This phenomenon is not limited to the 2×2 case. If A is a tridiagonal $n \times n$ Toeplitz matrix with $1 < a_{11} < \sqrt{2}$ and $a_{12} = -a_{21} \geq 1$ and there is at least one $x \in (b, c)$ such that $f'(x) = 1$ then there exists a $\mathbf{k}$ such that the system will violate the SBOP. To illustrate this, consider the concrete example of $a_{11} = 1.1$ and $a_{12} = 1$, giving

$$A = \begin{pmatrix} 1.1 & 1 & 0 & \dots & 0 \\ -1 & 1.1 & 1 & \dots & 0 \\ 0 & -1 & 1.1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1.1 \end{pmatrix}$$

and an operating point such that $x_1 \in (b, c)$ with $f'(x_1) = 1$ and x_i such that $f'(x_i) = 1/2$ for $i \neq 1$, giving $D = \mathrm{diag}(1, 1/2, 1/2, \dots, 1/2)$. This occurs, for example, if $f = f_\epsilon$ with $\epsilon = 1/\log(2)$, $\mathbf{x}^+ = (0, 2, \dots, 2)^T$, and $\mathbf{k} = \mathbf{x}^+ - Af_\epsilon(\mathbf{x}^+)$. The eigenvalues, λ_m, of $AD - I$ with largest real part in the $n \times n$ case for $n = 2, \dots, 10$ are shown in Table I. This indicates that the real part of λ_m converges toward $a_{11}/2 - 1$ from above as n increases, and so a CNN of arbitrary size may be constructed which does not have the SBOP. It is interesting to note that the matrix in this example is of the same form as those discussed in [18].

The remainder of this brief describes special forms of A which have the SBOP.

III. RECIPROCAL NETWORKS

The most important class of CNN's is reciprocal networks, in which A is Hermitian (or real symmetric). This corresponds to a cloning template which is centro-Hermitian ($a_{ij} = \bar{a}_{n-i+1,n-j+1}$) or

real centro-symmetric ($a_{ij} = a_{n-i+1,n-j+1}$). There is a large body of literature about matrices of this form [19]. In particular, many useful CNN's of this form have been developed [4], [6], [13]–[15]. Any reciprocal CNN has the SBOP. To prove this, the following known result [16] will be used.

Theorem 1: Let α and β be positive real numbers. If A is an $n \times n$ matrix whose diagonal elements a_{ii} are real and satisfy $a_{ii} > \alpha$, and $D = \text{diag}(\delta_1, \ldots, \delta_n)$ is a real diagonal matrix with $\delta_i = 0$ or $\delta_i \geq \beta$ for all i, and not all δ_i are zero, then AD has at least one eigenvalue λ with $\text{Re}(\lambda) > \alpha\beta$.

This will be combined with the following result.

Theorem 2: If A is an $n \times n$ Hermitian matrix, and $D = \text{diag}(\delta_1, \ldots, \delta_n)$ is a real diagonal matrix with $\delta_i \geq 0$ then for all nonzero eigenvalues λ of AD, $\partial\lambda/\partial\delta_i = k\lambda$, where $k \geq 0$ may depend on D and i.

Proof: Define the selector matrix, $S_i = (s_{jk})$ such that s_{jk} is 1 if $i = j = k$ and 0 otherwise. Let $\mathbf{v}$ be a right eigenvector of AD corresponding to some $\lambda \neq 0$, scaled such that $\mathbf{v}^* D\mathbf{v} = 1$ (which is possible since $\lambda \neq 0$ and D is nonnegative diagonal), so $AD\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{u}^* = (D\mathbf{v})^*$ is a left eigenvector of AD. Here * denotes the complex conjugate transpose. It can be shown that the nonzero eigenvalues of AD are nondefective, and so the well-known result (see, for example, [20, eq. 12.4])

$$\frac{\partial\lambda}{\partial\delta_i} = \frac{\mathbf{u}^*(\partial(AD)/\partial\delta_i)\mathbf{v}}{\mathbf{u}^*\mathbf{v}}$$

can be applied. Noting that $\mathbf{u}^*\mathbf{v} = 1$, it follows that

$$\begin{aligned}\frac{\partial\lambda}{\partial\delta_i} &= \mathbf{u}^* A\frac{\partial D}{\partial\delta_i}\mathbf{v} = \mathbf{u}^* AS_i\mathbf{v} = \mathbf{v}^* D^* AS_i\mathbf{v} \\ &= (AD\mathbf{v})^* S_i\mathbf{v} = \overline{\lambda}\mathbf{v}^* S_i\mathbf{v} = \lambda\mathbf{v}^* S_i\mathbf{v}\end{aligned}$$

where the last step uses the fact that λ is real, as will be shown below. Now let $k := \mathbf{v}^* S_i\mathbf{v} = \overline{v_i}v_i = |v_i|^2 \geq 0$. Then $\partial\lambda/\partial\delta_i = k\lambda$ with $k \geq 0$ as required. ∎

The above theorem used the fact that all of the eigenvalues λ of AD are real. In the case that D is nonsingular, this follows from the fact that K-Hermitian matrices have real eigenvalues [21, p. 111]. When D is singular, AD need not be K-hermitian, but λ may still simply be shown to be real as follows. If $\mathbf{v}^* D\mathbf{v} = 0$ then $\lambda = 0$ and is real. Otherwise $\lambda\mathbf{v}^* D\mathbf{v} = \mathbf{v}^* D(AD\mathbf{v}) = (\mathbf{v}^* D^* A^*)D\mathbf{v} = \overline{\lambda}\mathbf{v}^* D\mathbf{v}$, and so $\lambda = \overline{\lambda}$ and λ is real.

Corollary 1: Let A and D be $n \times n$ matrices. If A is Hermitian with $a_{ii} > \alpha > 0$ and D is positive semidefinite, real and diagonal with at least one element not less than $\beta > 0$, then AD has at least one eigenvalue $\lambda > \alpha\beta$.

Remark: The significance of this result is that it only requires that *at least one* diagonal element of D be not less than β. In the much more restricted case that *all* diagonal elements of D are not less than β, this follows from the elementary result that the sum of the eigenvalues is equal to the trace of AD [22, p. 95], [23, p. 1].

Proof of Corollary 1: Let $\mathcal{P}_j$ denote the set of all $n \times n$ positive semidefinite diagonal matrices with j elements in the interval $(0, \beta)$ and at least one element not less than β. Clearly $D \in \mathcal{P}_j$ for some j. Let $\mathcal{D}_j$ denote the set of all $P \in \mathcal{P}_j$ for which AP has at least one eigenvalue $\lambda > \alpha\beta$. It then suffices to show that $\mathcal{D}_j = \mathcal{P}_j$ for all j. Clearly $\mathcal{D}_0 = \mathcal{P}_0$ by Theorem 1. Make the inductive hypothesis that $\mathcal{D}_j = \mathcal{P}_j$ for some j. To see that $\mathcal{D}_{j+1} = \mathcal{P}_{j+1}$, consider an arbitrary $P = \text{diag}(\delta_1, \ldots, \delta_n) \in \mathcal{P}_{j+1}$ with $\delta_i \in (0, \beta)$. Consider $\hat{P}(t) = \text{diag}(\delta_1, \ldots, \delta_{i-1}, t, \delta_{i+1}, \ldots, \delta_n)$, and let $\lambda(t)$ denote the largest eigenvalue of $A\hat{P}(t)$. Now $\hat{P}(0) \in \mathcal{P}_j = \mathcal{D}_j$, so $\lambda(0) > \alpha\beta > 0$. But $d\lambda(t)/dt > 0$ by Theorem 2. Thus $\lambda(\delta_i) > \alpha\beta$ and $P = \hat{P}(\delta_i) \in \mathcal{D}_{j+1}$. Thus $\mathcal{D}_j = \mathcal{P}_j$ for all j, and the corollary is proved. ∎

It may now be proved that reciprocal networks have the SBOP.

Theorem 3: The class of CNN's with Hermitian weight matrix A has the SBOP.

Proof: For the output y_i to be in $(f(b), f(c))$ it is necessary that $x_i \in (b, c)$. Assume, with a view to obtaining a contradiction, that there exists a stable equilibrium with $x_i \in (b, c)$ for some i. Then the linearization about this point (2) satisfies the conditions of Corollary 1 with $\alpha = \beta = 1$, and so AD has an eigenvalue $\lambda > 1$, and $AD - I$ has a positive eigenvalue. Thus the equilibrium is unstable, contradicting the assumption, and the theorem is proved. ∎

It has been shown [1], [2] that a CNN with Hermitian feedback matrix converges for smooth activation functions as well as for the piecewise linear function. Thus, and because Theorem 3 shows that the only stable states are binary output states, such a system will almost surely converge to a binary output state.

IV. Diagonally Dominant Networks

Another class of CNN's which have the SBOP is those for which $A - I$ is transpose diagonally dominant (TDD). A matrix M is said to be TDD if M^* is strictly diagonally dominant with positive diagonal elements. This corresponds to a "centrally dominant" cloning template, in which the central element is at least one greater than the sum of the absolute values of the other elments.

Theorem 4: The class of CNN's with weight matrix A such that $A - I$ is TDD, has the SBOP.

Proof: By definition of transpose diagonal dominance, for all i

$$a_{ii} - 1 > \sum_{j \neq i} |a_{ji}|.$$

Now if there is an equilibrium with $y_i \in (f(b), f(c))$ for some i then $\delta_i = f'(x_i) \geq 1$, and so

$$a_{ii}\delta_i - 1 \geq \delta_i(a_{ii} - 1) > \delta_i\left(\sum_{j \neq i} |a_{ji}|\right).$$

Then by Gershgorin's theorem, $(AD - I)^*$ has an eigenvalue with positive real part, and hence so does $AD - I$. Thus the equilibrium is not stable. ∎

This result covers a wider range of dominant matrices than some previous results. For example, [24] shows that systems whose elements have a specific sign structure have a unique global attractor. In contrast, TDD matrices may give rise to multiple stable states. A simple example is $A = \alpha I$, where $\alpha > 1$ and $\mathbf{k} = 0$, in which all states with $|x_i| = \alpha$ for all i are stable. However, the result presented here is limited to simple CNN's, while that of [24] includes the wider class of delay type CNN's.

In many proposed templates, A is diagonally dominant but $A - I$ is not (for example [5], [8]). This is not in general sufficient for the CNN to have the SBOP. In the case $n = 2$ of the second example given in Section II A is diagonally dominant, but was shown there not to have the SBOP.

Theorems 3 and 4 are independent of the network's size and can be formulated in terms of replicated templates. A third sufficient condition for the SBOP to hold relates the self feedback to the size of the network rather than specifying a form for the cross feedback.

Theorem 5: The class of CNN's with n neurons and weight matrix, A, with diagonal elements satisfying $a_{ii} > n$ for all i and arbitrary off-diagonal elements, has the SBOP.

Proof: These networks satisfy $\text{Tr}(AD - I) = \left(\sum_i a_{ii}\delta_i\right) - n$. Each term of the sum will be nonnegative, since $\delta_i = f'(x_i) \geq 0$. If there is an equilibrium with $y_i \in (f(b), f(c))$ for some i then $\delta_i = f'(x_i) \geq 1$, and so $a_{ii}\delta_i > n$ giving $\text{Tr}(AD - I) > 0$, implying that at least one eigenvalue of $AD - I$ is positive. Thus the equilibrium is not stable. ∎

V. Conclusion

This brief has generalized the BOP to CNN's with an important class of realisable, smooth transfer functions. It has been shown that the cross feedback does affect the binary output of such CNN's. However in most important cases, namely reciprocal networks and transpose diagonally dominant networks, the precise cross feedback terms are not important.

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Approximating Delay Elements by Feedback

Alessandro Beghi, Antonio Lepschy, and Umberto Viaro

***Abstract*—A procedure for obtaining a proper rational approximant of the transfer function of a delayor is suggested. In particular, the step response of the unity-feedback system with the delayor in the direct path is first approximated by truncating the Fourier series expansion of its periodic component, and then the corresponding direct-path rational transfer function is derived, thus arriving at a stable Blaschke product.**

***Index Terms*— Approximation, delay elements, feedback systems, Fourier analysis.**

I. Introduction

Many approximation techniques aim at the construction of a simplified model of a plant to be controlled, in such a way that the design of the controller can be implemented more easily.

To this purpose, it is advisable to derive the model of the simplified plant by referring to the desired closed-loop system characteristics (cf., e.g., [1]). In fact, the feedback control system could even turn out to be unstable with the original plant if the controller were designed by referring to a reduced model of the plant obtained without consideration of the closed-loop specifications (bandwidth, resonance peak, etc.), as pointed out in [2].

Approximating a given system characterized by the transfer function $G(s)$ from an approximation of

$$W(s) = \frac{G(s)}{1 \pm G(s)} \tag{1}$$

(see Fig. 1) may be convenient for different reasons, too.

For instance, when $G(s)$ has no zeros and its poles cannot reasonably be separated into a set of dominant poles and a set of remote poles, a reduction procedure based on the retention of the dominant modes is not applicable to $G(s)$. Instead, in most cases a pole-retention technique can easily be applied to $W(s)$ because its poles diverge for increasing values of the loop gain magnitude (tending to the asymptotes of the related root locus) and, therefore, some of them become definitely dominant over the others.

By denoting with $\hat{W}(s)$ the approximation of $W(s)$, the required reduced model will then be obtained according to

$$\hat{G}(s) = \frac{\hat{W}(s)}{1 \mp \hat{W}(s)}. \tag{2}$$

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A. Beghi and A. Lepschy are with the Department of Electronics and Informatics, University of Padova, 35131 Padova, Italy.

U. Viaro is with the Department of Electrical, Managerial and Mechanical Engineering, University of Udine, 33100 Udine, Italy.

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