Comparative Study of Attenuation and Dispersion of Different Pressure Disturbances Propagating in Aerosols

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Abstract. This study aims to investigate the attenuation and dispersion of pressure waves propagating in aerosols. A numerical scheme based on Crank-Nicolson finite differencing and trapezoidal quadrature discretization methods is employed to simulate the evolutions of several pressure waves with different initial profiles (disturbances). The attenuation and dispersion are observed from the spectra of the wave profiles at different times of propagation obtained by performing FFT (Fast Fourier Transform) to the simulated wave profiles. Simulation results show distinctive behaviors of different pressure disturbances propagating through the same medium.

Keywords: Finite amplitude pressure wave, Aerosol, Crank-Nicolson, Wave profile change
PACS: 43.28.Js, 92.60.Mt, 47.11.Bc, 33.20.Ea

INTRODUCTION

Pressure disturbances in gaseous medium containing suspended liquid or solid particles have been studied by several authors [1–4]. These disturbances are referred to as pressure waves. As a pressure wave propagates through this medium, it interacts with the suspended particles by exchanging its energy. In general, the literature agrees that the most significant energy exchange processes between the pressure wave and the particles are thermal and momentum exchanges. It is well known that pressure wave attenuates (loses its energy) and disperses (separates into other frequencies) under such interactions.

Detailed studies on attenuation and dispersion of pressure waves in aerosols were performed by [3, 4]. The aforementioned works provide useful information for modelling pressure waves in audible amplitudes and frequencies. In this case, due to its low amplitude, the interaction between the wave and suspended particles is almost linear and therefore there is little change in the shape of the wave, although significant change in wave amplitude still exists. A long pressure wave with high amplitude, however, allows for energy exchange between phases to occur for an extensive period of time and given the non-linear nature of the exchange processes, the wave profile undergoes a major change during its propagation. To this end, this paper provides simulation and comparison of several pressure disturbances to visualize the attenuation and dispersion of long pressure waves in aerosols.

METHODOLOGY

The evolution of pressure wave profile propagating in aerosol is simulated by solving the following equation developed in [5] with periodic boundary conditions \( \hat{P}(\hat{x}+1,\hat{t}) = \hat{P}(\hat{x},\hat{t}) \).

\[
\hat{\frac{\partial \hat{P}}{\partial \hat{x}}} = \frac{\kappa^2 - \kappa^2 + 2}{\kappa^2 (\kappa + 1)} \hat{\frac{\partial \hat{P}}{\partial \hat{t}}} - \frac{9}{2} \frac{\partial^2 \hat{P}}{\partial \hat{y}^2} - 2 \frac{\partial \hat{P}}{\partial \hat{y}} \frac{9 \phi_{x}}{2} \frac{1}{\kappa + 1} \frac{T_c}{\tau_c} \hat{\frac{\partial \hat{P}}{\partial \hat{y}}} - \frac{9 \phi_{x}}{2} \frac{\kappa - 1}{\kappa + 1} \frac{T_c}{\tau_c} \hat{\frac{\partial^2 \hat{P}}{\partial \hat{y}^2}} - 3 \phi_{z} \rho_{x} \kappa - 1 \frac{C_{p}}{\tau_c} \frac{T_c}{\tau_c} \frac{\partial}{\partial \hat{y}} \frac{\hat{P}}{\sqrt{T_c} \hat{\frac{\partial \hat{P}}{\partial \hat{y}}}} 0, \exp \left( -\pi^2 \frac{T_c}{\tau_c} (\hat{t} - \hat{y}) \right) ) - 1 \right) d\hat{y} = 0. \tag{1}
\]

This model is developed by considering a pressure disturbance that is applied to governing equations of continuity, momentum conservation, energy conservation and the ideal gas law. By expanding the perturbation accurate to second order, the set of equations can be reduced into a single equation (Eq. (1)) which eliminates the need to solve several non-linear differential equations simultaneously. Here, \( \phi_{z}, \phi_{y} \) is Jacobi Theta function.
Equation (1) is in dimensionless form where \( \hat{P} = P/P_0 = (P - P_0)/P_0 \), \( \hat{x} = x/\lambda \) and \( \hat{\tau} = \tau/T \). Here, \( P_0 \) and \( \lambda \) are the undisturbed pressure of the gas in its steady state and the wavelength of the initial pressure disturbance. The wave propagates to the right (positive \( x \)-direction) with the speed of sound \( C \). \( P \) is the local gas pressure at the position \( x \) and retarded time \( \tau = t - x/C \). The coordinate system \((x, \tau)\) travels together with the wave with speed \( C \). The aerosol considered in this paper is a gaseous medium with suspended particles of uniform radius \( \delta \) and the fraction of volume occupied by these particles in the aerosol is described by \( \rho \).

During its propagation, the pressure wave loses its energy to the suspended particles through momentum and thermal exchanges. The rate at which the pressure wave loses its energy is related to the characteristic relaxation times of the momentum exchange process \( \tau_v = \delta^2/v \) and thermal exchange process \( \tau_P = \delta^2/a \) and these two parameters depend on the particle size and the physical properties of the aerosol. The properties involved are: \( C_V \) (specific heat of the gas), \( C_{part} \) (specific heat of the particle), \( a \) (thermal diffusivity of the gas), \( v \) (kinematic viscosity of the gas), \( \kappa \) (adiabatic constant of the gas), \( \rho_P \) (density of the particle) and \( \rho_0 \) (density of the gas).

The first term of Eq. (1) describes the stationary term of the wave that does not contribute to the change of wave shape during its propagation [6]. The second term, which is entirely dependent only on the adiabatic constant of the gas phase, accounts for the non-linearity of the gas phase of the medium. This allows shock formation (steepening of wave profile) to occur. The third to sixth terms describe the momentum relaxation process between the gas phase and suspended particles. The thermal relaxation process between the two phases can be seen from the last term of Eq. (1). These relaxation processes create asymmetry in the wave profile.

### Numerical Scheme

In order to solve the mathematical model of propagating pressure wave in aerosol, we propose a numerical scheme based on the Crank-Nicolson finite difference and semi-open trapezoidal quadrature. Due to its non-linear nature, the discretized form of the equation is solved by incorporating a certain root-finding algorithm.

![FIGURE 1. The computational grid setup used in simulating the evolution of pressure wave in aerosol.](image)

A computational grid was set up with the spatial step size of \( \Delta \hat{x} = 1/Z \), where \( Z \) denotes the number of spatial grids per wavelength and \( i = 1, 2, 3, ..., Z \). The temporal step size is \( \Delta \hat{\tau} = \Delta \hat{\nu} = 1/N \), where \( N \) is the number of temporal grids per wave period. We employ Crank-Nicolson method at the grid point of reference (Fig. 1). This grid point is an imaginary grid point located at \( \hat{x} = i\Delta \hat{x}, \hat{\tau} = (j + 1/2)\Delta \hat{\tau} \). Here, nodes at time \( \hat{\tau} \leq j\Delta \hat{\tau} \) are known from initial conditions or previous computational steps. The aim is to solve for the pressure wave profile at the next time step \( \hat{\tau} = (j + 1)\Delta \hat{\tau} \). The integral terms are discretized by semi-open trapezoidal quadrature across the temporal grids at \( \hat{x} = i\Delta \hat{x} \). We arrive at the final discretized form of Eq. (1) as an equation of three unknowns with quadratic non-linearity:

\[
S_i = K_i + \alpha P_{i-1,j+1} + \beta P_{i,j+1} + \gamma P_{i+1,j+1} + \Phi P_{i,j+1}^2 = 0.
\]  

By applying periodic boundary conditions \( P_{i,j+1} = P_{i+Z,j+1} = 0 \), we can construct the system of non-linear equations \( S \) from \( i = 1 \) to \( Z \).
The system of non-linear equations can be solved by employing the Newton-Raphson Method. This method approximates the roots of the system of equations by successively minimizing the differences between the approximated and the real roots. The algorithm begins by specifying the initial guess to the roots, \( \vec{p} \) (Eq. (4)). We use the wave profile from time step \( \hat{\tau} \) to be the initial approximation of \( \hat{\tau}_{j+1} \) due to the fact that the wave profile must evolve from previous time step. To proceed, the Jacobian matrix, \( J \) (matrix of partial derivatives) of the non-linear system is also needed. The Jacobian matrix that corresponds to the initial guess is \( J = J(p) \) and is a sparse matrix. The non-zero elements of \( J \) are shown in Eq. (5). The final step (Eq. (6)) computes the corrections to the current approximated roots, which are the elements of the deviation vector, \( \Delta = \vec{p} - \vec{p} \).

\[
\vec{p} = [\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_{Z-1}, \vec{p}_Z]^T = [P_{i,j}, P_{z,j}, \ldots, P_{Z-1,j}, P_{Z,j}]^T, \quad J_{i,j} = \beta + 2\delta \vec{p}_i, \quad J_{i,Z} = \alpha, \quad J_{i,j+1} = J_{Z,i} = \gamma,
\]

\[
\Delta = -S, \quad (J^{-1})(-S), \quad \vec{p} = \vec{p} + \Delta.
\]

The first Newton-Raphson approximated wave profile is stored in \( \vec{p} \). This process is performed iteratively until the maximum absolute value of the deviation vector is less than \( 10^{-15} \) which is the computer precision of a floating point number.

RESULTS AND DISCUSSIONS

Several simulations of pressure wave propagation in air-water aerosol are carried out. The initial temperature of the aerosol is 20°C and the physical properties of air and water are taken from [7] at 20°C. The aerosol contains particles with radius and volume fraction of \( \delta = 15\mu m \) and \( \rho = 6 \times 10^{-6} \), which are typical aerosol parameters found in automotive spray painting applications [8].

![Figure 2](image_url)

**FIGURE 2.** (a) The evolution of cosine wave at different times of propagation. (b) The corresponding one-sided magnitude spectra of the wave profiles.

We performed simulations on pressure disturbances of different profiles: Cosine with amplitude of \( \tilde{\rho} = 0.3 \) (Fig. 2) and Gaussian with Full-Width at Half Maximum (FWHM) = \( \lambda/8 \) and peak-to-peak magnitude \( \tilde{\rho} = 0.3 \) (Fig. 3). The period of both disturbances is \( T = 1s \) and the initial wave profiles are chosen to be zero-averaged. Figure 2(a) shows the evolution of a cosine wave in time-domain. Here, the wave profiles at different times (\( \hat{\tau} = 0, 5.0 \times 10^{-3} \) and \( 1.0 \times 10^{-2} \)) are plotted. The attenuation and change in shape of the wave profile are noticeable. Figure 2(b) shows the frequency-domain plots of the profiles in Fig. 2(a) and they are generated by performing FFT (Fast Fourier Transform). Due to the periodic boundary conditions, the spectrum of the wave is therefore discrete and only exists at the fundamental frequency and its harmonics. At time \( \hat{\tau} = 0 \), the magnitude spectrum of the cosine wave exists at the fundamental frequency component \( \hat{\xi} = 1 \). As the fundamental frequency component attenuates, it can be observed that the wave spreads into higher frequency harmonics and therefore the profile starts to steepen.
In similar fashion, Fig. 3 shows the evolution and spectra of a Gaussian wave. The spectrum of initial Gaussian profile is Gaussian in frequency-domain. Both the attenuation and dispersion of the Gaussian wave are evident. Unlike a sinusoidal wave, a Gaussian wave is not symmetrical about the steady state pressure \( \hat{P} = 0 \). Therefore, in order to attenuate to the steady state, the wave profile must widen as well (Fig. 3(a)). This means the higher frequency components are attenuating more rapidly than the lower frequency components (Fig. 3(b)).

**CONCLUSION**

The attenuation and dispersion of different pressure disturbances propagating in aerosols are generated using the numerical techniques described in this work. Simulation results show distinctive behaviors of different pressure disturbances. Frequency-domain analysis of the simulated profiles via FFT offers a convenient insight on how each individual frequency component of the wave behaves differently. This is particularly useful in studying the dispersion of waves that are not initially sinusoidal. The momentum and thermal interactions between the two phases allow the change of wave profile from its initial shape.

**ACKNOWLEDGMENTS**

The authors would like to express gratitude to Swinburne University of Technology for supporting this work through Swinburne Sarawak Research Grant (SSRG) and Swinburne University Postgraduate Research Award (SUPRA) for R. Siswoyo Jo.

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