Stabilization of T-S Fuzzy Systems Using LTV System Theory

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Abstract—Takage-Sugeno (T-S) fuzzy systems are equivalent to linear time-varying (LTV) systems, the theory of LTV systems can therefore be used to design global feedback controllers to stabilize T-S fuzzy systems. It is seen that the closed-loop T-S fuzzy system is first transformed into the canonical form by using a Lyapunov transform, a global feedback controller is then designed to assign all Parallel D-spectrum eigenvalues (PD-eigenvalues) of the closed-loop T-S fuzzy system to the desired locations whose real parts have negative extended means. The proposed control scheme not only guarantees asymptotic stability of the closed-loop T-S fuzzy systems, but also allows the designers to design a simple global controller instead of deriving a complex global controller by aggregating all local controllers using fuzzy membership functions. A simulation example is given in support of the proposed scheme.

I. INTRODUCTION

In reality, many physical systems are nonlinear and complex that may not be easily modeled mathematically. T-S fuzzy modeling technique is one of a few effective approaches to model complex nonlinear systems [1]-[5]. Three steps are often needed to model a complex system using T-S modeling technique: First, the whole state-space is decomposed into a few subspaces. Second, within each subspace, the closed-loop system is approximated using a linear time-invariant (LTI) model. Finally, the global T-S fuzzy model of the complex system is constructed using the weighted average fuzzy inference to aggregate all the subsystem matrices.

To stabilize T-S fuzzy systems, many control techniques have been developed in [6]-[13]. The common feature of these control schemes is that a local controller is first designed for each subsystem in the local region, the global controller is then designed by aggregating all local controllers in terms of fuzzy membership functions. The problem faced by these control schemes is that, in order to guarantee the stability of the closed-loop global T-S fuzzy systems, some strict conditions are assumed. For example, a common positive-definite matrix is often required to satisfy all local Riccati equations in [14]-[16], and a dominant subsystem is sometimes assumed in [17]. In practice, these strict constrains have greatly limited applications of the schemes for the control of complex nonlinear systems.

In this paper, we will develop a new feedback control scheme for T-S fuzzy systems using LTV system theory. It has been seen that T-S fuzzy systems are equivalent to LTV systems, it is therefore possible to use the theory of LTV systems to design global feedback controllers to stabilize T-S fuzzy systems. The main idea of the proposed control scheme is that the system matrix of a T-S system is first transformed into the canonical form by applying a Lyapunov transform, a global feedback controller is then designed to assign all PD-eigenvalues of the closed-loop T-S system to the desired locations whose real parts have negative extended means. According to the stability theory of LTV systems in [18]-[22] the closed-loop T-S system is then asymptotically stable. Unlike the schemes in [7]-[13], using this scheme, we are allowed to directly design a global feedback controller instead of deriving a global controller by aggregating all local controllers in terms of fuzzy membership functions.

It is known that the stability of LTV systems is not determined by system eigenvalues, but it is determined by the extended means of system Series D-spectrum eigenvalues (SD-eigenvalues) or Parallel D-spectrum eigenvalues (PD-eigenvalues) [18], [20]. If the real parts of all PD-eigenvalues (or SD-eigenvalues) have negative extended means, the LTV system is then asymptotically stable. This motivates the authors to use the LTV pole-placement method to design controllers for T-S fuzzy systems.

The paper is organized as follows: In section II, the T-S fuzzy systems, the stability of LTV systems and the related concepts are formulated. In section III, the design of a feedback controller to stabilize a general T-S fuzzy system is discussed in detail, and the keypoint to choose a Lyapunov transformation matrix for second-order T-S fuzzy systems is also addressed. In section IV, a simulation example is performed to show the validity of the proposed control scheme.

II. PROBLEM FORMULATION

In this paper, we consider complex nonlinear systems which can be modeled by the following T-S fuzzy models:

\[
R_i: \quad \text{IF } z_1(t) \text{ is } F_{i1} \text{ AND } z_2(t) \text{ is } F_{i2} \text{ AND} \ldots \text{ AND } z_n(t) \text{ is } F_{in} \text{ THEN }
\]

\[
\dot{x}(t) = A_i x(t) + B_i u(t) \]

\[
y(t) = C_i x(t) \quad (1)
\]

for \( i = 1, \ldots, m \)
where $R_i$ represents the $i$th fuzzy inference rule, $m$ the number of inference rules, $F_j$ ($j = 1, \ldots, n$) the fuzzy sets, $x(t)$ in $\mathbb{R}^{n \times 1}$ the state vector, $u(t)$ in $\mathbb{R}^{n \times 1}$ the system input, and $y(t)$ in $\mathbb{R}^{p \times 1}$ the system output. The matrices $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times n}$ and $C_i \in \mathbb{R}^{p \times n}$ are parameter matrices of the $i$th subsystem. $x(t) = (x_1(t), \ldots, x_n(t))^T$ represents the vector of some measurable system variables. Denote $\mu_i(x(t))$ as the normalized fuzzy membership function of the inferred fuzzy set $F_i$ where

$$F_i = \bigcap_{j=1}^{n} F_j^j \quad (2)$$

$$\sum_{i=1}^{m} \mu_i(t) = 1 \quad (3)$$

Using the weighted average fuzzy inferences, we obtain the following global T-S fuzzy state space model:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (4)$$

$$y(t) = C(t)x(t)$$

where

$$A(t) = \sum_{i=1}^{m} \mu_i(t)A_i \quad B(t) = \sum_{i=1}^{m} \mu_i(t)B_i \quad C(t) = \sum_{i=1}^{m} \mu_i(t)C_i \quad (5)$$

It is easy to see that the global T-S fuzzy model in (4) is an equivalent LTV system. For the further proceeding, we have the following assumptions:

**Assumption 2.1:** Each linear subsystem of the global fuzzy model is controllable, i.e. the matrices $M_i = \begin{bmatrix} B_i & A_iB_0 & A_i^2B_i & \cdots & A_i^{n-1}B_i \end{bmatrix}$, $i = 1, \ldots, m$, have full ranks.

**Assumption 2.2:** The global fuzzy model (4) is controllable in the state space, i.e. the matrix $M = \begin{bmatrix} B(t) & A(t)B(t) & A^2(t)B(t) & \cdots & A^{n-1}(t)B(t) \end{bmatrix}$ has full rank in the state space [23].

Because T-S fuzzy systems are equivalent to LTV systems, next, we briefly introduce the recently developed unified eigenvalue theory for LTV systems in [18], [20], which will be used for the design of a controller to stabilize the closed-loop T-S fuzzy system in (4).

First, the global T-S fuzzy system matrix $A(t)$ in (4) or (5) may be written as the following general form:

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \quad (6)$$

Using a proper Lyapunov transform $L(t) \in \mathbb{R}^{n \times n}$ [24], [25], we can transform $A(t)$ into the following canonical form with the preserved stability property [26], [27]:

$$A_c(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

where $A(t)$ and $A_c(t)$ satisfy the following relationship:

$$A_c(t) = L(t)A(t)L(t)^{-1} + L(t)L(t)^{-1} \quad (8)$$

Let $K$ be the differential ring (D-ring) of almost everywhere $C^\infty$ functions with $D = d/dt$ the derivative operator on $K$. Let $M = K^n$ be the $n$-dimensional differential module (D-module) over $K$. Then a scalar linear differential equation (SLDE) operator $D_a$ over the D-ring $K$ with parameters $a_i(t) \in K$ can be defined as:

$$D_a = D^n + a_n(t)D^{n-1} + \cdots + a_2(t)D + a_1(t) \quad (9)$$

According to [18], [20], the SLDE operator $D_a$ in (9) can be factorized as:

$$D_a = (D - p_1(t))(D - p_2(t))(D - p_3(t)) \cdots (D - p_m(t)) \quad (10)$$

where $p_1(t), \ldots, p_m(t)$ are non-unique in general.

**Remark 2.3:** For SLDE operator, the coefficients $a_i(t), i = 1, \ldots, n$ in (10) can be easily calculated if we know $p_i(t), i = 1, \ldots, n$.

For example, consider the following third-order SLDE operator:

$$D_a = (D - p_1(t))(D - p_2(t))(D - p_3(t)) \quad (11)$$

Parameters $a_i(t), i = 1, 2, 3$, may be calculated as follows:

$$a_1(t) = -p_1(t) + (p_2(t) + p_3(t))p_1(t) + p_1(t)p_2(t)p_3(t)$$

$$a_2(t) = -2p_1(t) - p_2(t) + p_1(t)p_2(t) + p_1(t)p_3(t)$$

$$a_3(t) = -p_1(t) + p_2(t) + p_3(t)$$

Using the factorization of the SLDE operator in (10), the concepts of SD-eigenvalues and PD-eigenvalues proposed in [18], [20] are summarized as follows:

**Definition 2.4:** Let $D_a$ in (9) be a SLDE operator with parameters $a_i(t) \in K$, $i = 1, \ldots, n$, and $A(t) \in K^{n \times n}$ in (7) be its canonical matrix. Then, any $n$ scalar functions $p_k(t) \in K$, $k = 1, \ldots, n$, satisfying the factorization in (10) are called Series D-eigenvalues (SD-eigenvalues) of $D_a$ and of $A(t)$. The particular element $\rho(t) = p_k(t)$ is called the Parallel D-eigenvalue (PD-eigenvalue) of $D_a$ and of $A(t)$.

**Definition 2.5:** A multi-set $T_a = \{p_i(t)\}_{i=1}^{n}$ is called a Series D-spectrum (SD-spectrum) for $D_a$ and for $A(t)$ if $p_i(t)$, $i = 1, \ldots, n$, satisfy (10).
Definition 2.6: A set \( \Phi_n = \{ \phi_i(t) \}_{i=1}^{n} \) is called a Parallel D-spectrum (PD-spectrum) for \( D_A \) and \( A(t) \), if: (i) \( \phi_i(t) \), \( i = 1, \ldots, n \), are \( n \) distinct PD-eigenvalues for \( D_A \) and \( A(t) \); and (ii) \( \{ y_i(t) = \exp \left\{ \sum_{i=1}^{n} \phi_i(t) dt \right\} \}_{i=1}^{n} \) constitutes a fundamental set of solutions to \( D_A \{ y_i(t) \} = 0 \).

In Definition 2.6, the distinct PD-eigenvalues can be calculated as follows:

\[
\rho_i(t) = \rho_i(t) + \int_{t_0}^{t} \phi_{i_1}(t) \phi_{i_2}(t) \cdots \phi_{i_k}(t) dt, \quad k = 2, \ldots, n
\]

with

\[
q_k(t) = \int \phi_{i_1}(t) \phi_{i_2}(t) \cdots \phi_{i_k}(t) \cdots \phi_{i_{n-k+1}}(t) dt
\]

\[
\phi_{ij}(t) = \exp \left( \int_{t_0}^{t} (p_i(t) - p_j(t)) dt \right)
\]

Lemma 2.7: Let \( D_A \) be an \( n \)th-order time-varying SLDE operator with locally Lebesgue-integrable coefficients \( a_i(t), \) \( i = 1, \ldots, n, \) and let \( \Phi_n = \{ \rho_i(t) \}_{i=1}^{n} \) be a Parallel D-spectrum (PD-spectrum) for \( D_A \). Then the null solution of \( D_A \{ y_i(t) \} \) is exponentially asymptotically stable on \( [t_0, \infty) \) in the sense of Lyapunov, if for every \( 1 \leq i \leq n \), \( \rho_i(t) \) has a bounded negative extended-mean \( em \{ Re \rho_i(t) \} \), which is defined as:

\[
em \{ Re \rho_i(t) \} = \lim_{T \to \infty} \sup_{t \geq t_0} \frac{1}{T} \int_{t_0}^{t+T} \{ Re \rho_i(t) \} dt,
\]

Remark 2.8: Lemma 2.7 is a main theorem of LTV system theory, which will be used to determine the stability of LTV systems. The above definitions and stability lemma for LTV systems are the fundamentals to design a global feedback controller in the next section to stabilize the global T-S fuzzy system in (4).

III. THE DESIGN OF GLOBAL FEEDBACK CONTROLLERS

Now we consider the global T-S fuzzy system in (4), and let system input be of the form:

\[
u(t) = K(t) x(t)
\]

where \( K(t) \in \mathbb{R}^{n \times n} \) is the time-varying feedback control gain matrix. Using (17) in (4), we write the closed-loop T-S system as:

\[
x_c(t) = (A(t) + B(t) K(t)) x(t)
\]

Before determining the control gain matrix, we first use a Lyapunov transformation matrix \( L(t) \in \mathbb{R}^{n \times n} \) to transform the closed-loop system (18) into the following canonical form with the preserved system stability property:

\[
x_c(t) = \left( A_c(t) + \hat{B}(t) \hat{K}(t) \right) x_c(t)
\]

where

\[
x_c(t) = L(t) x(t)
\]

\[
\hat{B}(t) = L(t) B(t)
\]

\[
\hat{K}(t) = K(t) L(t)^{-1}
\]

and \( A_c(t) \) is given in (8).

Remark 3.1: Lyapunov transformation matrix \( L(t) \) satisfies the following two conditions [26], [27]:

\[
\begin{align*}
\| L(t) \| & \leq M \\
\det L(t) & \geq m
\end{align*}
\]

for some \( M < \infty \) and \( m > 0 \). The details to obtain a Lyapunov transformation matrix for the closed-loop system (18) can be seen from [24], [25], [28].

If the desired PD-eigenvalues \( \rho_i(t) \) and the corresponding PD-eigenvector matrices \( V_a(t) \) are chosen in advance, the following equations can be used to find the parameters of the control gain matrix:

\[
\left( A_c(t) + \hat{B}(t) \hat{K}(t) \right) V_a(t) - \rho_i(t) V_a(t) = \dot{V}_a(t),
\]

\[
i = 1, \ldots, n
\]

The above equation can also be rewritten in the following matrix form:

\[
A_c(t) V_a(t) + \hat{B}(t) \hat{K}(t) V_a(t) - V_a(t) T(t) = \dot{V}_a(t)
\]

where

\[
T(t) = \text{diag} \{ \rho_1(t), \rho_2(t), \ldots, \rho_n(t) \}
\]

\[
V_a(t) = [ v_a(t), v_a(t), \ldots, v_a(t) ]
\]

We first let \( H(t) = \hat{K}(t) V_a(t) \), (26) can then be represented as the following differential Sylvester equation:

\[
A_c(t) V_a(t) + \hat{B}(t) H(t) - V_a(t) T(t) = \dot{V}_a(t)
\]

Remark 3.2: if \( \hat{B}(t) \) is a \( n \times n \) full rank matrix, \( H(t) \) can be calculated as:

\[
H(t) = -\hat{B}(t)^{-1} \left( A_c(t) V_a(t) - V_a(t) T(t) - \dot{V}_a(t) \right)
\]

and the feedback gain \( \hat{K}(t) \) of the transformed system is of the form:

\[
\hat{K}(t) = H(t) V_a^{-1}(t)
\]

However, if \( \hat{B}(t) \in \mathbb{R}^{n \times q} (n > q) \), a pseudo matrix \( \tilde{H}(t) \) needs to be obtained by solving

\[
\tilde{H}(t) = -\hat{B}(t)^{-1} \left( A_c(t) V_a(t) - V_a(t) T(t) - \dot{V}_a(t) \right)
\]

where \( \hat{B}(t)^{-1} \left( \hat{B}(t)^T \hat{B}(t) \right)^{-1} \hat{B}(t)^T \) denotes the pseudo-inverse of the matrix \( \hat{B}(t) \). Because of pseudo \( \tilde{H}(t) \) in (30), \( \tilde{H}(t) \) satisfies the following equation instead of (27):

\[
A_c(t) V_a(t) + \tilde{B}(t) \tilde{H}(t) - V_a(t) T(t) = \dot{V}_a(t)
\]

It is easy to see that the actual PD-eigenvector matrix \( V_a(t) \) is different from the desired PD-eigenvector matrix \( V_a(t) \) (in least square sense). However, if \( V_a(t) \) is close to \( V_a(t) \), the good closed-loop performance can also be obtained. The feedback gain \( \hat{K}(t) \) for the transformed system in (19) can be calculated as:

\[
\hat{K}(t) = \tilde{H}(t) V_a^{-1}(t)
\]
Finally, the control gain \( K(t) \) is computed as:

\[
K(t) = \dot{K}(t)L(t)
\]  

(Remark 3.3) It is seen that the first step to transform the closed-loop T-S fuzzy system into its canonical form is very important. Therefore, we need to choose a proper Lyapunov transformation matrix \( L(t) \) according to [24]. Especially, for the following general second-order T-S fuzzy systems:

\[
\dot{z}(t) = \begin{bmatrix}
\sum_{i=1}^{m} \mu_i(t) a_{11i} z(t) + \sum_{i=1}^{m} \mu_i(t) a_{12i} \dot{z}(t) \\
\sum_{i=1}^{m} \mu_i(t) a_{21i} z(t) + \sum_{i=1}^{m} \mu_i(t) a_{22i} \dot{z}(t)
\end{bmatrix}
+ \begin{bmatrix}
\sum_{i=1}^{m} \mu_i(t) b_{1i}
\sum_{i=1}^{m} \mu_i(t) b_{2i}
\end{bmatrix} u(t)
\]  

(34)

a \( 2 \times 2 \) Lyapunov transformation matrix \( L(t) \) is chosen in equation (35) on the next page, which will be used in the following simulation section to determine the Lyapunov transform matrix.

(Remark 3.4) In practice, the derivatives \( \frac{d}{dt} \left( \sum_{i=1}^{m} \mu_i(t) b_{1i} \right) \) and \( \frac{d}{dt} \left( \sum_{i=1}^{m} \mu_i(t) b_{2i} \right) \), \( j = 1 \) and 2, in (35) and (36) are computed numerically.

IV. A SIMULATION EXAMPLE

In this section, we consider a complex system which is approximated by the following LTI models in its subspaces:

\begin{align*}
R_1: & \quad \text{if } x_1(t) \text{ is about } 0, x_2(t) \text{ is about } 0 \\
& \quad \text{then } \dot{x}(t) = A_1 x(t) + B_1 u(t) \\
R_2: & \quad \text{if } x_1(t) \text{ is about } 0, x_2(t) \text{ is about } \pm 4 \\
& \quad \text{then } \dot{x}(t) = A_2 x(t) + B_2 u(t) \\
R_3: & \quad \text{if } x_1(t) \text{ is about } \pm 3/2, x_2(t) \text{ is about } 0 \\
& \quad \text{then } \dot{x}(t) = A_3 x(t) + B_3 u(t) \\
R_4: & \quad \text{if } x_1(t) \text{ is about } \pm 3/2, x_2(t) \text{ is about } \pm 4 \\
& \quad \text{or } x_1(t) \text{ is about } -\pi/3, x_2(t) \text{ is about } -4 \\
& \quad \text{then } \dot{x}(t) = A_4 x(t) + B_4 u(t) \\
R_5: & \quad \text{if } x_1(t) \text{ is about } \pm 3/2, x_2(t) \text{ is about } -4 \\
& \quad \text{or } x_1(t) \text{ is about } -\pi/3, x_2(t) \text{ is about } +4 \\
& \quad \text{then } \dot{x}(t) = A_5 x(t) + B_5 u(t)
\end{align*}

with:

\[
A_1 = \begin{bmatrix}
-0.05 & 1 \\
17.2941 & -0.05
\end{bmatrix} \quad B_1 = \begin{bmatrix}
-0.05 \\
-0.1765
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
-0.05 & 1 \\
14.4706 & -0.05
\end{bmatrix} \quad B_2 = \begin{bmatrix}
-0.05 \\
-0.1765
\end{bmatrix}
\]

\[
A_3 = \begin{bmatrix}
-0.05 & 1 \\
5.8512 & -0.05
\end{bmatrix} \quad B_3 = \begin{bmatrix}
-0.05 \\
-0.0779
\end{bmatrix}
\]

\[
A_4 = \begin{bmatrix}
-0.05 & 1 \\
7.2437 & 0.5399
\end{bmatrix} \quad B_4 = \begin{bmatrix}
-0.05 \\
-0.0779
\end{bmatrix}
\]

The global T-S fuzzy system can then be obtained by aggregating all the subsystems derived in the above using the fuzzy membership functions showing in Figure 1. Figure 2 shows the system states \( x_1 \) and \( x_2 \) of the global system with initial states \( x_1(0) = 1.5, x_2(0) = 0 \), but the control input is zero. In order to stabilize the system, a feedback controller is designed based on the algorithm developed in Section III. By choosing the PD-eigenvalues \( T(t) = \text{diag} \{ -1 -1 -1 -2 \} \) and the desired eigenvectors \( V_4 = \text{diag} \{ -1 -2 \} \), the system states \( x_1 \) and \( x_2 \) of the closed-loop global T-S fuzzy system with initial states \( x_1(0) = 1.5, x_2(0) = 0 \) are shown in Figure 3. It has been seen that, both PD-eigenvalues have negative extended means, and the global T-S fuzzy system has been stabilized. The control signal \( u(t) \) is shown in Figure 4.
V. CONCLUSION

In this paper, the controller design for T-S fuzzy system using LTV system theory has been investigated. The main contributions are that a general T-S fuzzy system is treated as a LTV system, and a global feedback controller is then designed to assign all PD-eigenvalues of the closed-loop T-S fuzzy system to the desired locations whose real parts have negative extended means. Unlike many other T-S fuzzy control system designs, the proposed control scheme not only guarantees asymptotic stability of the closed-loop T-S fuzzy systems, but also allows the designers to directly design a simple global controller instead of deriving a complex global controller by aggregating all local controllers using fuzzy membership functions. The simulation example has shown a good control performance using the developed scheme.

REFERENCES