Faculty of Information and Communication Technologies

Detecting Double Faults Related to Terms in Boolean Expression

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1 Introduction

Fault-based testing techniques have been proposed aiming at detection of hypothesized faults in a program. More precisely, if an hypothesized fault has been injected in the program, the test cases generated by the fault-based testing techniques will be able to reveal the corresponding failures of the faulty program. As a result, test sets generated aiming at the hypothesized fault will be able to detect the faulty program. Many techniques of fault-based testing have been developed and widely used [2, 4, 11, 12, 17]. Symbolic testing has been proposed as a fault-based testing strategy in [11], and it has been further applied in [2]. Moreover, mutation testing, a typical fault-based testing technique, has been used for many decades [4, 12, 17].

Recently, the detection conditions of hypothesized fault classes have been widely studied [1, 6, 9, 15]. The fault detection conditions of hypothesized faults are mainly used in two areas. First, they are used to develop test case selection strategies aiming at the detection of particular types of faults. Chen and Lau [1] proposed three test case selection strategies based on fault detection conditions of seven types of faults. Second, fault detection conditions are used to develop fault class hierarchies [6, 9, 15]. Fault class hierarchy establishes relationships between different types of faults. If any test cases that can be used to detect fault class \(A\) can be used to detect fault class \(B\), fault class \(A\) is put in the hierarchy lower than that of fault class \(B\). Kuhn [6] and Lau and Yu [9] used fault class hierarchies to explain the empirical results for existing fault-based testing methodologies.

Most previous studies on fault detection conditions assume that only one of these hypothesized faults occur in the software under test. However, investigations show that multiple faults occur more frequently in programs [10, 16]. Research on multiple faults is mainly focus on fault coupling of double fault. Double faults is a special instance of multiple faults. Fault coupling refers to the situation that two faults can be detected individually, but fail to be detected when they combined together.

Fault coupling have been studied previously [5, 12]. We will only briefly described these previous
work here. Detailed discussions can be found in Section 5. Offutt performed an empirical study on fault coupling with 3 small programs via mutation analysis [12]. Given a program, a 1-order (2-order) mutant is a program that differs from the original program by 1 syntactic change (2 syntactic changes). In [12], Offutt generated all possible 1-order mutants for the studied programs, discarded all the 1-order mutants that were equivalent to the original program, and generated all 2-order mutant based on those remaining 1-order mutants. Then, he generated test sets that can kill the remaining 1-order mutants and used these test sets to kill 2-order mutants. He found that most test sets that can kill 1-order mutants can also be used to kill a high percentage of 2-order mutants. However, there is no formal analysis related to high detection rate against the types of mutation operators being used to generate those mutants.

On the other hand, How Tai Wah study the fault coupling via theoretical study [5]. He modeled program as composition of functions while program with single fault (double faults, respectively) is denoted as a composition in which exactly one function is (two functions are, respectively) faulty. Test sets that detect individual faults of a double fault are called proper test sets. Among the proper test sets, those that cannot detect the double fault are called coupled test sets. He then calculates the coupling ratio, defined as the ratio of the number of coupled test sets to that of proper test sets. The mathematical analysis shows that the coupling ratio is very small, therefore How Tai Wah concluded that fault coupling rarely occurs. Since faulty program is modeled by incorrect use of one (respectively, two) of the functions, no fault types have been actually involved in this study.

Recently, Lau and Yu extended their study on using fault class hierarchy to study double faults related to terms [8]. They found that test case that can detect some particular classes of faults in the lower part of the hierarchy can detect double faults which involve these classes of faults and faults in the upper part of the hierarchy. For example, if a test case that can detect fault class $A$ which is lower than fault class $B$ in the fault class hierarchy, then the same test case can detect the double fault involving fault classes $A$ and $B$, denoted as $A \times B$.

In this report, our main target is to obtain the fault detection conditions of double faults related
to terms. In order to illustrate how these fault detection conditions can be used, we analyse the fault detection capabilities of some existing test case selection strategies, aimed at detecting single occurrence of particular types of fault, in detecting double faults studied in this report. Our analysis shows that test case selection strategies studied in this report can guarantee to detect the double faults related to terms. Hence, by using these test case selection strategies in detecting single faults, there is an additional benefit of detecting the double faults studied in this report.

The rest of report is organized as follows. Section 2 introduces the notation and fault classes studied in this report. Section 3 presents double fault classes and their corresponding faulty implementations. Section 4 proves the fault detection conditions of each faulty implementation and Section 5 analyses the effects of fault coupling between single and double faults. Section 6 analyses the fault detecting capabilities of some existing test case selection strategies in detecting double faults. Section 7 concludes the report and discusses further work.

2 Preliminary

2.1 Notation

In this report, we use ‘·’, ‘+’ and ‘¯’ to represent Boolean operators, AND, OR and NOT, respectively. Usually, ‘·’ is omitted whenever it is clear from the context. We use 1 and 0 to represent the truth values ‘TRUE’ and ‘FALSE’, respectively. The set of all truth values, that is \{0, 1\}, is denoted as \(\mathbb{B}\).

Let \(S\) be a Boolean expression in disjunctive normal form

\[ S = p_1 + \cdots + p_m \]

where \(m\) is the number of terms, \(p_i = x_{i1} \cdots x_{ik_i}\) is the \(i\)-th term of \(S\), \(x_{ij}\) is the \(j\)-th literal in \(p_i\), and \(k_i\) is the number of literals in \(p_i\). A Boolean expression is in irredundant disjunctive normal form (or, simply IDNF) if (1) none of its terms can be omitted from the expression; and (2) none of its literals
can be omitted from any term in the expression without affecting the function of the expression.

Let $S$ be a Boolean expression having $n$ variables, the input domain is the $n$-dimensional Boolean space $\mathbb{B}^n$. True points of $S$ are those points in $\mathbb{B}^n$ that make $S$ evaluate to $1$. The set of all true points of $S$ is denoted by $TP(S)$. A true point of the term $p_i$ in $S$ is a point that makes $p_i$ evaluate to $1$. The set of all true points of $p_i$ in $S$ is denoted by $TP_i(S)$. A unique true point of $p_i$ in $S$ is a true point of $S$ that makes (1) $p_i$ evaluate to $1$; and (2) all other terms evaluate to $0$. The set of all unique true points of $p_i$ in $S$ is denoted by $UTP_i(S)$. Hence, $TP(S) = \bigcup_i TP_i(S)$.

False points of $S$ are those points in $\mathbb{B}^n$ that make $S$ evaluate to $0$ and the set of all false points of $S$ is denoted by $FP(S)$. A near false point of the $j$-th literal, $x_j$, of the $i$-th term, $p_i$, in $S$ is a false point that makes (1) $x_j$ evaluate to $0$, and (2) all other literals in $p_i$ evaluate to $1$. The set of all near false points for the $j$-th literal $x_j$ of the $i$-th term, $p_i$, in $S$ is denoted by $NFP_i,j(S)$. The set of all near false points for the $i$-th term, $p_i$, in $S$ is denoted by $NFP_i(S)$. Therefore, $NFP_i(S) = \bigcup_j NFP_i,j(S)$. The set of all near false points of $S$ is denoted by $NFP(S)$ and $NFP(S) = \bigcup_i NFP_i(S)$.

### 2.2 Fault Types

In this report, we only consider five types of faults related to terms in Boolean expressions. Let $S$ be a Boolean expression. Suppose that a single fault $F$ changes a subexpression $E$ into a subexpression $E'$, the resulting faulty expression (or, implementation) is denoted as $IF(E\rightarrow E')$. A faulty implementation is referred to as single-fault expression if (1) it differs from the original expression by one syntactic change; and (2) it is not equivalent to the original expression. The following five fault types related to terms in a Boolean expression are studied in this report:

1. **Expression Negation Fault (ENF):** The entire Boolean expression or its sub-expression is implemented as its negation. The faulty implementation is $IF_{ENF}(S\rightarrow S') = \overline{S}$ or $IF_{ENF}(p_{i_1} + \cdots + p_{i_k}) = \overline{p_{i_1} + \cdots + p_{i_k}}$. 
For example, the Boolean expression $ab + cd + ef$ may be implemented as $ab + cd + ef$. The corresponding detection condition, denoted by $DC_{\text{ENF}}$, is “any point in $\left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1, \ldots, h_1}^{m} TP_i(S) \right)$ or any point in $FP(S)$”.

2. **Term Negation Fault (TNF)**: A particular term in the Boolean expression is implemented as its negation. The faulty implementation is $I_{\text{TNF}}(p_i \rightarrow \overline{p_i}) = p_1 + \cdots + p_{i-1} + p_i + p_{i+1} + \cdots + p_m$, where $1 \leq i_1 \leq m$ and $m > 1$. For example, the Boolean expression $ab + cd + ef$ may be implemented as $ab + \overline{cd} + ef$. The corresponding detection condition, denoted by $DC_{\text{TNF}}$, is “any point in $UTP_{i_1}(S)$ or any point in $FP(S)$”. As documented in [9], when $S$ contains just one term (that is $m = 1$), the negation fault is considered as an ENF.

3. **Term Omission Fault (TOF)**: A particular term in the Boolean expression is omitted in its implementation. The faulty implementation is $I_{\text{TOF}}(p_i \rightarrow) = p_1 + \cdots + p_{i-1} + p_{i+1} + \cdots + p_m$, where $1 \leq i_1 \leq m$ and $m > 1$. For example, the Boolean expression $ab + cd + ef$ may be implemented as $ab + cd$. The corresponding detection condition, denoted by $DC_{\text{TOF}}$, is “any point in $UTP_{i_1}(S)$”.

4. **Disjunctive Operator Reference Fault (DORF)**: A Boolean operator ‘+’ is implemented as ‘.’. The faulty implementation is $I_{\text{DORF}}(p_i + p_{i+1}) = p_1 + \cdots + p_{i-1} + p_i + p_{i+1} + \cdots + p_m$, where $1 \leq i_1 < m$. For example, the Boolean expression $ab + cd + ef$ may be implemented as $ab + cdef$. The corresponding detection condition, denoted by $DC_{\text{DORF}}$, is “any point in $UTP_{i_1}(S)$ or any point in $UTP_{i_1+1}(S)$”.

5. **Conjunctive Operator Reference Fault (CORF)**: A Boolean operator ‘.’ is implemented as ‘+’. The faulty implementation is $I_{\text{CORF}}(p_i \cdot p_{i+1}) = p_1 + \cdots + p_{i-1} + p_i + p_{i+1} + \cdots + p_m$, where $1 \leq i_1 \leq m$, $m > 1$, $1 \leq j_1 < k_1$, $p_{i_1, j_1} = x_{j_1}^{i_1}$, $p_{i_1, j_1+1} = x_{j_1+1}^{i_1}$ and $p_i = p_{i_1, j_1} \cdot p_{i_1, j_1+1, k_1}$. For example, the Boolean expression $ab + cd + ef$ may be implemented as $ab + c + d + ef$. The corresponding detection condition, denoted by $DC_{\text{DORF}}$, is “any point in $FP(S)$ such that $p_{i_1, j_1} + p_{i_1, j_1+1, k_1} = 1$.”
3 Double Fault Models

Similar to previous studies of double fault [5, 12], a double fault is defined as the occurrence of two single faults in this report. When a double fault occurs in a Boolean expression, the resulting expression may be equivalent to the original expression or a faulty expression with a single fault. Lau and Liu [7] defined a double-fault expression is one which differs from the original expression by two syntactic changes and is equivalent to neither the original expression nor any expression that results from any single fault.

When a double fault occurs, it is possible that the order of occurrence of the two single faults may result in different faulty expressions. Such a double fault is referred to as double fault with ordering. However, there is also a chance that the two faults may result in two faulty expressions that are equivalent to each other no matter which fault occurs first. This type of double fault is referred to as double fault without ordering.

As reported by Lau and Liu [7], based on the five single fault classes studied in this report, there are 15 double fault classes in double faults without ordering resulting in 27 possible distinct faulty expressions. For the case of double faults with ordering, there are 25 double fault classes resulting in 53 possible faulty expressions. Lau and Liu also found that 49 out of these 53 double-fault expressions have their equivalent double-fault expressions in double faults without ordering. Only 4 double-fault expressions based on double faults with ordering do not have their equivalent counterparts in double faults without ordering. Together with 27 double-fault expressions of double faults without ordering, there are 31 distinct double-fault expressions to be studied in this report. In this section, we introduce all 31 double-fault expressions. The detection conditions of all these double-fault expressions will be studied in Section 4.

For any two single fault classes A and B, we use the notation $A \bowtie B$ to denote the double fault class formed from A and B, that is, the class of faults due to the occurrences of two faults: one fault of class A and another fault of class B. Given a Boolean expression $S$, suppose two faults
Table 1: Double fault, double-fault expression and detection condition \((S = p_1 + \ldots + p_m)\)

(a) Double-fault expressions \((1 - 27)^a\) due to double fault without ordering

<table>
<thead>
<tr>
<th>Fault class</th>
<th>Double-fault expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENF × ENF</td>
<td>Case 1 ((i_1 &lt; h_1 &lt; i_2 &lt; h_2): p_1 + \ldots + p_{i_1} + \ldots + p_{h_2} + \ldots + p_{i_2} + \ldots + p_m) (1) Case 2 ((i_1 \leq i_2 &lt; h_2 \leq h_1) and ((i_2, \ldots, h_2) \subseteq {i_1, \ldots, h_1}): (p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_2} + \ldots + p_m) (2)</td>
</tr>
<tr>
<td>ENF × TNF</td>
<td>Case 1 ((i_1 &lt; h_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{h_2} + \ldots + p_{i_2} + \ldots + p_m) (3) Case 2 ((i_1 &lt; i_2 \leq h_1) and (i_1 &lt; h_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_1} + \ldots + p_m) (4)</td>
</tr>
<tr>
<td>ENF × TOF</td>
<td>Case 1 ((i_1 &lt; h_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{h_2} + \ldots + p_{i_2} + \ldots + p_m) (5) Case 2 ((i_1 \leq i_2 \leq h_1) and (i_1 &lt; h_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_1} + \ldots + p_m) (6)</td>
</tr>
<tr>
<td>ENF × DORF</td>
<td>Case 1 ((i_1 &lt; h_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{h_2} + \ldots + p_{i_2} + \ldots + p_m) (7) Case 2 ((i_1 &lt; h_1 &lt; m): p_1 + \ldots + p_{i_1} + \ldots + p_{h_1} + \ldots + p_m) (8)</td>
</tr>
<tr>
<td>ENF × CORF</td>
<td>Case 3 ((i_1 &lt; m): p_1 + \ldots + p_{i_1} + \ldots + p_m) (9) Case 4 ((i_1 \leq i_2 &lt; h_1): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_1} + \ldots + p_m) (10)</td>
</tr>
<tr>
<td>TNF × TNF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (11) Case 2 ((i_1 &lt; i_2 \leq h_1): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_1} + \ldots + p_m) (12)</td>
</tr>
<tr>
<td>TNF × TOF</td>
<td>Case 1 ((i_1 &lt; i_2 \leq m): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (13) Case 2 ((i_1 &lt; i_2 \leq h_1): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_1} + \ldots + p_m) (14)</td>
</tr>
<tr>
<td>TNF × DORF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (15) Case 2 ((i_1 &lt; i_2 \leq h_1): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_1} + \ldots + p_m) (16)</td>
</tr>
<tr>
<td>TNF × CORF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_1} + \ldots + p_m) (17) Case 2 ((i_1 &lt; i_2 \leq h_1): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_1} + \ldots + p_m) (18)</td>
</tr>
<tr>
<td>TOF × TOF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (19) Case 2 ((i_1 &lt; i_2 \leq m): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (20)</td>
</tr>
<tr>
<td>TOF × DORF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (21) Case 2 ((i_1 &lt; i_2 \leq m): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (22)</td>
</tr>
<tr>
<td>TOF × CORF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (23) Case 2 ((i_1 &lt; i_2 \leq m): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (24)</td>
</tr>
<tr>
<td>DORF × DORF</td>
<td>Case 1 ((i_1 &lt; i_2 \leq h_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_2} + \ldots + p_m) (25) Case 2 ((i_1 &lt; i_2 \leq m): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (26)</td>
</tr>
<tr>
<td>DORF × CORF</td>
<td>Case 1 ((i_1 &lt; i_2 \leq h_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_{h_2} + \ldots + p_m) (27) Case 2 ((i_1 &lt; i_2 \leq m): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (28)</td>
</tr>
</tbody>
</table>

(b) Four double-fault expressions \((53, 70, 73,\) and \(76)^a\) due to double fault with ordering

<table>
<thead>
<tr>
<th>Fault class</th>
<th>Double-fault expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>TOF × DORF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (53)</td>
</tr>
<tr>
<td>CORF × ENF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (70)</td>
</tr>
<tr>
<td>CORF × TNF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (73)</td>
</tr>
<tr>
<td>CORF × TOF</td>
<td>Case 1 ((i_1 &lt; i_2): p_1 + \ldots + p_{i_1} + \ldots + p_{i_2} + \ldots + p_m) (76)</td>
</tr>
</tbody>
</table>

\(^a\)For ease of cross-reference, the numbering of the faulty expressions follows that of [7].
A and B are committed on the expression changing \( E_1 \) and \( E_2 \) in the expression to \( E'_1 \) and \( E'_2 \), respectively, the resulting faulty expression (or, implementation) is denoted as \( I_{A(E_1\rightarrow E'_1)\times B(E_2\rightarrow E'_2)} \).

Table 1 lists all these double fault classes and their corresponding double-fault expressions. We use the same expression numbers of those double-fault expressions reported in [7] for consistency purposes. Let us consider the third row of Table 1, which is concerned with \( \text{ENF} \bowtie \text{TOF} \). Let \( S \) be a Boolean expression in IDNF. There are two subcases. First, the term \( p_{i_2} \) is not contained in the subexpression \( p_{i_1} + \cdots + p_{h_1} \). Without loss of generality, we may assume that \( h_1 < i_2 \). The double-fault expression is equivalent to \( p_1 + \cdots + p_{i_1} + \cdots + p_{h_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_m \). Second, the term \( p_{i_2} \) is contained in the subexpression \( p_{i_1} + \cdots + p_{h_1} \), that is, \( i_1 \leq i_2 \leq h_1 \) and \( i_1 < h_1 \). The faulty expression is then equivalent to \( p_1 + \cdots + p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1} + \cdots + p_m \).

## 4 Detection Conditions of Various Double Fault Classes

The detection conditions of hypothesized faults have been studied recently [1, 6, 9, 15]. Let \( S \) be a specification and \( I \) be the expression which differs from \( S \) by several syntactic changes. Whenever \( S \) and \( I \) evaluate to different values, they can be distinguished from each other. The detection condition of \( I \) with respect to \( S \) is a condition that makes \( S \) and \( I \) evaluate to different values. As a result, the Boolean exclusive-or operator \( \text{XOR} \), denoted as \( \oplus \), can be used to find the detection conditions. In short, the detection condition can be derived from \( S \oplus I \).

In this section, we prove the detection conditions of all 31 double-fault expressions listed in Table 1. The detection conditions of the 27 faulty expressions of double fault without ordering are proved in Section 4.1 whereas the detection conditions of the remaining 4 faulty expressions are proved in Section 4.2. Instead of simply presenting the Boolean expression \( S \oplus I \) as detection conditions, we present them as conditions satisfied by test cases in \( \mathbb{B}^n \). Since such categorization is based on certain properties of test sets, it helps in identifying and developing test case selection strategy to detect such double faults in Table 1.
4.1 Detection Conditions on Double Term Faults Without Ordering

4.1.1 ENF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is an ENF.

**Theorem 4.1 (ENF × ENF - Case 1)**

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two subexpressions $(p_{i_1} + \cdots + p_{h_1})$ and $(p_{i_2} + \cdots + p_{h_2})$ in $S$ are negated where $1 \leq i_1 < h_1 < i_2 < h_2 \leq m$, the resulting expression denoted as $I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} \rightarrow \overline{p_{i_1} + \cdots + p_{h_1}}) \times I_{\text{ENF}}(p_{i_2} + \cdots + p_{h_2} \rightarrow \overline{p_{i_2} + \cdots + p_{h_2}})$ is equivalent to that given by Expression (1) in Table 1. Then, we have $S \neq I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} \rightarrow \overline{p_{i_1} + \cdots + p_{h_1}}) \times I_{\text{ENF}}(p_{i_2} + \cdots + p_{h_2} \rightarrow \overline{p_{i_2} + \cdots + p_{h_2}})$ if and only if there is a test case $\bar{t}$ that satisfies any of the following conditions:

1. $\bar{t} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \cap \left( \bigcup_{i=i_2}^{h_2} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)$, or
2. $\bar{t} \in FP(S)$.

**Proof:** First, we observe that $S \oplus I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} \rightarrow \overline{p_{i_1} + \cdots + p_{h_1}}) \times I_{\text{ENF}}(p_{i_2} + \cdots + p_{h_2} \rightarrow \overline{p_{i_2} + \cdots + p_{h_2}})$

\[\equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}) \oplus (p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}) \right) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_2-1} \cdots \bar{p}_{h_2+1} \cdots \bar{p}_m \]

\[\equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}) \cdot \overline{(p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2})} \right) \]

\[+ \left( p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2} \right) \cdot \left( p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2} \right) \]

\[\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_2-1} \cdots \bar{p}_{h_2+1} \cdots \bar{p}_m \]

\[\equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}) \cdot \overline{(p_{i_1} + \cdots + p_{h_1} \cdot p_{i_2} + \cdots + p_{h_2})} \right) \]

\[+ \left( p_{i_1} + \cdots + p_{h_1} \cdot p_{i_2} + \cdots + p_{h_2} \right) \cdot (p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}) \]

\[\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_2-1} \cdots \bar{p}_{h_2+1} \cdots \bar{p}_m \]
\begin{align*}
\equiv \left( (p_{i_1} + \cdots + p_{h_1}) \cdot (p_{i_2} + \cdots + p_{h_2}) \right) \\
+ \left( (p_{i_1} + \cdots + p_{h_1}) \cdot (p_{i_2} + \cdots + p_{h_2}) \right) \\
\cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2-1} \cdot \cdots \cdot \bar{p}_m \\
\equiv \left( (p_{i_1} + \cdots + p_{h_1}) \cdot (p_{i_2} + \cdots + p_{h_2}) \right) \\
+ \left( (p_{i_1} + \cdots + p_{h_1}) \cdot (p_{i_2} + \cdots + p_{h_2}) \right) \\
\cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2-1} \cdot \cdots \cdot \bar{p}_m \\
\equiv \left( (p_{i_1} + \cdots + p_{h_1}) \cdot (p_{i_2} + \cdots + p_{h_2}) \right) \\
\cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2-1} \cdot \cdots \cdot \bar{p}_m + S
\end{align*}

Now, \( S(\bar{t}) \neq I_{ENF(p_{i_1} + \cdots + p_{h_1}) \lor ENF(p_{i_2} + \cdots + p_{h_2})(\bar{t})} \)

if and only if \( S(\bar{t}) \land ENF(p_{i_1} + \cdots + p_{h_1}) \lor ENF(p_{i_2} + \cdots + p_{h_2})(\bar{t}) = 1 \)

if and only if \( (p_{i_1} + \cdots + p_{h_1}) \cdot (p_{i_2} + \cdots + p_{h_2}) \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2-1} \cdot \cdots \cdot \bar{p}_m + S \)

evaluates to 1 on \( \bar{t} \)

if and only if \( \bar{t} \) satisfies any of the following conditions

1. \( \bar{t} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \cap \left( \bigcup_{i=i_2}^{h_2} TP_i(S) \right) \) \( \cup \) \( \bigcup_{i=1}^{m} TP_i(S) \), or

2. \( \bar{t} \in FP(S) \).

Hence, the result follows. \qed

**Theorem 4.2** (\( ENF \Join ENF - \text{Case 2} \))

Let \( S = \sum_{i=1}^{n} p_i \) be a Boolean specification in irredundant disjunctive normal form. Suppose that two subexpressions \( (p_{i_1} + \cdots + p_{h_1}) \) and \( (p_{i_2} + \cdots + p_{h_2}) \) in \( S \) are negated where \( 1 \leq i_1 \leq i_2 < h_2 \leq h_1 \leq m \) and \( \{i_2, \ldots, h_2\} \subseteq \{i_1, \ldots, h_1\} \), the resulting expression denoted as \( I_{ENF(p_{i_1} + \cdots + p_{h_1}) \lor ENF(p_{i_2} + \cdots + p_{h_2}) \lor \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_2-1} \cdot \cdots \cdot \bar{p}_m} \) is equivalent to that given by Expression (2) in Table 1. Then, \( S \neq I_{ENF(p_{i_1} + \cdots + p_{h_1}) \lor ENF(p_{i_2} + \cdots + p_{h_2}) \lor \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_2-1} \cdot \cdots \cdot \bar{p}_m} \) if and only if there is a test case \( \bar{t} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \cup \left( \bigcup_{i=1}^{m} TP_i(S) \right) \).
Proof: First, we observe that \( S \oplus I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow \bar{p}_1 + \cdots + \bar{p}_{h_1}) \times \text{ENF}(p_{i_1} + \cdots + p_{h_2} \rightarrow \bar{p}_{i_1} + \cdots + \bar{p}_{h_2}) \equiv ((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_2} + \cdots + p_{h_1}) \oplus (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_2} + \cdots + p_{h_1})) \cdot \bar{p}_{i_1} \cdots \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m \)

\[
\equiv ((p_{i_1} + \cdots + p_{h_1})(p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_2} + \cdots + p_{h_1}) \\
+ (p_{i_1} + \cdots + p_{h_1})(p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_2} + \cdots + p_{h_1})) \cdot \bar{p}_{i_1} \cdots \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
\equiv ((p_{i_1} + \cdots + p_{i_2-1} + p_{h_2+1} + \cdots + p_{h_1}) + 0) \cdot \bar{p}_{i_1} \cdots \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

(By making use of \((A + B)(A + \overline{B}) \equiv A\) and \((A + B)(A + \overline{B}) \equiv 0\))

\[
\equiv (p_{i_1} + \cdots + p_{i_2-1} + p_{h_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_{i_1} \cdots \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m.
\]

Now, \( S(\overline{r}) \neq I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow \bar{p}_1 + \cdots + \bar{p}_{h_1}) \times \text{ENF}(p_{i_1} + \cdots + p_{h_2} \rightarrow \bar{p}_{i_1} + \cdots + \bar{p}_{h_2})(\overline{r}) \) if and only if \( S(\overline{r}) \oplus I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow \bar{p}_1 + \cdots + \bar{p}_{h_1}) \times \text{ENF}(p_{i_1} + \cdots + p_{h_2} \rightarrow \bar{p}_{i_1} + \cdots + \bar{p}_{h_2})(\overline{r}) = 1 \) if and only if \((p_{i_1} + \cdots + p_{i_2-1} + p_{h_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_{i_1} \cdots \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m\) evaluates to 1 on \( \overline{r} \)

if and only if \( \overline{r} \in \bigcup_{i=1}^{h_1} TP_i(S) \setminus \bigcup_{\overline{r} \neq i=1 \ldots h_1} TP_i(S) \).

Hence, the result follows. \( \square \)

As a reminder, Theorem 4.2 excludes the case when \( S \) contains just 1 term or 2 terms. When \( S \) contains just 1 term (that is \( S = p_1 \)), negating it twice is equivalent to \( S \). When \( S \) contains just 2 terms (that is \( S = p_1 + p_2 \)), both ENFs require that \( i_1 < h_1 \) and \( i_2 < h_2 \). The net result is to negate \( S \) twice, which is then equivalent to \( S \).

**Theorem 4.3 (\( \text{ENF} \times \text{TNF} \) - Case 1)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression \((p_{i_1} + \cdots + p_{h_1})\) and the \( i_2 \)-th term, \( p_{i_2} \), in \( S \) are negated where \( 1 \leq i_1 < h_1 < i_2 \leq m \), the resulting expression denoted as \( I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow \bar{p}_1 + \cdots + \bar{p}_{h_1}) \times \text{ENF}(p_{i_1} + \cdots + p_{h_2} \rightarrow \bar{p}_{i_1} + \cdots + \bar{p}_{h_2}) \) is equivalent to that given by Expression (3) in Table 1. Then, \( S \neq I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow \bar{p}_1 + \cdots + \bar{p}_{h_1}) \times \text{ENF}(p_{i_1} + \cdots + p_{h_2} \rightarrow \bar{p}_{i_1}) \) if
and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cap \left( TP_{t_2}(S) \right) \left( \bigcup_{i=1}^{m} \left( TP_i(S) \right) \right)$, or

2. $\vec{t} \in FP(S)$.

**Proof:** First, we observe that $S \oplus I_{ENF(p_1 + \cdots + p_{h_1}) \oplus TNF(p_{t_2} \rightarrow \vec{p}_{t_2})}$

$\equiv ((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \oplus (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2})) \cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{m}$

$\equiv ((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}) + (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}))$

$\cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{m}$

$\equiv ((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}) + (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}))$

$\cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{m}$

$\equiv ((p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}) + (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1} + \vec{p}_{i_2}))$

$\cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{m}$

$\equiv (p_{i_1} + \cdots + p_{h_1}) \cdot \vec{p}_{i_2} \cdot (p_{i_1} + \cdots + p_{h_1}) \cdot \vec{p}_{i_2} \cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{m}$

$\equiv (p_{i_1} + \cdots + p_{h_1}) \cdot \vec{p}_{i_2} \cdot (p_{i_1} + \cdots + p_{h_1}) \cdot \vec{p}_{i_2} \cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{m} + \overline{S}$

Now, $S(\vec{t}) \neq I_{ENF(p_1 + \cdots + p_{h_1}) \oplus TNF(p_{t_2} \rightarrow \vec{p}_{t_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_1 + \cdots + p_{h_1}) \oplus TNF(p_{t_2} \rightarrow \vec{p}_{t_2})}(\vec{t}) = 1$

if and only if $(p_{i_1} + \cdots + p_{h_1})p_{i_2} \cdot \vec{p}_{i_2} \cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_{m} + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cap \left( TP_{t_2}(S) \right) \left( \bigcup_{i=1}^{m} \left( TP_i(S) \right) \right)$, or

2. $\vec{t} \in FP(S)$. 
Hence, the result follows.

\[ \square \]

**Theorem 4.4** (ENF \( \times \) TNF - Case 2)

Let \( S=p_1 + \cdots + p_m \) be a Boolean specification in irredudant disjunctive normal form. Suppose that the subexpression \( (p_{i_1} + \cdots + p_{i_h}) \) and the \( i_2 \)-th term, \( p_{i_2} \), in \( S \) are negated where \( 1 \leq i_1 \leq i_2 \leq h_1 \leq m \) and \( i_1 \neq h_1 \), the resulting expression denoted as \( I_{\text{ENF}}(p_{i_1} + \cdots + p_{i_h})\text{\(\times\)TNF}(p_{i_2}) \) is equivalent to Expression (4) in Table 1. Then, \( S \neq I_{\text{ENF}}(p_{i_1} + \cdots + p_{i_h})\text{\(\times\)TNF}(p_{i_2}) \) if and only if there is a test case \( \bar{t} \in \bigcup_{i=i_1}^{h_1} \mathop{\mathcal{T}P}_i(S) \setminus \bigcup_{i \neq i_1, \ldots, h_1}^{i=h_1} \mathop{\mathcal{T}P}_i(S) \).

**Proof:** First, we observe that \( S \oplus I_{\text{ENF}}(p_{i_1} + \cdots + p_{i_h})\text{\(\times\)TNF}(p_{i_2}) \equiv \left( (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{i_1}) \oplus \left( p_{i_1} + \cdots + p_{i_2} + \cdots + p_{i_1} \right) \right) \cdot p_{i_1} \cdots p_{i_1-1} \cdot p_{h_1+1} \cdots p_m \equiv \left( \left( p_{i_1} + \cdots + p_{i_2} + \cdots + p_{i_1} \right) \cdot \left( p_{i_1} + \cdots + p_{i_2} + \cdots + p_{i_1} \right) \right) \cdot p_{i_1} \cdots p_{i_1-1} \cdot p_{h_1+1} \cdots p_m \equiv \left( \left( p_{i_1} + \cdots + p_{i_2} + \cdots + p_{i_1} \right) \cdot \left( p_{i_1} + \cdots + p_{i_2} + \cdots + p_{i_1} \right) \right) \cdot p_{i_1} \cdots p_{i_1-1} \cdot p_{h_1+1} \cdots p_m \equiv \left( p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{i_1} \right) \cdot p_{i_1} \cdots p_{i_1-1} \cdot p_{h_1+1} \cdots p_m \) (By making use of \( (A+B)(A+B) \equiv A \) and \( (A+B)(A+B) \equiv 0 \))

Now, \( S(\bar{t}) \neq I_{\text{ENF}}(p_{i_1} + \cdots + p_{i_h})\text{\(\times\)TNF}(p_{i_2}) \) if and only if \( S(\bar{t}) \oplus I_{\text{ENF}}(p_{i_1} + \cdots + p_{i_h})\text{\(\times\)TNF}(p_{i_2}) \) \(\bar{t} \equiv 1 \) if and only if \( \left( p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{i_1} \right) \cdot p_{i_1} \cdots p_{i_1-1} \cdot p_{h_1+1} \cdots p_m \) evaluates to \( 1 \) on \( \bar{t} \) if and only if \( \bar{t} \in \bigcup_{i=i_1}^{h_1} \mathop{\mathcal{T}P}_i(S) \setminus \bigcup_{i \neq i_1, \ldots, h_1}^{i=h_1} \mathop{\mathcal{T}P}_i(S) \).

Hence, the result follows.

\[ \square \]

**Theorem 4.5** (ENF \( \times \) TOF - Case 1)

Let \( S=p_1 + \cdots + p_m \) be a Boolean specification in irredudant disjunctive normal form. Suppose
that the subexpression \((p_{i_1} + \cdots + p_{h_1})\) in \(S\) is negated and the \(i_2\)-th term, \(p_{i_2}\), in \(S\) is omitted where \(1 \leq i_1 < h_1 < i_2 \leq m\), the resulting expression denoted as \(I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{h_1}) \times \text{TOF}(p_{i_2} - )\) is equivalent to Expression (5) in Table 1. Then, \(S \neq I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{h_1}) \times \text{TOF}(p_{i_2} - )\) if and only if there is a test case \(\bar{t}\) that satisfies any of the following conditions:

1. \(\bar{t} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)\), or

2. \(\bar{t} \in FP(S)\).

**Proof**: First, we observe that \(S \oplus I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{h_1}) \times \text{TOF}(p_{i_2} - )\)

\[ \equiv ((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \oplus (p_{i_1} + \cdots + p_{h_1})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]

\[ \equiv ((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1}) + (p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]

\[ \equiv ((p_{i_1} + \cdots + p_{h_1}) + (p_{i_1} + \cdots + p_{h_1} + p_{i_2})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]

\[ \equiv (p_{i_1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + S \]

Now, \(S(\bar{t}) \neq I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{h_1}) \times \text{TOF}(p_{i_2} - ) (\bar{t})\) if and only if \(S(\bar{t}) \oplus I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{h_1}) \times \text{TOF}(p_{i_2} - ) (\bar{t}) = 1\) if and only if \((p_{i_1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + S\) evaluates to 1 on \(\bar{t}\) if and only if \(\bar{t}\) satisfies any of the following conditions:

1. \(\bar{t} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)\), or

2. \(\bar{t} \in FP(S)\).
Hence, the result follows. □

**Theorem 4.6** (ENF × TOF - Case 2)

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + \cdots + p_{h_1})$ in $S$ is negated and the $i_2$-th term, $p_{i_2}$, in $S$ is omitted where $1 \leq i_1 \leq i_2 \leq h_1 \leq m$ and $i_1 \neq h_1$, the resulting expression denoted as $I_{ENF}(p_1 + \cdots + p_{h_1} \rightarrow \overline{p_1 + \cdots + p_{h_1}}) \times TOF(p_{i_2} \rightarrow )$ is equivalent to that given by Expression (6) in Table 1. Then, $S \neq I_{ENF}(p_1 + \cdots + p_{h_1} \rightarrow \overline{p_1 + \cdots + p_{h_1}}) \times TOF(p_{i_2} \rightarrow )$ if and only if there is a test case $\bar{I}$ that satisfies any of the following conditions:

1. $\bar{I} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right)$, or

2. $\bar{I} \in FP(S)$.

**Proof**: First, we observe that

$$S \oplus I_{ENF}(p_1 + \cdots + p_{h_1} \rightarrow \overline{p_1 + \cdots + p_{h_1}}) \times TOF(p_{i_2} \rightarrow )$$

$$\equiv \left( (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \oplus (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \right)$$

$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$$

$$\equiv \left( (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) + (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \right)$$

$$\cdot (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$$

$$\equiv \left( (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) + (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \right)$$

$$\cdot (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$$

$$\equiv \left( (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \cdot (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \right) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$$

$$\equiv (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$$

$$\equiv (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + \bar{S}$$
Now, \[ S(\overline{t}) \neq I_{ENF}(p_1 + \cdots + p_{h_1} - \overline{p_{i_1}} + \cdots + \overline{p_{i_l}}) \times TOF(p_{i_2} \rightarrow) (\overline{t}) \]
if and only if \[ S(\overline{t}) \oplus I_{ENF}(p_1 + \cdots + p_{h_1} - \overline{p_{i_1}} + \cdots + \overline{p_{i_l}}) \times TOF(p_{i_2} \rightarrow) (\overline{t}) = 1 \]
if and only if \[ (p_{i_1} + \cdots + p_{i_2} - 1 + p_{i_2+1} + \cdots + p_{h_1}) \cdot \overline{p_1} \cdot \overline{p_{h_1+1}} \cdot \overline{p_m} + S \]
evaluates to 1 on \( \overline{t} \)
if and only if \( \overline{t} \) satisfies any of the following conditions:

1. \( \overline{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \), or
2. \( \overline{t} \in FP(S) \).

Hence, the result follows.

\[ \square \]

**Theorem 4.7 (ENF \times DORF - Case 1)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression \( (p_{i_1} + \cdots + p_{h_1}) \) in \( S \) is negated and the subexpression \( (p_{i_2} + p_{i_2+1}) \) in \( S \) is implemented as \( p_{i_2} \cdot p_{i_2+1} \) where \( 1 \leq i_1 < h_1 < i_2 < m \), the resulting expression denoted as

\[ I_{ENF}(p_1 + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{i_l}) \times DORF(p_{i_2} + p_{i_2+1} - p_{i_2+1}) \]
is equivalent to that given in Expression (7) in Table 1. Then, \( S \neq I_{ENF}(p_1 + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{i_l}) \times DORF(p_{i_2} + p_{i_2+1} - p_{i_2+1}) \) if and only if there is a test case \( \overline{t} \) that satisfies any of the following conditions:

1. \( \overline{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup \left( \bigcup_{i=1}^{m} TP_i(S) \right) \), or
2. \( \overline{t} \in FP(S) \).

**Proof:** First, we observe that

\[ S \oplus I_{ENF}(p_1 + \cdots + p_{h_1} - \overline{p_{i_1}} + p_{i_2} + p_{i_2+1}) \times DORF(p_{i_2} + p_{i_2+1} - p_{i_2+1}) \]
\[ \equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{i_2} + p_{i_2+1}) \oplus (p_{i_1} + \cdots + p_{h_1} + p_{i_2} + p_{i_2+1}) \right) \]
\[ = \overline{p_1} \cdot \overline{p_{h_1+1}} \cdot \overline{p_{i_2+2}} \cdots \overline{p_m} \]
\[
\begin{align*}
\equiv & \left( (p_i + \cdots + p_{h_1} + p_{i2} + p_{i2+1}) \cdot (\overline{p_i} + \cdots + p_{h_1} + p_{i2}p_{i2+1}) + (p_i + \cdots + p_{h_1} + p_{i2} + p_{i2+1}) \right) \\
& \cdot \left( (p_i + \cdots + p_{h_1} + p_{i2}p_{i2+1}) \right) \cdot \bar{p}_{i1} \cdots \bar{p}_{i_{h_1-1}} \cdot \bar{p}_{i_{h_1+1}} \cdots \bar{p}_{i_{i2-1}} \cdot \bar{p}_{i_{i2+2}} \cdots \bar{p}_m \\
\equiv & \left( (p_i + \cdots + p_{h_1} + p_{i2} + p_{i2+1}) \cdot (\overline{p_i} + \cdots + p_{h_1} \cdot p_{i2}p_{i2+1}) + (p_i + \cdots + p_{h_1} + p_{i2} + p_{i2+1}) \right) \\
& \cdot \left( (p_i + \cdots + p_{h_1} + p_{i2}p_{i2+1}) \right) \cdot \bar{p}_{i1} \cdots \bar{p}_{i_{h_1-1}} \cdot \bar{p}_{i_{h_1+1}} \cdots \bar{p}_{i_{i2-1}} \cdot \bar{p}_{i_{i2+2}} \cdots \bar{p}_m \\
\equiv & \left( (p_i + \cdots + p_{h_1})p_{i2}p_{i2+1} \cdot \bar{p}_{i1} \cdots \bar{p}_{i_{h_1-1}} \cdot \bar{p}_{i_{h_1+1}} \cdots \bar{p}_{i_{i2-1}} \cdot \bar{p}_{i_{i2+2}} \cdots \bar{p}_m \\
+ & (p_i + \cdots + p_{h_1} + p_{i2} + p_{i2+1}) \cdot \bar{p}_{i1} \cdots \bar{p}_{i_{h_1-1}} \cdot \bar{p}_{i_{h_1+1}} \cdots \bar{p}_{i_{i2-1}} \cdot \bar{p}_{i_{i2+2}} \cdots \bar{p}_m \\
= & \left( (p_i + \cdots + p_{h_1})p_{i2}p_{i2+1} \cdot \bar{p}_{i1} \cdots \bar{p}_{i_{h_1-1}} \cdot \bar{p}_{i_{h_1+1}} \cdots \bar{p}_{i_{i2-1}} \cdot \bar{p}_{i_{i2+2}} \cdots \bar{p}_m + \overline{S} \right)
\end{align*}
\]

Now, \( S(\bar{t}) \neq 1 \) if and only if \( S(\bar{t}) \oplus I_{\text{ENF}}(p_{i1} + \cdots + p_{h_1} \rightarrow \overline{p_{i1}} + \cdots + \overline{p_{h_1}}) \times DORF(p_{i2} + p_{i2+1} \rightarrow \overline{p_{i2}p_{i2+1}})(\bar{t}) \)

if and only if \( (p_i + \cdots + p_{h_1})p_{i2}p_{i2+1} \cdot \bar{p}_{i1} \cdots \bar{p}_{i_{h_1-1}} \cdot \bar{p}_{i_{h_1+1}} \cdots \bar{p}_{i_{i2-1}} \cdot \bar{p}_{i_{i2+2}} \cdots \bar{p}_m + \overline{S} \) evaluates to 1 on \( \bar{t} \)

if and only if \( \bar{t} \) satisfies any of the following conditions:

1. \( \bar{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \cup (TP_{i2}(S) \cap TP_{i2+1}(S)) \right) \),

2. \( \bar{t} \in FP(S) \).

Hence, the result follows.

\[\square\]

**Theorem 4.8 (ENF \times DORF - Case 2)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression \( (p_{i1} + \cdots + p_{h_1}) \) in \( S \) is negated and the subexpression \( (p_{i2} + p_{i2+1}) \) in \( S \)
is implemented as $p_{i_1}p_{i_2+1}$ where $1 \leq i_1 < (h_1 = i_2) < m$, the resulting expression denoted as $I_{ENF}(p_{i_1} + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{h_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})$ is equivalent to that given by Expression (8) in Table 1. Then, $S \neq I_{ENF}(p_{i_1} + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{h_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})$ if and only if there is a test case $i \in \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)$. 

Proof: Since $i_2 = h_1$, we observe that $S \oplus I_{ENF}(p_{i_1} + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{h_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})$

$\equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{h_1+1}) \oplus (p_{i_1} + \cdots + p_{h_1}p_{h_1+1}) \right) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1} \cdots \bar{p}_{h_1} \cdots \bar{p}_m$

$\equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{h_1+1}) \cdot (p_{i_1} + \cdots + p_{h_1}p_{h_1+1}) + (p_{i_1} + \cdots + p_{h_1} + p_{h_1+1}) \right) \cdot (p_{i_1} + \cdots + p_{h_1}p_{h_1+1}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1} \cdots \bar{p}_{h_1} \cdots \bar{p}_m$

$\equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{h_1+1}) \cdot (p_{i_1} + \cdots + p_{h_1}p_{h_1+1}) + (p_{i_1} + \cdots + p_{h_1} + p_{h_1+1}) \right) \cdot (p_{i_1} + \cdots + p_{h_1}p_{h_1+1}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{h_1} \cdots \bar{p}_m$

$\equiv (p_{i_1} + \cdots + p_{h_1}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1} \cdots \bar{p}_{h_1} \cdots \bar{p}_m$

(By making use of $(A + B)(A + \bar{B}) \equiv A$ and $(A\bar{B})(AB) \equiv 0$)

Now, $S(\bar{r}) \neq I_{ENF}(p_{i_1} + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{h_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})(\bar{r})$

if and only if $S(\bar{r}) \oplus I_{ENF}(p_{i_1} + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{h_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})(\bar{r}) = 1$

if and only if $(p_{i_1} + \cdots + p_{h_1}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1} \cdots \bar{p}_m$ evaluates to 1 on $\bar{r}$

if and only if $\bar{r} \in \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)$.

Hence, the result follows. \qed

Theorem 4.9 (ENF $\times$ DORF - Case 3)

Let $S=p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + \cdots + p_{h_1})$ in $S$ is negated and the subexpression $(p_{i_2} + p_{i_2+1})$ in $S$ is implemented as $p_{i_2}p_{i_2+1}$ where $1 \leq (i_1 = i_2) < (h_1 = i_2 + 1) \leq m$, the resulting expression denoted as $I_{ENF}(p_{i_1} + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{h_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})$ is equivalent to that given by Expression (9)
in Table 1. Then, \( S \not= I_{\text{ENF}}(p_1 + \ldots + p_{n_1} - p_{i_1} - \ldots - p_{n_1}) \times DORF(p_{i_2} + p_{i_2+1} - p_{i_2} p_{i_2+1}) \) if and only if there is a test case \( \tilde{t} \) that satisfies any of the following conditions:

1. \( \tilde{t} \in (TP_{i_1}(S) \cap TP_{i_1+1}(S)) \setminus \left( \bigcup_{i \neq i_1, i_1+1} TP_i(S) \right), \) or

2. \( \tilde{t} \in FP(S). \)

**Proof:** Since \( i_1 = i_2 \) and \( h_1 = i_1 + 1 \), we have \( h_1 = i_1 + 1 \). Then, we observe that

\[
S \oplus I_{\text{ENF}}(p_1 + \ldots + p_{n_1} - p_{i_1} - \ldots - p_{n_1}) \times DORF(p_{i_2} + p_{i_2+1} - p_{i_2} p_{i_2+1}) \\
\equiv \left( (p_{i_1} + p_{i_1+1}) \oplus (\overline{p_{i_1} p_{i_1+1}}) \right) \cdot \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m \\
\equiv \left( (p_{i_1} + p_{i_1+1}) \cdot (\overline{p_{i_1} p_{i_1+1}}) \right) \cdot \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m \\
\equiv \left( (p_{i_1} + p_{i_1+1}) \cdot (p_{i_1} p_{i_1+1}) \right) \cdot \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m \\
\equiv \left( p_{i_1} p_{i_1+1} + (p_{i_1} + p_{i_1+1}) \right) \cdot \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m
\]

(By making use of \((A + B)(AB) = AB \) and \((A + B)(AB) = A + B)\)

\[
\equiv p_{i_1} p_{i_1+1} \cdot \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m + (p_{i_1} + p_{i_1+1}) \cdot \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m \\
\equiv p_{i_1} p_{i_1+1} \cdot \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m + \overline{\tilde{S}}
\]

Now, \( S(\tilde{t}) \not= I_{\text{ENF}}(p_1 + \ldots + p_{n_1} - p_{i_1} - \ldots - p_{n_1}) \times DORF(p_{i_2} + p_{i_2+1} - p_{i_2} p_{i_2+1})(\tilde{t}) \) if and only if \( S(\tilde{t}) \oplus I_{\text{ENF}}(p_1 + \ldots + p_{n_1} - p_{i_1} - \ldots - p_{n_1}) \times DORF(p_{i_2} + p_{i_2+1} - p_{i_2} p_{i_2+1})(\tilde{t}) = 1 \) if and only if \( p_{i_1} p_{i_1+1} \cdot \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m + \overline{\tilde{S}} \) evaluates to 1 on \( \tilde{t} \) if and only if \( \tilde{t} \) satisfies any of the following conditions:

1. \( \tilde{t} \in (TP_{i_1}(S) \cap TP_{i_1+1}(S)) \setminus \left( \bigcup_{i \neq i_1, i_1+1} TP_i(S) \right), \) or

2. \( \tilde{t} \in FP(S). \)

Hence, the result follows. \( \square \)

**Theorem 4.10 (ENF \( \times \) DORF - Case 4)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose
that the subexpression \((p_i + \cdots + p_{i_1})\) in \(S\) is negated and the subexpression \((p_{i_2} + p_{i_2+1})\) in \(S\) is implemented as \(p_{i_2}p_{i_2+1}\) where \(1 \leq i_1 \leq i_2 < h_1 \leq m, i_1 \neq i_2\) and \(h_1 \neq i_2 + 1\), the resulting expression denoted as \(I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow p_1 + \cdots + p_{h_1}) \ast DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})\) is equivalent to that given by Expression (10) in Table 1. Then, \(S \neq I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow p_1 + \cdots + p_{h_1}) \ast DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})\) if and only if there is a test case \(\tilde{i}\) that satisfies any of the following conditions:

1. \(\tilde{i} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \setminus \left( \bigcup_{i=i_1, \ldots, h_1} TP_i(S) \right), \) or

2. \(\tilde{i} \in FP(S).\)

**Proof:** First, we observe that \(S \oplus I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow p_1 + \cdots + p_{h_1}) \ast DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})\)

\[
\equiv \left( (p_i + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}) \oplus (p_i + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) \right) \\
\cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{h_1+1} \cdots \tilde{p}_m \\
\equiv \left( (p_i + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}) \cdot (p_i + \cdots + p_{i_2p_{i_2+1} + \cdots + p_{h_1}} \right) \\
+ (p_i + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}) \cdot (p_i + \cdots + p_{i_2p_{i_2+1} + \cdots + p_{h_1}} \right) \\
\cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{h_1+1} \cdots \tilde{p}_m \\
\equiv \left( (p_i + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) + (p_i + \cdots + p_{i_2p_{i_2+1} + \cdots + p_{h_1}} \right) \\
\cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{h_1+1} \cdots \tilde{p}_m \\
\text{(By making use of } (A + B + C)(AB + C) \equiv AB + C \text{ and } (A + B + C)(AB + C) \equiv A + B + C) \\
\equiv (p_i + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) \cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{h_1+1} \cdots \tilde{p}_m \\
+ (p_i + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) \cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{h_1+1} \cdots \tilde{p}_m \\
\equiv (p_i + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) \cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{h_1+1} \cdots \tilde{p}_m + \bar{S}
\]
Now, \[ S(\vec{t}) \neq I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{i_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1}) \]

if and only if \[ S(\vec{t}) \oplus I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{i_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1}) = 1 \]

if and only if \[ (p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) \cdot \vec{p}_{i_1} \cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_m + \vec{S} \text{ evaluates to 1 on } \vec{t} \]

if and only if \( \vec{t} \) satisfies any of the following conditions:

1. \( \vec{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \setminus \left( \bigcup_{i \neq i_1, i_2+1}^{m} TP_i(S) \right), \) or
2. \( \vec{t} \in FP(S). \)

Hence, the result follows. \qed

**Theorem 4.11 (ENF ⊨ CORF - Case 1)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression \( (p_{i_1} + \cdots + p_{h_1}) \) in \( S \) is negated and the \( i_2 \)-th term, \( p_{i_2} \), in \( S \) is implemented as \( p_{i_2,1}.j_2 + p_{i_2,2+1}.k_2 \) where \( 1 \leq i_1 < h_1 < i_2 \leq m, p_{i_2} = p_{i_2,1}.j_2 \cdot p_{i_2,2+1}.k_2 \) and \( 1 \leq j_2 < k_2 \), the resulting expression denoted as \( I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{i_1}) \times CORF(p_{i_2} \rightarrow p_{i_2,1}.j_2 + p_{i_2,2+1}.k_2) \) is equivalent to Expression (11) in Table 1. Then, \( S \not\equiv I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{i_1}) \times CORF(p_{i_2} \rightarrow p_{i_2,1}.j_2 + p_{i_2,2+1}.k_2) \) if and only if there is a test case \( \vec{t} \) that satisfies any of the following conditions:

1. \( \vec{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1, i_2+1}^{m} TP_i(S) \right) \) such that \( p_{i_2,1}.j_2 + p_{i_2,2+1}.k_2 = 0 \), or
2. \( \vec{t} \in FP(S). \)

**Proof:** First, we observe that \( S \oplus I_{\text{ENF}}(p_1 + \cdots + p_{h_1} \rightarrow p_{i_1} + \cdots + p_{i_1}) \times CORF(p_{i_2} \rightarrow p_{i_2,1}.j_2 + p_{i_2,2+1}.k_2) \)

\[ \equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \oplus (p_{i_1} + \cdots + p_{h_1} + p_{i_2,1}.j_2 + p_{i_2,2+1}.k_2) \right) \]

\[ \times \vec{p}_{i_1} \cdot \vec{p}_{i_1+1} \cdot \vec{p}_{i_2} \cdot \vec{p}_{i_2+1} \cdot \vec{p}_m \]

\[ \equiv \left( (p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1} + p_{i_2,1}.j_2 + p_{i_2,2+1}.k_2) \right) + \left( \vec{p}_{i_1} + \cdots + p_{h_1} + p_{i_2} \right) \]
\[
(\bar{p}_1 + \cdots + p_{h_1} + p_{i_1,1,j_2} + p_{i_2,1,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m
\equiv ( (p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1}) \cdot (p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_{i_2}}) + (p_{i_1} + \cdots + p_{h_1} + p_{i_2})
\cdot (p_{i_1} + \cdots + p_{h_1} + p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_{i_2}}) ) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m
\equiv ( (p_{i_1} + \cdots + p_{h_1}) (p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_{i_2}}) + (p_{i_1} + \cdots + p_{h_1} + p_{i_2})
\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m
\]  

(By making use of \((A + B)A \equiv A\) and \((\overline{A + BC}) (A + B + C) \equiv \overline{A + BC}\))

\[
\equiv \left( p_{i_1} + \cdots + p_{h_1} \right) (p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2} \bar{p}_{i_2+1} \cdots \bar{p}_m
\;+
\; (p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m
\equiv \left( p_{i_1} + \cdots + p_{h_1} \right) (p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + S
\]  

Now, \(S(\bar{t}) \neq I_{ENF}(p_{i_1} + \cdots + p_{h_1} \rightarrow \overline{p_{i_1} + \cdots + p_{h_1}}) \otimes CORF(p_{i_2,1,j_2} \rightarrow p_{i_2,1,j_2+1,k_{i_2}})(\bar{t})\) if and only if \(S(\bar{t}) \oplus I_{ENF}(p_{i_1} + \cdots + p_{h_1} \rightarrow \overline{p_{i_1} + \cdots + p_{h_1}}) \otimes CORF(p_{i_2,1,j_2} \rightarrow p_{i_2,1,j_2+1,k_{i_2}})(\bar{t}) = 1\) if and only if \((p_{i_1} + \cdots + p_{h_1}) (p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + S\) evaluates to 1 on \(\bar{t}\) if and only if \(\bar{t}\) satisfies any of the following conditions:

1. \(\bar{t} \in \bigcup_{i=i_1}^{h_1} TP_i(S) \setminus \bigcup_{i=1}^{m} TP_i(S)\) such that \(p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_{i_2}}\) is negated and the \(i_2\)-th term, \(p_{i_2}\), in \(S\) is implemented

Hence, the result follows. \(\square\)

**Theorem 4.12** (\(ENF \otimes CORF - Case 2\))

Let \(S=p_1 + \cdots + p_m\) be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression \((p_{i_1} + \cdots + p_{h_1})\) in \(S\) is negated and the \(i_2\)-th term, \(p_{i_2}\), in \(S\) is implemented...
as \(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}\) where \(1 \leq i_1 \leq i_2 \leq h_1 \leq m, i_1 \neq h_1, p_{i_2} = p_{i_2,1,j_2}, p_{i_2,j_2+1,k_{i_2}}\) and \(1 \leq j_2 < k_{i_2}\), the resulting expression denoted as \(I_{\text{ENF}}(p_1 + \ldots + p_{h_1} - p_1 + \ldots + p_{h_1}) \times \text{CORF}(p_2 - p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\) is equivalent to that given by Expression (12) in Table 1. Then, we have \(S \neq I_{\text{ENF}}(p_1 + \ldots + p_{h_1} - p_1 + \ldots + p_{h_1}) \times \text{CORF}(p_2 - p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\) if and only if there is a test case \(\vec{t}\) that satisfies any of the following conditions:

1. \(\vec{t} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)\), or

2. \(\vec{t} \in \text{FP}(S)\) such that \(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\) on \(\vec{t}\).

**Proof**: First, we observe that \(S \oplus I_{\text{ENF}}(p_1 + \ldots + p_{h_1} - p_1 + \ldots + p_{h_1}) \times \text{CORF}(p_2 - p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\)

\[
\equiv \left( (p_1 + \ldots + p_{i_2} + \ldots + p_{h_1}) \oplus (p_1 + \ldots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \ldots + p_{h_1}) \right) \\
\ 
\cdot \vec{p}_1 \cdot \vec{p}_{i_1-1} \cdot \vec{p}_{h_1+1} \cdot \ldots \cdot \vec{p}_m \\
\equiv \left( (p_1 + \ldots + p_{i_2} + \ldots + p_{h_1}) \cdot (p_1 + \ldots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \ldots + p_{h_1}) \\
+ (p_1 + \ldots + p_{i_2} + \ldots + p_{h_1}) \cdot (p_1 + \ldots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \ldots + p_{h_1}) \\
\cdot \vec{p}_1 \cdot \vec{p}_{i_1-1} \cdot \vec{p}_{h_1+1} \cdot \ldots \cdot \vec{p}_m \\
\equiv \left( (p_1 + \ldots + p_{i_2} + \ldots + p_{h_1}) + (p_1 + \ldots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \ldots + p_{h_1}) \\
\cdot \vec{p}_1 \cdot \vec{p}_{i_1-1} \cdot \vec{p}_{h_1+1} \cdot \ldots \cdot \vec{p}_m \\
\equiv (A + BC)(A + B + C) \equiv A + BC
\]

and \((A + BC)(A + B + C) \equiv A + B + C\)

\[
\equiv (p_1 + \ldots + p_{i_2} + \ldots + p_{h_1}) \cdot \vec{p}_1 \cdot \vec{p}_{i_1-1} \cdot \vec{p}_{h_1+1} \cdot \ldots \cdot \vec{p}_m \\
\equiv (p_1 + \ldots + p_{i_2} + \ldots + p_{h_1}) \cdot \vec{p}_1 \cdot \vec{p}_{i_1-1} \cdot \vec{p}_{h_1+1} \cdot \ldots \cdot \vec{p}_m \\
\equiv (p_1 + \ldots + p_{i_2} + \ldots + p_{h_1}) \cdot \vec{p}_1 \cdot \vec{p}_{i_1-1} \cdot \vec{p}_{h_1+1} \cdot \ldots \cdot \vec{p}_m \\
\equiv (p_1 + \ldots + p_{i_2} + \ldots + p_{h_1}) \cdot \vec{p}_1 \cdot \vec{p}_{i_1-1} \cdot \vec{p}_{h_1+1} \cdot \ldots \cdot \vec{p}_m
\]
\[ + \bar{p}_{i_2, j_2} \cdot \bar{p}_{i_2, j_2+1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2, j_2} + \bar{p}_{i_2, j_2+1, k_2}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]

(By rewriting \( AB \) as \( A(B + B) \) because they are equivalent)

\[ \equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m \]
\[ + \bar{p}_{i_2, j_2} \cdot \bar{p}_{i_2, j_2+1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2, j_2} + \bar{p}_{i_2, j_2+1, k_2}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]

\[ \equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_2, j_2} \cdot \bar{p}_{i_2, j_2+1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m \]
\[ \equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_2, j_2} \cdot \bar{p}_{i_2, j_2+1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m \]
\[ \equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + (\bar{p}_{i_2, j_2} + \bar{p}_{i_2, j_2+1, k_2}) \cdot \bar{S} \]

Now,

\[ S(\bar{t}) \neq I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{h_1}) \times \text{CORF}(p_{i_2, j_2} - p_{i_2, j_2} + p_{i_2, j_2+1, k_2})(\bar{t}) \]

if and only if \[ S(\bar{t}) \oplus I_{\text{ENF}}(p_{i_1} + \cdots + p_{h_1} - p_{i_1} + \cdots + p_{h_1}) \times \text{CORF}(p_{i_2, j_2} - p_{i_2, j_2} + p_{i_2, j_2+1, k_2})(\bar{t}) = 1 \]

if and only if \[ (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + (\bar{p}_{i_2, j_2} + \bar{p}_{i_2, j_2+1, k_2}). \bar{S} \]
evaluates to 1 on \( \bar{t} \)

if and only if \( \bar{t} \) satisfies any of the following conditions:

1. \( \bar{t} \in \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \), or

2. \( \bar{t} \in FP(S) \) such that \( p_{i_2, j_2} + p_{i_2, j_2+1, k_2} = 0 \) on \( \bar{t} \).

Hence, the result follows. \( \square \)

### 4.1.2 TNF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is a TNF. Since ENF \( \equiv \) TNF has been discussed in previous section, we do not repeat the discussion here.

**Theorem 4.13** \( (\text{TNF} \equiv \text{TNF}) \)

*Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that*
two different terms, $p_{i_1}$ and $p_{i_2}$, in $S$ are negated where $1 \leq i_1 < i_2 \leq m$, the resulting expression denoted as $I_{\text{TNF}}(p_{i_1} \rightarrow \overline{p_{i_1}}) \times \text{TNF}(p_{i_2} \rightarrow \overline{p_{i_2}})$ is equivalent to that given by Expression (13) in Table 1. Then, $S \neq I_{\text{TNF}}(p_{i_1} \rightarrow \overline{p_{i_1}}) \times \text{TNF}(p_{i_2} \rightarrow \overline{p_{i_2}})$ if and only if there is a test case $\overline{t}$ that satisfies any of the following conditions:

1. $\overline{t} \in \left( TP_{i_1}(S) \cap TP_{i_2}(S) \right) \backslash \left( \bigcup_{i=1, i \neq i_1, i_2}^{m} TP_i(S) \right)$, or
2. $\overline{t} \in FP(S)$.

Proof: First, we observe that $S \oplus I_{\text{TNF}}(p_{i_1} \rightarrow \overline{p_{i_1}}) \times \text{TNF}(p_{i_2} \rightarrow \overline{p_{i_2}})$

\[
\equiv \left( (p_{i_1} + p_{i_2}) \oplus (\overline{p_{i_1}} + \overline{p_{i_2}}) \right) \cdot \overline{p_{i_1}} \cdot \overline{p_{i_1} + 1} \cdot \overline{p_{i_1} + 2} \cdots \overline{p_{i_1} \cdot \overline{p_{i_2} - 1}} \cdot \overline{p_{i_2} + 1} \cdots \overline{p_m}
\equiv \left( (p_{i_1} + p_{i_2}) \cdot (\overline{p_{i_1} + p_{i_2}}) \right) \cdot \overline{p_{i_1}} \cdot \overline{p_{i_1} + 1} \cdot \overline{p_{i_1} + 2} \cdots \overline{p_{i_1} + \overline{p_{i_2} - 1}} \cdot \overline{p_{i_2} + 1} \cdots \overline{p_m}
\equiv \left( (p_{i_1} + p_{i_2}) \cdot (p_{i_1} + p_{i_2}) \right) \cdot \overline{p_{i_1}} \cdot \overline{p_{i_1} + 1} \cdot \overline{p_{i_1} + 2} \cdots \overline{p_{i_1} + \overline{p_{i_2} - 1}} \cdot \overline{p_{i_2} + 1} \cdots \overline{p_m}
\] (By making use of $(A + B)(AB) = AB$ and $(\overline{A + B})(AB) = \overline{A + B}$)

\[
\equiv p_{i_1} p_{i_2} \cdot \overline{p_{i_1} - 1} \cdot \overline{p_{i_1} + 1} \cdot \overline{p_{i_2} - 1} \cdot \overline{p_{i_2} + 1} \cdots \overline{p_m} + (p_{i_1} + p_{i_2}) \cdot \overline{p_{i_1} - 1} \cdot \overline{p_{i_1} + 1} \cdots \overline{p_{i_1} + \overline{p_{i_2} - 1}} \cdot \overline{p_{i_2} + 1} \cdots \overline{p_m}
\equiv p_{i_1} p_{i_2} \cdot \overline{p_{i_1} - 1} \cdot \overline{p_{i_1} + 1} \cdot \overline{p_{i_2} - 1} \cdot \overline{p_{i_2} + 1} \cdots \overline{p_m} + \overline{S}
\]

Now, $S(\overline{t}) \neq I_{\text{TNF}}(p_{i_1} \rightarrow \overline{p_{i_1}}) \times \text{TNF}(p_{i_2} \rightarrow \overline{p_{i_2}})(\overline{t})$

if and only if $S(\overline{t}) \oplus I_{\text{TNF}}(p_{i_1} \rightarrow \overline{p_{i_1}}) \times \text{TNF}(p_{i_2} \rightarrow \overline{p_{i_2}})(\overline{t}) = 1$

if and only if $p_{i_1} p_{i_2} \cdot \overline{p_{i_1} - 1} \cdot \overline{p_{i_1} + 1} \cdot \overline{p_{i_2} - 1} \cdot \overline{p_{i_2} + 1} \cdots \overline{p_m} + \overline{S}$ evaluates to 1 on $\overline{t}$

if and only if $\overline{t}$ satisfies any of the following conditions:

1. $\overline{t} \in \left( TP_{i_1}(S) \cap TP_{i_2}(S) \right) \backslash \left( \bigcup_{i=1, i \neq i_1, i_2}^{m} TP_i(S) \right)$, or
2. $\overline{t} \in FP(S)$.

Hence, the result follows. \qed
Theorem 4.14 \((T NF \nsim TOF)\)

Let \(S=p_1 + \cdots + p_m\) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \(i_1\)-th term, \(p_{i_1}\), in \(S\) is negated and the \(i_2\)-th term, \(p_{i_2}\), in \(S\) is omitted where \(1 \leq i_1 < i_2 \leq m\), the resulting expression denoted as \(I_{T NF}(p_{i_1} \rightarrow \bar{p}_{i_1}) \nsim I_{TOF}(p_{i_2} \rightarrow )\) is equivalent to that given by Expression (14) in Table 1. Then, \(S \not\equiv I_{T NF}(p_{i_1} \rightarrow \bar{p}_{i_1}) \nsim I_{TOF}(p_{i_2} \rightarrow )\) if and only if there is a test case \(\vec{t}\) that satisfies any of the following conditions:

1. \(\vec{t} \in \left( TP_{i_1}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_{i}(S) \right), \) or
2. \(\vec{t} \in FP(S). \)

Proof: First, we observe that \(S \oplus I_{T NF}(p_{i_1} \rightarrow \bar{p}_{i_1}) \nsim I_{TOF}(p_{i_2} \rightarrow )\)

\[
\equiv \left( (p_{i_1} + p_{i_2}) \oplus (\bar{p}_{i_1}) \right) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
\equiv \left( (p_{i_1} + p_{i_2}) \cdot (\bar{p}_{i_1}) + (\bar{p}_{i_1} + p_{i_2}) \cdot (\bar{p}_{i_1}) \right) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
\equiv \left( (p_{i_1} + p_{i_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
\equiv \left( p_{i_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{S} \right) \]

Now, \(S(\vec{t}) \not\equiv I_{T NF}(p_{i_1} \rightarrow \bar{p}_{i_1}) \nsim I_{TOF}(p_{i_2} \rightarrow )\(\vec{t}\)\)

if and only if \(S(\vec{t}) \oplus I_{T NF}(p_{i_1} \rightarrow \bar{p}_{i_1}) \nsim I_{TOF}(p_{i_2} \rightarrow )\(\vec{t}\) = 1\)

if and only if \(p_{i_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{S}\) evaluates to 1 on \(\vec{t}\)

if and only if \(\vec{t}\) satisfies any of the following conditions:

1. \(\vec{t} \in \left( TP_{i_1}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_{i}(S) \right), \) or
2. \(\vec{t} \in FP(S). \)

Hence, the result follows. \(\square\)
Theorem 4.15 \((TNF \bowtie DORF - \text{Case 1})\)

Let \(S = p_1 + \cdots + p_m\) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \(i_1\)-th term, \(p_{i_1}\), in \(S\) is negated and the subexpression \((p_{i_2} + p_{i_2+1})\) in \(S\) is implemented as \(p_{i_2}p_{i_2+1}\) where \(1 \leq i_1 < i_2 < m\), the resulting expression denoted as \(I_{\text{TNF}}(p_{i_1} \rightarrow \bar{p}_{i_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})\) is equivalent to that given by Expression (15) in Table 1.

Then, \(S \not\equiv I_{\text{TNF}}(p_{i_1} \rightarrow \bar{p}_{i_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})\) if and only if there is a test case \(\bar{t}\) that satisfies any of the following conditions:

1. \(\bar{t} \in TP_{i_2}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \cup (TP_{i_2}(S) \cap TP_{i_2+1}(S)) \right)\), or
2. \(\bar{t} \in FP(S)\).

**Proof:** First, we observe that \(S \oplus I_{\text{TNF}}(p_{i_1} \rightarrow \bar{p}_{i_1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})\)

\[
\equiv \left( (p_{i_1} + p_{i_2} + p_{i_2+1}) \oplus (\bar{p}_{i_1} + p_{i_2}p_{i_2+1}) \right) \cdot \bar{p}_{i_2} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2} \cdots \bar{p}_m
\]

\[
\equiv \left( (p_{i_1} + p_{i_2} + p_{i_2+1}) \cdot (\bar{p}_{i_1}p_{i_2}p_{i_2+1}) + (p_{i_1} + p_{i_2} + p_{i_2+1}) \cdot (\bar{p}_{i_2}p_{i_2+1}) \right) \cdot \bar{p}_{i_2} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2} \cdots \bar{p}_m
\]

\[
\equiv \left( (p_{i_1} + p_{i_2} + p_{i_2+1}) \cdot (\bar{p}_{i_1}p_{i_2}p_{i_2+1}) + (p_{i_1} + p_{i_2} + p_{i_2+1}) \cdot (\bar{p}_{i_2}p_{i_2+1}) \right) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2} \cdots \bar{p}_m
\]

\[
\equiv \left( (p_{i_1}p_{i_2}p_{i_2+1}) + p_{i_1} + p_{i_2} + p_{i_2+1} \right) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2} \cdots \bar{p}_m
\]

(By making use of \((A + B + C)(\bar{A} + BC) \equiv A + B + C\))

\[
\equiv p_{i_1} (p_{i_2}p_{i_2+1}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2} \cdots \bar{p}_m
\]

\[
+ (p_{i_1} + p_{i_2} + p_{i_2+1}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2} \cdots \bar{p}_m
\]

\[
\equiv p_{i_1} (p_{i_2}p_{i_2+1}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2} \cdots \bar{p}_m + S
\]
Now, \( S(\overline{t}) \neq I_{\text{TNF}}(p_1 \overline{\bar{p}}_i) \times DORF(p_2 + p_2 + 1 \overline{\bar{p}}_2 p_2 + 1)(\overline{t}) \)

if and only if \( S(\overline{t}) \oplus I_{\text{TNF}}(p_1 \overline{\bar{p}}_i) \times DORF(p_2 + p_2 + 1 \overline{\bar{p}}_2 p_2 + 1)(\overline{t}) = 1 \)

if and only if \( p_i(\overline{\bar{p}}_i p_2 p_2 + 1) \cdot \overline{\bar{p}}_1 \cdots \overline{\bar{p}}_{i-1} \cdot \overline{\bar{p}}_{i+1} \cdots \overline{\bar{p}}_{i+2} \cdots \overline{\bar{p}}_m + S \) evaluates to 1 on \( \overline{t} \)

if and only if \( \overline{t} \) satisfies any of the following conditions:

1. \( \overline{t} \in TP_{i_1}(S) \setminus \left( \bigcup_{i = 1}^{m} TP_i(S) \right) \) or \( \overline{t} \notin TP_{i_1}(S) \)

2. \( \overline{t} \in FP(S) \).

Hence, the result follows. \( \square \)

**Theorem 4.16 (TNF \times DORF - Case 2)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \( i_1 \)-th term, \( p_{i_1} \), is negated and the subexpression \( (p_i + p_{i+2}) \) is implemented as \( p_{i_2} p_{i_2 + 1} \) where \( 1 \leq (i_1, i_2) < m \), the resulting expression denoted as \( I_{\text{TNF}}(p_1 \overline{\bar{p}}_i) \times DORF(p_2 + p_2 + 1 \overline{\bar{p}}_2 p_2 + 1) \) is equivalent to that given by Expression (16) in Table 1. Then, \( S \neq I_{\text{TNF}}(p_1 \overline{\bar{p}}_i) \times DORF(p_2 + p_2 + 1 \overline{\bar{p}}_2 p_2 + 1) \) if and only if there is a test case \( \overline{t} \in TP_{i_1}(S) \setminus \left( \bigcup_{i = 1}^{m} TP_i(S) \right) \).

**Proof:** Since \( i_1 = i_2 \), we observe that \( S \oplus I_{\text{TNF}}(p_1 \overline{\bar{p}}_i) \times DORF(p_2 + p_2 + 1 \overline{\bar{p}}_2 p_2 + 1) \)

\[ \equiv ((p_i + p_{i+1}) \oplus (\overline{\bar{p}}_i p_{i+1})) \cdot \overline{\bar{p}}_1 \cdots \overline{\bar{p}}_{i-1} \overline{\bar{p}}_{i+2} \cdots \overline{\bar{p}}_m \]

\[ \equiv ((p_i + p_{i+1}) \cdot (\overline{\bar{p}}_i p_{i+1}) + (p_i + p_{i+1}) \cdot (\overline{\bar{p}}_i p_{i+1})) \cdot \overline{\bar{p}}_1 \cdots \overline{\bar{p}}_{i-1} \overline{\bar{p}}_{i+2} \cdots \overline{\bar{p}}_m \]

\[ \equiv ((p_i + p_{i+1}) \cdot (\overline{\bar{p}}_i + \overline{\bar{p}}_{i+1}) + (p_i + p_{i+1}) \cdot (\overline{\bar{p}}_i p_{i+1})) \cdot \overline{\bar{p}}_1 \cdots \overline{\bar{p}}_{i-1} \overline{\bar{p}}_{i+2} \cdots \overline{\bar{p}}_m \]

\[ \equiv ((p_i + p_{i+1}) \cdot (\overline{\bar{p}}_i + \overline{\bar{p}}_{i+1}) + bar p_{i+1}) + (p_i + p_{i+1}) \cdot (\overline{\bar{p}}_i p_{i+1})) \cdot \overline{\bar{p}}_1 \cdots \overline{\bar{p}}_{i-1} \overline{\bar{p}}_{i+2} \cdots \overline{\bar{p}}_m \]

\[ \equiv (p_i + 0) \cdot \overline{\bar{p}}_1 \cdots \overline{\bar{p}}_{i-1} \overline{\bar{p}}_{i+2} \cdots \overline{\bar{p}}_m \]

(By making use of \((A + B)(A + \overline{\overline{\overline{B}}}) \equiv A \) and \((\overline{\overline{\overline{A}} + B})(A \overline{\overline{\overline{B}}}) \equiv 0 \))

\[ \equiv p_i \cdot \overline{\bar{p}}_1 \cdots \overline{\bar{p}}_{i-1} \overline{\bar{p}}_{i+2} \cdots \overline{\bar{p}}_m \]
Now, \[ S(\vec{r}) \neq I_{\text{TNF}}(p_1 \rightarrow \bar{p}_1) \ltimes \text{DORF}(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})(\vec{r}) \]
if and only if \[ S(\vec{r}) \oplus I_{\text{TNF}}(p_1 \rightarrow \bar{p}_1) \ltimes \text{DORF}(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})(\vec{r}) = 1 \]
if and only if \[ p_1 \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+2} \cdots \bar{p}_m \text{ evaluates to } 1 \text{ on } \vec{r} \]
if and only if \[ \vec{r} \in TP_{i_1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right). \] Hence, the result follows. \hfill \Box

**Theorem 4.17 (TNF \ltimes CORF - Case 1)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \( i_1 \)-th term, \( p_{i_1} \), in \( S \) is negated and the \( i_2 \)-th term, \( p_{i_2} \), in \( S \) is implemented as \( p_{i_2+1} + p_{i_2+2} + k_{i_2} \) where \( 1 \leq i_1 < i_2 \leq m \), \( p_{i_2} = p_{i_2+1} \cdot p_{i_2+2} + k_{i_2} \) and \( 1 \leq j_2 < k_{i_2} \), the resulting expression denoted as \( I_{\text{TNF}}(p_1 \rightarrow \bar{p}_1) \ltimes \text{CORF}(p_{i_2} \rightarrow p_{i_2+1}, j_2 + p_{i_2+2} + k_{i_2}) \) is equivalent to that given by Expression (17) in Table 1.

Then, \( S \neq I_{\text{TNF}}(p_1 \rightarrow \bar{p}_1) \ltimes \text{CORF}(p_{i_2} \rightarrow p_{i_2+1}, j_2 + p_{i_2+2} + k_{i_2}) \) if and only if there is a test case \( \vec{r} \) that satisfies any of the following conditions:

1. \( \vec{r} \in UTP_{i_1}(S) \) such that \( p_{i_2+1} + p_{i_2+2} + k_{i_2} = 0 \), or

2. \( \vec{r} \in FP(S) \).

**Proof:** First, we observe that \( S \oplus I_{\text{TNF}}(p_1 \rightarrow \bar{p}_1) \ltimes \text{CORF}(p_{i_2} \rightarrow p_{i_2+1}, j_2 + p_{i_2+2} + k_{i_2}) \)

\[ \equiv ((p_{i_1} + p_{i_2}) \oplus (\bar{p}_{i_1} + p_{i_1+1} + p_{i_1+2} + k_{i_1} - 1) \cdot \bar{p}_{i_1} + 1 \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m) \]

\[ \equiv ((p_{i_1} + p_{i_2}) \cdot (\bar{p}_{i_1} + p_{i_1+1} + p_{i_1+2} + k_{i_1}) + (p_{i_1} + p_{i_2}) \cdot (\bar{p}_{i_1} + p_{i_1+1} + p_{i_1+2} + k_{i_1})) \]

\[ \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1+1} \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]

\[ \equiv ((p_{i_1} + p_{i_2}) \cdot (\bar{p}_{i_1} + p_{i_1+1} + p_{i_1+2} + k_{i_1}) + (p_{i_1} + p_{i_2}) \cdot (\bar{p}_{i_1} + p_{i_1+1} + p_{i_1+2} + k_{i_1})) \]

\[ \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1+1} \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]

\[ \equiv ((p_{i_1+1} + p_{i_2+1}) \cdot (\bar{p}_{i_1+1} + p_{i_1+1} + p_{i_1+2} + k_{i_1}) + (p_{i_1+1} + p_{i_2+1}) \cdot (\bar{p}_{i_1+1} + p_{i_1+1} + p_{i_1+2} + k_{i_1})) \]

\[ \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_1+1+1} \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]

\[ \equiv ((p_{i_1+1} + p_{i_2+1}) \cdot (\bar{p}_{i_1+1} + p_{i_1+1} + p_{i_1+2} + k_{i_1}) + (p_{i_1+1} + p_{i_2+1}) \cdot (\bar{p}_{i_1+1} + p_{i_1+1} + p_{i_1+2} + k_{i_1})) \]

\[ \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_1+1+1} \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \]
\[(\text{By making use of } (A+BC)(A+B+C) \equiv A+BC)\]
\[= p_i \cdot (p_{i_1,j_2} + p_{i_2,j_2+1,k_2}) \cdot \tilde{p}_{i_1} \cdot \tilde{p}_{i_1+1} \cdot \tilde{p}_{i_2-1} \cdot \tilde{p}_{i_2} \cdot \tilde{p}_{m} + S\]
\[= (p_i + p_{i_2}) \cdot \tilde{p}_{i_1} \cdot \tilde{p}_{i_1+1} \cdot \tilde{p}_{i_2-1} \cdot \tilde{p}_{m} + S\]
\[= p_i \cdot \tilde{p}_{i_2} \cdot \tilde{p}_{i_2+1} + (p_i + p_{i_2}) \cdot \tilde{p}_{i_1} \cdot \tilde{p}_{i_1+1} \cdot \tilde{p}_{i_2-1} \cdot \tilde{p}_{m} + S\]
\[= p_i \cdot \tilde{p}_{i_2} \cdot \tilde{p}_{i_2+1} \cdot \tilde{p}_{m} + S\]
\[= p_i \cdot \tilde{p}_{i_2} \cdot \tilde{p}_{m} + S\]

\[\text{Now, } S(\tilde{t}) \neq I_{\text{TNF}}(p_i \rightarrow \neg p_1) \otimes I_{\text{CORF}}(p_2 \rightarrow p_{i_2} + p_{i_2,j_2+1,k_2})(\tilde{t})\]
\[\text{if and only if } S(\tilde{t}) \oplus I_{\text{TNF}}(p_i \rightarrow \neg p_1) \otimes I_{\text{CORF}}(p_2 \rightarrow p_{i_2} + p_{i_2,j_2+1,k_2})(\tilde{t}) = 1\]
\[\text{if and only if } p_i \cdot (p_{i_1,j_2} + p_{i_2,j_2+1,k_2}) \cdot \tilde{p}_{i_1} \cdot \tilde{p}_{i_1+1} \cdot \tilde{p}_{m} + S\]
\[\text{evaluates to } 1 \text{ on } \tilde{t}\]
\[\text{if and only if } \tilde{t} \text{ satisfies any of the following conditions:}\]

1. \(\tilde{t} \in UTP_{i_1}(S) \) such that \(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0\), or
2. \(\tilde{t} \in FP(S)\).

Hence, the result follows. \(\square\)

**Theorem 4.18 (TNF \(\otimes\) CORF - Case 2)**

Let \(S = p_1 + \cdots + p_m\) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \(i_1\)-th term, \(p_{i_1}\), in \(S\) is negated and the \(i_2\)-th term, \(p_{i_2}\), in \(S\) is implemented as \(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2}\) where \(1 \leq (i_1 = i_2) \leq m\), \(p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_2}\) and \(1 \leq j < k_2\), the resulting expression denoted as \(I_{\text{TNF}}(p_i \rightarrow \neg p_1) \otimes I_{\text{CORF}}(p_2 \rightarrow p_{i_2} + p_{i_2,j_2+1,k_2})\) is equivalent to that given by Expression (18) in Table 1. Then, \(S \neq I_{\text{TNF}}(p_i \rightarrow \neg p_1) \otimes I_{\text{CORF}}(p_2 \rightarrow p_{i_2} + p_{i_2,j_2+1,k_2})\) if and only if there is a test case \(\tilde{t}\) that satisfies any of the following conditions:
Hence, the result follows. □

Proof: Since $i_1 = i_2$, we observe that $S \oplus I_{\text{TNF}}(p_1 \rightarrow \bar{p}_i) \times \text{CORS}(p_2 \rightarrow p_{i_2,1,j_2} + p_{i_2,2,j_2})$

$\equiv ((p_i) \oplus (p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2})) \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv ((p_i) \cdot (p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2}) + (\bar{p}_i) \cdot (p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2})) \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv ((p_i) \cdot (p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2}) + (\bar{p}_i) \cdot (p_{i_2,1,j_2} \cdot p_{i_2,2,j_2} + 1, k_{i_2})) \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv (p_i + (p_{i_2,1,j_2} \cdot p_{i_2,2,j_2} + 1, k_{i_2}) \cdot \bar{p}_i \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $(AB)(\bar{A} \cdot \bar{B}) \equiv \bar{A} \cdot \bar{B})$

$\equiv p_i \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_2,1,j_2} \cdot p_{i_2,2,j_2} + 1, k_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By rewriting $AB$ as $AB(A + B)$ because they are equivalent)

$\equiv p_i \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_2,1,j_2} \cdot p_{i_2,2,j_2} + 1, k_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot (p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_i \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_2,1,j_2} \cdot p_{i_2,2,j_2} + 1, k_{i_2} \cdot S$

$\equiv p_i \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2}) \cdot S$

Now, $S(\bar{t}) \neq I_{\text{TNF}}(p_i \rightarrow \bar{p}_i) \times \text{CORS}(p_2 \rightarrow p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2})$

if and only if $S(\bar{t}) \oplus I_{\text{TNF}}(p_i \rightarrow \bar{p}_i) \times \text{CORS}(p_2 \rightarrow p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2})(\bar{t}) = 1$

if and only if $p_i \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2}) \cdot S$ evaluates to 1 on $\bar{t}$

if and only if $\bar{t}$ satisfies any of the following conditions:

1. $\bar{t} \in UTP_{i_2}(S)$, or

2. $\bar{t} \in FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,2,j_2} + 1, k_{i_2} = 0$ on $\bar{t}$.

Hence, the result follows.
4.1.3 TOF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is a TOF. Since ENF $\bowtie$ TOF and TNF $\bowtie$ TOF have been discussed in previous sections, we do not repeat the discussions here.

\textbf{Theorem 4.19} \((\text{TOF} \bowtie \text{TOF})\)

Let \(S = p_1 + \cdots + p_m\) be a Boolean specification in irredundant disjunctive normal form. Suppose that two different terms, \(p_{i_1}\) and \(p_{i_2}\), in \(S\) are omitted where \(1 \leq i_1 < i_2 \leq m\), the resulting expression denoted as \(I_{\text{TOF}(p_{i_1} \rightarrow ) \bowtie \text{TOF}(p_{i_2} \rightarrow )}\) is equivalent to that given by Expression (19) in Table 1. Then, \(S \not\equiv I_{\text{TOF}(p_{i_1} \rightarrow ) \bowtie \text{TOF}(p_{i_2} \rightarrow )}\) if and only if there is a test case \(\vec{t} \in (TP_{i_1}(S) \cup TP_{i_2}(S)) \setminus \bigcup_{i=1}^{m} TP_i(S)\).

\textbf{Proof} : First, we observe that \(S \oplus I_{\text{TOF}(p_{i_1} \rightarrow ) \bowtie \text{TOF}(p_{i_2} \rightarrow )}\)

\[\equiv (p_1 + \cdots + p_{i_1-1} + p_{i_1+1} + p_{i_2-1} + p_{i_2+1} + \cdots + p_m)\]

\[\equiv (p_{i_1} + p_{i_2}) \cdot \bar{p}_1 \cdot \ldots \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdot \ldots \cdot \bar{p}_m\]

(By making use of \((A + B) \oplus B \equiv A \cdot \overline{B}\))

Now, \(S(\vec{t}) \neq I_{\text{TOF}(p_{i_1} \rightarrow ) \bowtie \text{TOF}(p_{i_2} \rightarrow )}(\vec{t})\) if and only if \(S(\vec{t}) \oplus I_{\text{TOF}(p_{i_1} \rightarrow ) \bowtie \text{TOF}(p_{i_2} \rightarrow )}(\vec{t}) = 1\)

if and only if \((p_{i_1} + p_{i_2}) \cdot \bar{p}_1 \cdot \ldots \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdot \ldots \cdot \bar{p}_m\) evaluates to 1 on \(\vec{t}\)

if and only if \(\vec{t} \in (TP_{i_1}(S) \cup TP_{i_2}(S)) \setminus \bigcup_{i=1}^{m} TP_i(S)\).

Hence, the result follows. \(\square\)

\textbf{Theorem 4.20} \((\text{TOF} \bowtie \text{DORF})\)

Let \(S = p_1 + \cdots + p_m\) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \(i_1\)-th term, \(p_{i_1}\), in \(S\) is omitted and the subexpression \((p_{i_2} + p_{i_2+1})\) in \(S\) is implemented as \(p_{i_2}p_{i_2+1}\) where \(1 \leq i_1 < i_2 < m\), the resulting expression denoted as
$I_{TOF(p_1 \rightarrow ) \times DORF(p_2+p_2+1-p_2p_2+1)}$ is equivalent to that given in Expression (20) in Table 1. Then, $S \not= I_{TOF(p_1 \rightarrow ) \times DORF(p_2+p_2+1-p_2p_2+1)}$ if and only if there is a test case $\bar{i} \in \left( \left( TP_{I_1}(S) \cup TP_{I_2}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \setminus \left( \bigcup_{i=1}^{m} \left( \bigcup_{i \neq i_1,i_2+1} TP_i(S) \right) \right)$.

**Proof:** First, we observe that $S \oplus I_{TOF(p_1 \rightarrow ) \times DORF(p_2+p_2+1-p_2p_2+1)}$

$\equiv \left( (p_i + p_{i_2} + p_{i_2+1}) \oplus (p_{i_2}p_{i_2+1}) \right) \cdot \bar{p}_1 \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdot \bar{p}_m$

$\equiv \left( (p_i + p_{i_2} + p_{i_2+1}) \cdot (p_{i_2}p_{i_2+1}) + (p_i + p_{i_2} + p_{i_2+1}) \cdot (p_{i_2}p_{i_2+1}) \right)$

$\cdot \bar{p}_1 \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdot \bar{p}_m$

$\equiv \left( (p_i + p_{i_2} + p_{i_2+1}) \cdot (p_{i_2}p_{i_2+1}) \right) \cdot \bar{p}_1 \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdot \bar{p}_m$

$\quad \quad \quad \text{(By making use of } \left( A+B+C \right) \left( BC \right) \equiv 0)$

$\equiv \left( p_{i_2} (p_i + p_{i_2} + p_{i_2+1}) + p_{i_2+1} (p_i + p_{i_2}) \right) \cdot \bar{p}_1 \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdot \bar{p}_m$

$\equiv (p_i + p_{i_2}) \cdot \bar{p}_1 \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdot \bar{p}_m$

$+ (p_i + p_{i_2+1}) \cdot \bar{p}_1 \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdot \bar{p}_m$

Now, $S(\bar{i}) \not= I_{TOF(p_1 \rightarrow ) \times DORF(p_2+p_2+1-p_2p_2+1)}(\bar{i})$

if and only if $S(\bar{i}) \oplus I_{TOF(p_1 \rightarrow ) \times DORF(p_2+p_2+1-p_2p_2+1)}(\bar{i}) = 1$

if and only if $(p_i + p_{i_2}) \cdot \bar{p}_1 \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdot \bar{p}_m$

$+ (p_i + p_{i_2+1}) \cdot \bar{p}_1 \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdot \bar{p}_m$

evaluates to 1 on $\bar{i}$

if and only if $\bar{i} \in \left( \left( TP_{I_1}(S) \cup TP_{I_2}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \setminus \left( \bigcup_{i=1}^{m} \left( \bigcup_{i \neq i_1,i_2+1} TP_i(S) \right) \right)$

Hence, the result follows. \[\Box\]

**Theorem 4.21** \((TOF \times CORF)\)

Let $S=p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the $i_1$-th term, $p_{i_1}$, in $S$ is omitted and the $i_2$-th term, $p_{i_2}$, in $S$ is implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2}$ where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_2 < k_2$ and $p_{i_2} = p_{i_2,1,j_2}p_{i_2,j_2+1,k_2}$, the resulting expression denoted as
\( I_{TOF}(p_1 \rightarrow) \times CORF(p_2 \rightarrow p_{i_1, j_2} + p_{i_2, j_2 + 1, k_2}) \) is equivalent to that given by Expression (21) in Table 1.

Then, \( S \not= I_{TOF}(p_1 \rightarrow) \times CORF(p_2 \rightarrow p_{i_1, j_2} + p_{i_2, j_2 + 1, k_2}) \) if and only if there is a test case \( \bar{t} \) that satisfies any of the following conditions:

1. \( \bar{t} \in UTP_{i_1}(S) \) such that \( p_{i_2, 1, j_2} + p_{i_2, j_2 + 1, k_2} = 0 \) on \( \bar{t} \), or

2. \( \bar{t} \in FP(S) \) such that \( p_{i_2, 1, j_2} + p_{i_2, j_2 + 1, k_2} = 1 \) on \( \bar{t} \).

**Proof:** First, we observe that \( S \oplus I_{TOF}(p_1 \rightarrow) \times CORF(p_2 \rightarrow p_{i_1, j_2} + p_{i_2, j_2 + 1, k_2}) \)

\[
\begin{align*}
&\equiv ((p_{i_1} + p_{i_2}) \oplus (p_{i_1, j_2} + p_{i_2, j_2 + 1, k_2})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&\equiv ((p_{i_1} + p_{i_2}) \cdot (\bar{p}_{i_1, j_2} + p_{i_2, j_2 + 1, k_2}) + (p_{i_1} + p_{i_2}) \cdot (p_{i_1, j_2} + p_{i_2, j_2 + 1, k_2})) \\
&\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&\equiv (p_{i_1} \cdot \bar{p}_{i_1, j_2} + \bar{p}_{i_1} \cdot p_{i_1, j_2} + p_{i_2, j_2 + 1, k_2} + \bar{p}_{i_1} \cdot p_{i_1, j_2} + p_{i_2, j_2 + 1, k_2}) \\
&\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m
\end{align*}
\]

(By making use of \((A + B \cdot C) \cdot (\bar{B} \cdot \bar{C}) \equiv (A \cdot B \cdot C) + (A + B \cdot \bar{C}) \equiv \bar{A} \cdot B \cdot C + \bar{A} \cdot \bar{B} \cdot C\)

\[
\begin{align*}
&\equiv p_{i_1} \cdot \bar{p}_{i_1, j_2} \cdot \bar{p}_{i_1, j_2 + 1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&+ (p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_2, j_2 + 1, k_2} + p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_2, j_2 + 1, k_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&\equiv p_{i_1} \cdot \bar{p}_{i_1, j_2} \cdot \bar{p}_{i_1, j_2 + 1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot (p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_2, j_2 + 1, k_2}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&+ (p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_2, j_2 + 1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m
\end{align*}
\]

(By rewriting \(AB\) as \(AB(A + B)\) because they are equivalent;
and \((AB + \bar{A}B)\) as \((A + B)\bar{A}B\) because they are equivalent)

\[
\begin{align*}
&\equiv p_{i_1} \cdot \bar{p}_{i_1, j_2} \cdot \bar{p}_{i_1, j_2 + 1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot (p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_2, j_2 + 1, k_2}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&+ (p_{i_2, j_2 + 1, k_2} \cdot p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&\equiv p_{i_1} \cdot \bar{p}_{i_1, j_2} \cdot \bar{p}_{i_1, j_2 + 1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&+ (p_{i_2, j_2 + 1, k_2} \cdot p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&\equiv p_{i_1} \cdot (p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_1, j_2} \cdot \bar{p}_{i_1, j_2 + 1, k_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_2, j_2 + 1, k_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&\equiv p_{i_1} \cdot (p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_1, j_2} \cdot \bar{p}_{i_1, j_2 + 1, k_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2, j_2 + 1, k_2} \cdot \bar{p}_{i_2, j_2 + 1, k_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m
\end{align*}
\]
Now, 
\[ S(\vec{t}) \neq I_{TOF(p_1 \to \neg \text{CORF}(p_2 \to p_{i_2,j_2} + p_{i_2,j_2+1,k_2}))}(\vec{t}) \]
if and only if 
\[ S(\vec{t}) \oplus I_{TOF(p_1 \to \neg \text{CORF}(p_2 \to p_{i_2,j_2} + p_{i_2,j_2+1,k_2}))}(\vec{t}) = 1 \]
if and only if 
\[ p_1(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} \cdot p_{i_1} \cdots p_{i_1-1} \cdot p_{i_1+1} \cdots p_m + (p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2}) \cdot \vec{S}) \]
evaluates to 1 on \( \vec{t} \)
if and only if \( \vec{t} \) satisfies any of the following conditions:

1. \( \vec{t} \in UTP_{i_1}(S) \) such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \) on \( \vec{t} \), or
2. \( \vec{t} \in FP(S) \) such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \) on \( \vec{t} \).

Hence, the result follows. \( \square \)

### 4.1.4 DORF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is a DORF. Since ENF \( \bowtie \) DORF, TNF \( \bowtie \) DORF and TOF \( \bowtie \) DORF have been discussed in previous sections, we do not repeat the discussions here.

**Theorem 4.22** (DORF \( \bowtie \) DORF - Case 1)

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that two different subexpressions \( (p_{i_1} + p_{i_1 + 1}) \) and \( (p_{i_2} + p_{i_2 + 1}) \) in \( S \) are implemented as \( p_{i_1}p_{i_1 + 1} \) and \( p_{i_2}p_{i_2 + 1} \) respectively, where \( 1 \leq i_1 < i_1 + 1 < i_2 < m \), the resulting expression denoted as \( I_{DORF(p_1 + p_{i_1 + 1} \to p_{i_1}p_{i_1 + 1}) \bowtie \text{DORF}(p_{i_2} + p_{i_2 + 1} \to p_{i_2}p_{i_2 + 1})} \) is equivalent to that given by Expression (22) in Table 1. Then, \( S \neq I_{DORF(p_1 + p_{i_1 + 1} \to p_{i_1}p_{i_1 + 1}) \bowtie \text{DORF}(p_{i_2} + p_{i_2 + 1} \to p_{i_2}p_{i_2 + 1})} \) if and only if there is a test case \( \vec{t} \in \bigcup_{i = i_1,i_1 + 1,j_2,j_2 + 1} \left( \left( \bigcup_{i = 1}^{m} TP_i(S) \right) \cup \left( TP_{i_1}(S) \cap TP_{i_1 + 1}(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2 + 1}(S) \right) \right) \).
Proof: First, we observe that $S \oplus I_{DORF}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1}p_{i_1+1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})$
\[
\equiv ((p_{i_1} + p_{i_1+1} + p_{i_2} + p_{i_2+1}) \oplus (p_{i_1}p_{i_1+1} + p_{i_2}p_{i_2+1})) \cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m \\
\equiv ((p_{i_1} + p_{i_1+1} + p_{i_2} + p_{i_2+1}) \cdot (p_{i_1}p_{i_1+1} + p_{i_2}p_{i_2+1}) + (p_{i_1} + p_{i_1+1} + p_{i_2} + p_{i_2+1})) \\
\cdot (p_{i_1}p_{i_1+1} + p_{i_2}p_{i_2+1})) \cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m \\
\equiv (p_{i_1} + p_{i_1+1} + p_{i_2} + p_{i_2+1})(p_{i_1}p_{i_1+1} + p_{i_2}p_{i_2+1}) \cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_m \\
\text{(By making use of } (A + B + C + D)(AB + CD) \equiv 0) \\
\text{Now, } S(\vec{i}) \neq I_{DORF}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1}p_{i_1+1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})(\vec{i}) \\
\text{if and only if } S(\vec{i}) \oplus I_{DORF}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1}p_{i_1+1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})(\vec{i}) = 1 \\
\text{if and only if } (p_{i_1} + p_{i_1+1} + p_{i_2} + p_{i_2+1})(p_{i_1}p_{i_1+1} + p_{i_2}p_{i_2+1}) \cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+2} \cdots \tilde{p}_{i_2-1} \cdot \\
\tilde{p}_{i_2+2} \cdots \tilde{p}_m \text{ evaluates to } 1 \text{ on } \vec{i} \\
\text{if and only if } \vec{i} \in \bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S) \setminus \left( \bigcup_{i=i_1}^m TP_i(S) \cup (TP_{i_1}(S) \cap TP_{i+1}(S) \cap TP_{i_2+1}(S)) \right). \\
\text{Hence, the result follows.} \hspace{1cm} \square

Theorem 4.23 (DORF \times DORF - Case 2)
Let $S=p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose
that the two subexpressions $(p_{i_1} + p_{i_1+1})$ and $(p_{i_2} + p_{i_2+1})$ in $S$ are implemented as $p_{i_1}p_{i_1+1}$ and
$p_{i_2}p_{i_2+1}$, respectively, where $1 \leq i_1 \leq m - 2$ and $i_2 = i_1 + 1$, the resulting expression denoted as
$I_{DORF}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1}p_{i_1+1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})$ is equivalent to that given by Expression (23) in
Table 1. Then, $S \neq I_{DORF}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1}p_{i_1+1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})$ if and only if there is a test

\text{case } \vec{i} \in \left( \bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S) \setminus \left( \bigcup_{i=i_1}^m TP_i(S) \cup (TP_{i_1}(S) \cap TP_{i+1}(S) \cap TP_{i_2+1}(S)) \right) \right).

Proof: Since $i_2 = i_1 + 1$, we observe that $S \oplus I_{DORF}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1}p_{i_1+1}) \times DORF(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2}p_{i_2+1})$
\[
\equiv ((p_{i_1} + p_{i_1+1} + p_{i_1+2}) \oplus (p_{i_1}p_{i_1+1}p_{i_1+2})) \cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+3} \cdots \tilde{p}_m \\
\equiv ((p_{i_1} + p_{i_1+1} + p_{i_1+2}) \cdot (p_{i_1}p_{i_1+1}p_{i_1+2}) + (p_{i_1} + p_{i_1+1} + p_{i_1+2}) \cdot (p_{i_1}p_{i_1+1}p_{i_1+2})) \\
\cdot \tilde{p}_1 \cdots \tilde{p}_{i_1-1} \cdot \tilde{p}_{i_1+3} \cdots \tilde{p}_m
\begin{align*}
\equiv (p_{i_1} + p_{i_1+1} + p_{i_1+2}) (p_{i_1} p_{i_1+1} p_{i_1+2}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m \\
\text{(By making use of } (A + B + C)(ABC) \equiv 0 \text{)}
\end{align*}

Now, $S(\bar{t}) \neq I_{\text{DORF}}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \times \text{DORF}(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2} p_{i_2+1})(\bar{t})$

if and only if $S(\bar{t}) \oplus I_{\text{DORF}}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \times \text{DORF}(p_{i_2} + p_{i_2+1} \rightarrow p_{i_2} p_{i_2+1})(\bar{t}) = 1$

if and only if $(p_{i_1} + p_{i_1+1} + p_{i_1+2}) (p_{i_1} p_{i_1+1} p_{i_1+2}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m$ evaluates to 1 on $\bar{t}$

if and only if $\bar{t} \in \left( \bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S) \right) \setminus \left( \bigcup_{i=i_1,i_1+1,i_1+2} ^m TP_i(S) \right) \cup TP_i(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S) \right)$. 

Hence, the result follows. \hfill \Box

\textbf{Theorem 4.24 (DORF \times CORF - Case 1)}

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + p_{i_1+1})$ is implemented as $p_{i_1} p_{i_1+1}$ and the $i_2$-th term, $p_{i_2}$, is implemented as $p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}$, where $1 \leq i_1 < i_1 + 1 < i_2 \leq m$, $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,2,j_2+1,k_2}$ and $1 \leq j_2 < k_2$, the resulting expression denoted as $I_{\text{DORF}}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \times \text{CORF}(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2})$ is equivalent to Expression (24) in Table 1. Then, $S \neq I_{\text{DORF}}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \times \text{CORF}(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2})$ if and only if there is a test case $\bar{t}$ that satisfies any of the following conditions:

1. $\bar{t} \in UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2} = 0$ on $\bar{t}$, or

2. $\bar{t} \in FP(S)$, such that such that $p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2} = 1$ on $\bar{t}$.

\textbf{Proof :} First, we observe that $S \oplus I_{\text{DORF}}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \times \text{CORF}(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2})$

\begin{align*}
\equiv \left( (p_{i_1} + p_{i_1+1} + p_{i_2}) \oplus (p_{i_1} p_{i_1+1} + p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \right) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m \\
\equiv \left( (p_{i_1} + p_{i_1+1} + p_{i_2}) (p_{i_1} p_{i_1+1} + p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) + (p_{i_1} + p_{i_1+1} + p_{i_2}) \cdot (p_{i_1} p_{i_1+1} + p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \right) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m \\
\equiv \left( (p_{i_1} + p_{i_1+1} + p_{i_2}) (p_{i_1} + p_{i_1+1}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m \\
+ (p_{i_1} + p_{i_1+1} + p_{i_2}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m \right)
\end{align*}
\[
\begin{align*}
&\cdot(p_{i_1} p_{i_1+1} + p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&\equiv \left((p_{i_1} + p_{i_2}) \bar{p}_{i_1+1} \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,2,j_2+1,k_2} + (p_{i_1+1} + p_{i_2}) \bar{p}_{i_1} \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,2,j_2+1,k_2} + (p_{i_1} \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2})\right) \\
&\cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdot \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m \\
&\equiv (p_{i_1} \bar{p}_{i_1+1} + p_{i_1+1} \bar{p}_{i_1}) \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,2,j_2+1,k_2} \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdot \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2,1,j_2} \bar{p}_{i_2,2,j_2+1,k_2}) \\
&\cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2} \cdots \bar{p}_m \\
&(\text{By making use of } (C + A \cdot B) \bar{A} \cdot B \equiv C \cdot \bar{A} \cdot B \text{ and } \bar{A} \cdot B (A \cdot B + C) \equiv \bar{A} \cdot B \cdot C) \\
&\equiv (p_{i_1} \bar{p}_{i_1+1} + p_{i_1+1} \bar{p}_{i_1}) \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,2,j_2+1,k_2} \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdot \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2,1,j_2} \bar{p}_{i_2,2,j_2+1,k_2}) \\
&\cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot S \\
&(\text{By rewriting } AB \text{ as } A B (A + B) \text{ because they are equivalent}) \\
&\equiv (p_{i_1} \bar{p}_{i_1+1} + p_{i_1+1} \bar{p}_{i_1}) \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,2,j_2+1,k_2} \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdot \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2,1,j_2} \bar{p}_{i_2,2,j_2+1,k_2}) \\
&\cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot S \\
&\equiv (p_{i_1} \bar{p}_{i_1+1} + p_{i_1+1} \bar{p}_{i_1}) (\bar{p}_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdot \bar{p}_{i_2} \cdots \bar{p}_m \\
&+ (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot S \\
&\equiv p_{i_1} (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + p_{i_1+1} (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot \bar{p}_{i_1} \cdot \bar{p}_{i_1} \\
&\cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot S \\
&\text{Now, } S(\bar{t}) \neq I_{DORF}(p_{i_1} \cdot p_{i_1+1} \to p_{i_1} p_{i_1+1}) \times CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2})(\bar{t}) \\
&\text{if and only if } S(\bar{t}) \oplus I_{DORF}(p_{i_1} \cdot p_{i_1+1} \to p_{i_1} p_{i_1+1}) \times CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2})(\bar{t}) = 1 \\
&\text{if and only if } p_{i_1} (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1+1} (p_{i_2,1,j_2} \\
&+ p_{i_2,2,j_2+1,k_2}) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) \cdot S \text{ evaluates to} \\
&1 \text{ on } \bar{t} \\
&\text{if and only if } \bar{t} \text{ satisfies any of the following conditions: } \\
&\quad 1. \bar{t} \in UTP_{i_1}(S) \cup UTP_{i_1+1}(S) \text{ such that } p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2} = 0 \text{ on } \bar{t}, \text{ or} \\
&\quad 2. \bar{t} \in FP(S), \text{ such that such that } p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2} = 1 \text{ on } \bar{t}. \\
&\text{Hence, the result follows.} \ 
\]
Theorem 4.25 (DORF \( \times \) CORF - Case 2)

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression \( (p_i + p_{i+1}) \) in \( S \) is implemented as \( p_{i_1}p_{i+1} \) and the \( i_1 \)-th term, \( p_{i_1} \), in \( S \) is implemented as \( p_{i_1,j_1} + p_{i_1,j_1 + 1,k_i} \), where \( 1 \leq i_1 < m \), \( p_{i_1} = p_{i_1,j_1} \cdot p_{i_1,j_1 + 1,k_i} \) and \( 1 \leq j_1 < k_i \), the resulting expression denoted as \( I_{DORF}(p_{i_1}p_{i+1} \to p_{i_1}p_{i+1}) \times CORF(p_{i_2}p_{i_2 + 1} + p_{i_2 + 1}p_{i_2 + 1}) \) is equivalent to Expression (25) in Table 1. Then, \( S \neq I_{DORF}(p_{i_1}p_{i+1} \to p_{i_1}p_{i+1}) \times CORF(p_{i_2}p_{i_2 + 1} + p_{i_2 + 1}p_{i_2 + 1}) \) if and only if there is a test case \( \bar{t} \) that satisfies any of the following conditions:

1. \( \bar{t} \in UTP_{i_1+1}(S) \), such that \( p_{i_1,j_1} + p_{i_1,j_1 + 1,k_i} = 0 \) on \( \bar{t} \), or
2. \( \bar{t} \in FP(S) \), such that \( p_{i_1,j_1} = 1 \) on \( \bar{t} \).

Proof: Since \( i_2 = i_1 \), we observe that \( S \oplus I_{DORF}(p_{i_1}p_{i+1} \to p_{i_1}p_{i+1}) \times CORF(p_{i_2}p_{i_2 + 1} + p_{i_2 + 1}p_{i_2 + 1}) \)

\[
\equiv ((p_i + p_{i+1}) \oplus (p_i,j_1 + p_{i_1,j_1 + 1,k_i}p_{i+1})) \cdot \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m
\]

\[
\equiv ((p_i + p_{i+1})(p_i,j_1 + p_{i_1,j_1 + 1,k_i}p_{i+1}) + (p_i + p_{i+1})(p_{i_1,j_1} + p_{i_1,j_1 + 1,k_i}p_{i+1}))
\]

\[
\cdot \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m
\]

\[
\equiv (\bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m)
\]

(By making use of \( (A \cdot B + C)(A + B \cdot C) \equiv \bar{A} \cdot B \cdot C \) and \( (A \cdot B + C)(A + B \cdot C) \equiv A \cdot B \cdot \bar{C} \))

\[
\equiv \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m + p_{i1,1,j_1}j_1k_i \cdot \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m
\]

\[
\equiv \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m
\]

(By rewriting \( AB \) as \( AB(A + B) \) because they are equivalent; and \( AB \) as \( A \cdot \bar{B} \) because they are equivalent)

\[
\equiv \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m + p_{i1,1,j_1} \cdot \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m + p_{i1,1,j_1} \cdot \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m
\]

\[
\equiv p_{i1,1,j_1} \cdot \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m + p_{i1,1,j_1} \cdot \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m + p_{i1,1,j_1} \cdot \bar{p}_{i1} \cdot \bar{p}_{i2} \cdot \cdots \cdot \bar{p}_m
\]
Now, \( S(\overline{r}) \neq I_D\text{DORF}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1}p_{i_1+1}) \times CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) (\overline{r}) \)

if and only if

\[ S(\overline{r}) \oplus I_D\text{DORF}(p_{i_1} + p_{i_1+1} \rightarrow p_{i_1}p_{i_1+1}) \times CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2}) (\overline{r}) = 1 \]

if and only if

\[ p_{i_1+1}(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1}) \cdot \overline{p}_1 \cdot \overline{p}_i \cdot \overline{p}_{i+2} \cdot \overline{p}_m + p_{i_1,1,j_1} \cdot \overline{S} \text{ evaluates to } 1 \]
on \( \overline{r} \)

if and only if \( \overline{r} \) satisfies any of the following conditions:

1. \( \overline{r} \in UTP_{i_1+1}(S) \) such that \( p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 0 \) on \( \overline{r} \), or

2. \( \overline{r} \in FP(S) \), such that \( p_{i_1,1,j_1} = 1 \) on \( \overline{r} \).

Hence, the result follows. \( \square \)

### 4.1.5 CORF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is a CORF. Since ENF \( \ncong \) CORF, TNF \( \ncong \) CORF, TOF \( \ncong \) CORF and DORF \( \ncong \) CORF have been discussed in previous sections, we do not repeat the discussions here.

**Theorem 4.26** (\( CORF \ncong \) CORF - Case 1)

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that two different terms \( p_{i_1} \) and \( p_{i_2} \) in \( S \) are implemented as \( p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} \) and \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} \) respectively, where \( 1 \leq i_1 < i_2 \leq m, p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_1}, p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_2}, 1 \leq j_1 < k_1 \) and \( 1 \leq j_2 < k_2 \), the resulting expression denoted as \( I_{CORF}(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1}) \times CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2}) \) is equivalent to that given by Expression (26) in Table 1. Then, \( S \neq I_{CORF}(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1}) \times CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2}) \) if and only if there is a test case \( \overline{r} \in FP(S) \) such that \( p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \) on \( \overline{r} \).
Proof: First, we observe that  

\[ S \oplus I_{\text{CORF}}(p_1 \rightarrow p_{1,j_1} + p_{1,j_1+1,k_1}) \otimes \text{CORF}(p_2 \rightarrow p_{2,j_2} + p_{2,j_2+1,k_2}) \]

\[ \equiv \left( (p_{i_1} + p_{i_2}) \oplus (p_{i_1,j_1} + p_{i_1,j_1+1,k_1} + p_{i_2,j_2} + p_{i_2,j_2+1,k_2}) \right) \]

\[ \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1} - 1 \cdot \bar{p}_{i_1} + 1 \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2} + 1 \cdots \bar{p}_m \]

\[ \equiv \left( (p_{i_1} + p_{i_2}) \cdot \bar{p}_{i_1,j_1} \cdot p_{i_1,j_1+1,k_1} \cdot p_{i_2,j_2} \cdot p_{i_2,j_2+1,k_2} + \left( p_{i_1} + p_{i_2} \right) \right) \]

\[ \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1} - 1 \cdot \bar{p}_{i_1} + 1 \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2} + 1 \cdots \bar{p}_m \]

\[ \equiv \left( (p_{i_1} + p_{i_2}) \cdot \bar{p}_{i_1,j_1} \cdot p_{i_1,j_1+1,k_1} \cdot p_{i_2,j_2} \cdot p_{i_2,j_2+1,k_2} + \left( p_{i_1} \cdot p_{i_2} \right) \right) \]

\[ \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1} - 1 \cdot \bar{p}_{i_1} + 1 \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2} + 1 \cdots \bar{p}_m \]

\[ \equiv \left( (p_{i_1} + p_{i_2}) \cdot \bar{p}_{i_1,j_1} \cdot p_{i_1,j_1+1,k_1} \cdot p_{i_2,j_2} \cdot p_{i_2,j_2+1,k_2} \right) \bar{p}_{i_1} \cdot \bar{p}_{i_1} - 1 \cdot \bar{p}_{i_1} + 1 \cdots \bar{p}_{i_2} - 1 \cdot \bar{p}_{i_2} + 1 \cdots \bar{p}_m \]

(Prove using the property of \((AB + CD)(\bar{A} \cdot \bar{B} \cdot \bar{C} \cdot \bar{D}) \equiv 0\))

\[ \equiv \left( p_{i_1,j_1} + p_{i_1,j_1+1,k_1} + p_{i_2,j_2} + p_{i_2,j_2+1,k_2} \right) \cdot \bar{S} \]

Now, 

\[ S(\bar{t}) \neq I_{\text{CORF}}(p_1 \rightarrow p_{1,j_1} + p_{1,j_1+1,k_1}) \otimes \text{CORF}(p_2 \rightarrow p_{2,j_2} + p_{2,j_2+1,k_2})(\bar{t}) \]

if and only if 

\[ S(\bar{t}) \oplus I_{\text{CORF}}(p_1 \rightarrow p_{1,j_1} + p_{1,j_1+1,k_1}) \otimes \text{CORF}(p_2 \rightarrow p_{2,j_2} + p_{2,j_2+1,k_2})(\bar{t}) = 1 \]

if and only if 

\[ (p_{i_1,j_1} + p_{i_1,j_1+1,k_1} + p_{i_2,j_2} + p_{i_2,j_2+1,k_2}) \cdot \bar{S} \text{ evaluates to } 1 \text{ on } \bar{t} \]

if and only if 

\[ \bar{t} \in FP(S) \text{ such that } p_{i_1,j_1} + p_{i_1,j_1+1,k_1} + p_{i_2,j_2} + p_{i_2,j_2+1,k_2} = 1 \text{ on } \bar{t} \]

Hence, the result follows. \(\square\)

Theorem 4.27 (\text{CORF} \otimes \text{CORF} - Case 2)

Let \(S = p_1 + \cdots + p_m\) be a Boolean specification in irredundant disjunctive normal form. Suppose that two different CORFs are committed at a particular term in \(S\). That is, the \(i_1\)-th term, \(p_{i_1}\), in \(S\) is implemented as \(p_{i_1,j_1} + p_{i_1,j_1+1,k_1}\) where \(p_{i_1} = p_{i_1,j_1} \cdot p_{i_1,j_1+1,k_1} \cdot p_{i_1,j_1+2,k_1} \) and \(1 < j_1 < j_2 < k_1\), the resulting expression denoted as 

\[ I_{\text{CORF}}(p_1 \rightarrow p_{1,j_1} + p_{1,j_1+1,k_1}) \otimes \text{CORF}(p_2 \rightarrow p_{2,j_2} + p_{2,j_2+1,k_2}) \]

is equivalent to Expression (27) in Table 1. Then, \(S \neq I_{\text{CORF}}(p_1 \rightarrow p_{1,j_1} + p_{1,j_1+1,k_1}) \otimes \text{CORF}(p_2 \rightarrow p_{2,j_2} + p_{2,j_2+1,k_2})\) if and only if there is a test case \(\bar{t} \in FP(S)\) such that \(p_{i_1,j_1} + p_{i_1,j_1+1,k_1} + p_{i_2,j_2} + p_{i_2,j_2+1,k_2} = 1 \text{ on } \bar{t}\).

Proof: First, we observe that 

\[ S \oplus I_{\text{CORF}}(p_1 \rightarrow p_{1,j_1} + p_{1,j_1+1,k_1}) \otimes \text{CORF}(p_2 \rightarrow p_{2,j_2} + p_{2,j_2+1,k_2}) \]

\[ \equiv \left( (p_{i_1}) \oplus (p_{i_1,j_1} + p_{i_1,j_1+1,k_1} + p_{i_2,j_2} + p_{i_2,j_2+1,k_2}) \right) \cdot \bar{p}_{i_1} \cdots \bar{p}_{i_1} - 1 \cdot \bar{p}_{i_1} + 1 \cdots \bar{p}_m \]
\[
\equiv (p_i (p_{i1, j_1} + p_{i1, j_1+1, j_2} + p_{i1, j_2+1, k_1}) + \overline{p_i} (p_{i1, j_1} + p_{i1, j_1+1, j_2} + p_{i1, j_2+1, k_1})) \cdot \bar{p}_1 \cdots \bar{p}_{i-1} \cdot \bar{p}_{i+1} \cdots \bar{p}_m \\
\equiv (p_{i1, j_1} + p_{i1, j_1+1, j_2} + p_{i1, j_2+1, k_1}) \bar{p}_i \cdot \bar{p}_1 \cdots \bar{p}_{i-1} \cdot \bar{p}_{i+1} \cdots \bar{p}_m \\
\text{(By making use of } (ABC) (A + B + C) \equiv 0) \\
\equiv (p_{i1, j_1} + p_{i1, j_1+1, j_2} + p_{i1, j_2+1, k_1}) \cdot \bar{S}
\]

Now, \( S(t) \neq I_{\text{CORF}}(p_i \rightarrow p_{i1, j_1} + p_{i1, j_1+1, j_2}) \times I_{\text{CORF}}(p_i \rightarrow p_{i1, j_1} + p_{i1, j_1+1, j_2}) (t) \)
if and only if \( S(t) \oplus I_{\text{CORF}}(p_i \rightarrow p_{i1, j_1} + p_{i1, j_1+1, j_2}) \times I_{\text{CORF}}(p_i \rightarrow p_{i1, j_1} + p_{i1, j_1+1, j_2}) (t) = 1 \)
if and only if \( (p_{i1, j_1} + p_{i1, j_1+1, j_2} + p_{i1, j_2+1, k_1}) \cdot \bar{S} \) evaluates to 1 on \( t \)
if and only if \( t \in FP(S) \) such that \( p_{i1, j_1} + p_{i1, j_1+1, j_2} + p_{i1, j_2+1, k_1} = 1 \) on \( t \).

Hence, the result follows. \( \square \)

### 4.2 Detection Conditions of 4 Remaining Faulty Implementations

In this section, we study the detection conditions of 4 double-fault expressions in double faults with ordering which do not have their equivalent counterparts in double faults without ordering.

**Theorem 4.28** \((\text{TOF} \propto \text{DORF})\)

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \( i_1 \)-th term, \( p_{i1} \), is omitted and then the subexpression \( (p_{i1-1} + p_{i1+1}) \) is implemented as \( p_{i1-1} \oplus p_{i1+1} \) where \( 1 < i_1 < m \), the resulting expression denoted as \( I_{\text{TOF}}(p_i \rightarrow) \propto I_{\text{DORF}}(p_{i1-1} + p_{i1+1} \rightarrow p_{i1-1} p_{i1+1}) \) is equivalent to that given by Expression (53) in Table 1.

Then, we have \( S \neq I_{\text{TOF}}(p_i \rightarrow) \propto I_{\text{DORF}}(p_{i1-1} + p_{i1+1} \rightarrow p_{i1-1} p_{i1+1}) \) if and only if there is a test case \( t \in \left( \bigcup_{i=1}^{m} TP_{i1} (S) \right) \setminus \left( \bigcup_{i1-1 \neq i \neq i1+1} TP_{i1} (S) \right) \).

**Proof** : First, we observe that
\[
S \oplus I_{\text{TOF}}(p_i \rightarrow) \propto I_{\text{DORF}}(p_{i1-1} + p_{i1+1} \rightarrow p_{i1-1} p_{i1+1})
\]
= ((p_{i-1} + p_i + p_{i+1}) \oplus (p_{i-1} \cdot p_{i+1})) \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-2} \cdot \bar{p}_{i+2} \cdot \bar{p}_m

= ((p_{i-1} + p_i + p_{i+1}) (p_{i-1} \cdot p_{i+1}) + (p_{i-1} + p_i + p_{i+1}) (p_{i-1} \cdot p_{i+1}))

\cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-2} \cdot \bar{p}_{i+2} \cdot \bar{p}_m

= (p_{i-1} + p_{i+1}) \cdot (p_{i-1} \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-2} \cdot \bar{p}_{i+2} \cdot \bar{p}_m + (p_{i-1} + p_i) \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-2} \cdot \bar{p}_{i+2} \cdot \bar{p}_m

= (p_{i-1} + p_{i+1}) \cdot (p_{i-1} \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-2} \cdot \bar{p}_{i+1} + (p_{i+1} + p_{i+1}) \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-1} \cdot \bar{p}_{i+2} \cdot \bar{p}_m

(By making use of \((A + B + C) (\overline{AC}) \equiv \overline{A} (B + C) + \overline{A} (B + C) \equiv 0\)

= (p_{i-1} + p_{i+1}) \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-1} \cdot \bar{p}_{i+1} + (p_{i+1} + p_{i+1}) \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-1} \cdot \bar{p}_{i+2} \cdot \bar{p}_m

Now, \( S(\bar{t}) \neq I_{TOF(p_{i-1} \rightarrow DORF(p_{i-1} \rightarrow p_{i-1} + p_{i+1}))}\)

if and only if \( S(\bar{t}) \oplus I_{TOF(p_{i-1} \rightarrow p_{i+1} + p_{i+1} \rightarrow p_{i-1} + p_{i+1})}\) = 1

if and only if \( (p_{i-1} + p_i) \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-1} \cdot \bar{p}_{i+1} + (p_{i+1} + p_{i+1}) \cdot \bar{p}_1 \cdot \bar{p}_2 \cdot \bar{p}_{i-1} \cdot \bar{p}_{i+2} \cdot \bar{p}_m \)

evaluates to 1 on \( \bar{t} \)

if and only if \( \bar{t} \in \left( \bigcup_{i=1}^{m} TP_{i-1}(S) \right) \setminus \left( \bigcup_{i \neq i-1, i+1} \bigcup_{i=1}^{m} TP_i(S) \right) \)

Hence, the result follows.

\[ \square \]

**Theorem 4.29 (CORF with ENF)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose the \( i_1 \)-th term, \( p_{i_1} \), in \( S \) is implemented as \( p_{i_1,1,j_1} + p_{i_1,1,j_1+1} \) where \( 1 \leq i_1 < h_1 < m, p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,1,j_1+1} \) and \( 1 \leq j_1 < k_{i_1} \), and the subexpression \( p_{i_1,1,j_1+1} \) is then negated, the resulting expression denoted as \( I_{CORF(p_{i-1} \rightarrow p_{i-1} + p_{i-1} + + p_{i+1})} \) is equivalent to that given by Expression (70) in Table 1. Then, we have \( S \neq I_{CORF(p_{i-1} \rightarrow p_{i-1} + p_{i-1} + + p_{i+1})} \) if and only if there is a test case \( \bar{t} \) that satisfies any of the following conditions

1. \( \bar{t} \in \left( \bigcup_{i=i_1+1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \) such that \( p_{i_1,1,j_1} = 0 \) on \( \bar{t} \), or
2. \( \tilde{t} \in FP(S) \) such that \( p_{l_1,j_1+1,k_1} = 0 \) on \( \tilde{t} \).

**Proof**: First, we observe that

\[
S \oplus I_{\text{CORF}}(p_{i_1} \oplus p_{i_1,1,j_1} \oplus p_{i_1,1,j_1+1,k_1}) \times \text{ENF}(p_{i_1,j_1+1,k_1} \oplus \cdots \oplus p_{h_1} \oplus p_{i_1,1,j_1+1,k_1} \oplus \cdots \oplus p_{h_1})
\]

\[
\equiv ((p_{i_1} + \cdots + p_{h_1}) \oplus (p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1} + \cdots + p_{h_1})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
\equiv ((p_{i_1} + \cdots + p_{h_1})(p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1} + \cdots + p_{h_1}) + (p_{i_1} + \cdots + p_{h_1})(p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1} + \cdots + p_{h_1})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
\equiv ((p_{i_1} + \cdots + p_{h_1})(p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1} + \cdots + p_{h_1}) + p_{i_1} \cdot p_{i_1+1} + \cdots + p_{h_1} (p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1} + \cdots + p_{h_1})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
(\text{By making use of } (AB+C)\bar{A} \equiv \bar{A}C)
\]

\[
\equiv (p_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) + p_{i_1,1,j_1+1,k_1} \cdot p_{i_1+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
(\text{By making use of } (\bar{A} + B)(A + BC) \equiv \bar{B}C)
\]

\[
\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
+ (p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1}) \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
+ \bar{p}_{i_1,1,j_1+1,k_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1}) \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
(\text{By rewriting } A \text{ as } A(A + B) \text{ because they are equivalent; and } B \text{ as } B(A + B) \text{ because they are equivalent})
\]

\[
\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1}) \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
+ \bar{p}_{i_1,1,j_1+1,k_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_1}) \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m
\]

\[
\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_1,1,j_1+1,k_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m
\]

\[
\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_1,1,j_1+1,k_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m
\]
Now, \[ S(\overline{t}) \neq I_{\text{CORF}}(p_i \rightarrow p_{i+1} \wedge p_{i+1} \cdot p_{i+1} + p_{i+1} \cdot p_{i+1} + p_{i+1}) \] if and only if \[ S(\overline{t}) \oplus I_{\text{CORF}}(p_i \rightarrow p_{i+1} \wedge p_{i+1} \cdot p_{i+1} + p_{i+1} \cdot p_{i+1} + p_{i+1}) \] 
\[ \times ENF(p_{i+1} + p_{i+1} \cdot p_{i+1} + p_{i+1}) = 1 \] if and only if \[ \bar{p}_{i,j_1}(p_{i+1} + \cdots + p_{k_1}) \bar{p}_1 \cdots \bar{p}_{i+1} \cdots \bar{p}_m + \bar{p}_{i,j_1} \bar{p}_{i,j_1} S \) evaluates to 1 on \( \overline{t} \) if and only if \( \overline{t} \) satisfies any of the following conditions

1. \( \overline{t} \in \bigcup_{i=1}^{h_1} TP_i(S) \setminus \bigcup_{i \neq 1, \ldots, h_1} TP_i(S) \) such that \( p_{i,j_1} = 0 \) on \( \overline{t} \), or

2. \( \overline{t} \in FP(S) \) such that \( p_{i,j_1} = 0 \) on \( \overline{t} \).

Hence, the result follows. \( \square \)

**Theorem 4.30 (CORF with TNF)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \( i \)-th term, \( p_{i_1} \), in \( S \) is implemented as \( p_{i_1} + p_{i_1} \cdot p_{i_1} \) where \( 1 \leq i_1 \leq m \), \( p_{i_1} = p_{i_1} \cdot p_{i_1} \cdot p_{i_1} \) and \( 1 \leq j_1 < k_1 \), and the term \( p_{i_1} \cdot p_{i_1} \cdot p_{i_1} \) is then negated, the expression denoted as \( I_{\text{CORF}}(p_{i_1} \rightarrow p_{i_1} + p_{i_1} \cdot p_{i_1} + p_{i_1}) \times TNF(p_{i_1} \cdot p_{i_1} \cdot p_{i_1} \rightarrow p_{i_1} \cdot p_{i_1} \cdot p_{i_1}) \) is equivalent to that given by Expression (73) in Table 1. Then, \( S \neq I_{\text{CORF}}(p_{i_1} \rightarrow p_{i_1} + p_{i_1} \cdot p_{i_1} + p_{i_1}) \times TNF(p_{i_1} \cdot p_{i_1} \cdot p_{i_1} \rightarrow p_{i_1} \cdot p_{i_1} \cdot p_{i_1}) \) if and only if there is a test case \( \overline{t} \in FP(S) \) such that \( p_{i_1} \cdot p_{i_1} + p_{i_1} \cdot p_{i_1} \cdot p_{i_1} = 0 \) on \( \overline{t} \).

**Proof:** First, we observe that

\[
S \oplus I_{\text{CORF}}(p_{i_1} \rightarrow p_{i_1} + p_{i_1} \cdot p_{i_1} + p_{i_1}) \times TNF(p_{i_1} + p_{i_1} + p_{i_1} \rightarrow p_{i_1} + p_{i_1} + p_{i_1})
\]
\[
\equiv (p_{i_1} \oplus (p_{i_1} + p_{i_1} + p_{i_1})) \bar{p}_1 \cdots \bar{p}_1 - \bar{p}_1 + \cdots \bar{p}_m
\]
\[
\equiv ((p_{i_1}) (p_{i_1} + p_{i_1} + p_{i_1})) \bar{p}_1 \cdots \bar{p}_1 - \bar{p}_1 + \cdots \bar{p}_m
\]
\[
\equiv (((p_{i_1}) \bar{p}_1 + p_{i_1} + p_{i_1} \cdot p_{i_1} + p_{i_1})) \bar{p}_1 \cdots \bar{p}_1 - \bar{p}_1 + \cdots \bar{p}_m
\]
\[
\equiv (p_{i_1} + p_{i_1} + p_{i_1} \cdot p_{i_1} + p_{i_1}) \bar{p}_1 \cdots \bar{p}_1 - \bar{p}_1 + \cdots \bar{p}_m
\]
\[
\equiv (p_{i_1} + p_{i_1} + p_{i_1} \cdot p_{i_1} + p_{i_1} \cdot p_{i_1} + p_{i_1} \cdot p_{i_1} \cdot p_{i_1}) \bar{p}_1 \cdots \bar{p}_1 - \bar{p}_1 + \cdots \bar{p}_m
\]

(By making use of \( AB = 0 \) and \( (AB)(A + B) = B \))
\[ \equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})\bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\bar{p}_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}}) \cdot \bar{p}_{i_1} \cdots \bar{p}_m \]

(By rewriting \( A \) as \( A(A + B) \) because they are equivalent)
\[ \equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})\bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\bar{p}_{i_1,1,j_1} \cdot \bar{p}_{i_1,j_1+1,k_{i_1}}) \cdot \bar{p}_{i_1} \cdots \bar{p}_m \]
\[ \equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})\bar{p}_1 \cdots \bar{p}_i \cdots \bar{p}_m \]
\[ \equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})S \]

Now, \( S(\bar{t}) \neq I_{CORF}(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \times TOF(p_{i_1,j_1+1,k_{i_1}} \rightarrow \bar{p}_{i_1,j_1+1,k_{i_1}})(\bar{t}) \)
if and only if \( S(\bar{t}) \oplus I_{CORF}(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \times TOF(p_{i_1,j_1+1,k_{i_1}} \rightarrow \bar{p}_{i_1,j_1+1,k_{i_1}})(\bar{t}) = 1 \)
if and only if \( (\bar{p}_{i_1,j_1+1,k_{i_1}})S \) evaluates to 1 on \( \bar{t} \)
if and only if \( \bar{t} \in FP(S) \) such that \( p_{i_1,j_1+1,k_{i_1}} = 0 \) on \( \bar{t} \).

Hence, the result follows.

\[ \square \]

**Theorem 4.31 (CORF with TOF)**

Let \( S = p_1 + \cdots + p_m \) be a Boolean specification in irredundant disjunctive normal form. Suppose that the \( i_1 \)-th term, \( p_{i_1} \), in \( S \) is implemented as \( p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} \) where \( 1 \leq i_1 \leq m, \ p_{i_1} = p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} \) and \( 1 \leq j_1 < k_{i_1} \), and then the term \( p_{i_1,j_1+1,k_{i_1}} \) in \( S \) is omitted from the expression, the resulting expression denoted as \( I_{CORF}(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \times TOF(p_{i_1,j_1+1,k_{i_1}} \rightarrow \bar{p}_{i_1,j_1+1,k_{i_1}}) \) is equivalent to that given by Expression (76) in Table 1. Then, \( S \neq I_{CORF}(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \times TOF(p_{i_1,j_1+1,k_{i_1}} \rightarrow \bar{p}_{i_1,j_1+1,k_{i_1}}) \) if and only if there is a test case \( \bar{t} \in FP(S) \) such that \( p_{i_1,1,j_1} = 1 \) on \( \bar{t} \).

**Proof:** First, we observe that
\[ S \oplus I_{CORF}(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \times TOF(p_{i_1,j_1+1,k_{i_1}} \rightarrow \bar{p}_{i_1,j_1+1,k_{i_1}}) \]
\[ \equiv ((p_{i_1}) \oplus (p_{i_1,1,j_1}))\bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \cdots \bar{p}_m \]
\[ \equiv (p_{i_1})((\bar{p}_{i_1}) (p_{i_1,1,j_1}))\bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \cdots \bar{p}_m \]
\[ \equiv (0 + (\bar{p}_{i_1}) (p_{i_1,1,j_1}))\bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \cdots \bar{p}_m \]
\[ \equiv \bar{p}_{i_1} p_{i_1,1,j_1} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \cdots \bar{p}_m \]
\[ \equiv p_{i_1,1,j_1} \bar{S} \]
Now, \[ S(\vec{t}) \neq I_{\text{CORF}}(p_{i_1} \to p_{i_1,j_1} \oplus p_{i_1,j_1+1,k_1}) \otimes TOF(p_{i_1,j_1+1,k_1} \to) ) \]
if and only if \[ S(\vec{t}) \oplus I_{\text{CORF}}(p_{i_1} \to p_{i_1,j_1} \oplus p_{i_1,j_1+1,k_1}) \otimes TOF(p_{i_1,j_1+1,k_1} \to) ) \approx 1 \]
if and only if \[ p_{i_1,1,j_1} \overline{S} \text{ evaluates to } 1 \text{ on } \vec{t} \]
if and only if \[ \vec{t} \in FP(S) \text{ such that } p_{i_1,1,j_1} \text{ evaluate to } 1. \]

Hence, the result follows. \( \square \)

For ease of reading and understanding, we list all double fault classes, the corresponding double-fault expression numbers and their corresponding fault detection conditions in Table 2. Let us consider the third row of Table 2, which presents the detection conditions of two double-fault expressions of \( \text{ENF} \otimes \text{TOF} \). For double-fault expression (5) (please refer to Table 1 for the actual double-fault expression), the detection condition shows that any true point of \( S \) in \( \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1, \ldots, h_1,j_1}^{m} TP_i(S) \right) \) or any false point of \( S \) can distinguish \( S \) and the expression. While for double-fault expression (6) in Table 1, the detection condition shows that any true point of \( S \) in \( \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_2}^{m} TP_i(S) \right) \) or any false point of \( S \) can distinguish \( S \) and the expression.

5 Fault Coupling

Most fault-based testing strategies derive their test cases based on the assumption that at most one of the hypothesized faults is committed by programmers [11]. A fundamental question in fault-based testing is whether test cases that detect programs with single fault in isolation are also able to detect programs with multiple faults in combination.

Fault coupling has been studied for years [5, 12], but so far it has no universally agreed definition. In this report, faults are said to be \textit{coupled} together if they can be detected in isolation but not
### Table 2: Double fault, double-fault expression and detection condition \( S = p_1 + \ldots + p_m \)

(a) Double-fault expressions (1)–(12) due to double faults without ordering

<table>
<thead>
<tr>
<th>Fault Class</th>
<th>(Expression No.) : Detection Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENF ⋊ ENF</td>
<td>(1) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \cap \bigcup_{i=i_2}^{h_2} T P_i(S) ) ( \setminus \bigcup_{i=1}^{m} T P_i(S) ), or (C2) any point in ( FP(S) )</td>
</tr>
<tr>
<td>ENF ⋊ TNF</td>
<td>(2) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) )</td>
</tr>
<tr>
<td>ENF ⋊ TOF</td>
<td>(3) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) ), or (C2) any point in ( FP(S) )</td>
</tr>
<tr>
<td>ENF ⋊ DORF</td>
<td>(4) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) )</td>
</tr>
<tr>
<td>ENF ⋊ CORF</td>
<td>(5) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) )</td>
</tr>
<tr>
<td>ENF ⋊ DORF</td>
<td>(6) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) ), or (C2) any point in ( FP(S) )</td>
</tr>
<tr>
<td>ENF ⋊ CORF</td>
<td>(7) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) )</td>
</tr>
<tr>
<td>ENF ⋊ DORF</td>
<td>(8) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) )</td>
</tr>
<tr>
<td>ENF ⋊ CORF</td>
<td>(9) : (C1) any point in ( TP_i(S) \cap TP_{i+1}(S) ) ( \setminus \bigcup_{i=1}^{m} T P_i(S) ), or (C2) any point in ( FP(S) )</td>
</tr>
<tr>
<td>ENF ⋊ DORF</td>
<td>(10) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) ) ( \setminus \bigcup_{i=1}^{m} T P_i(S) ), or (C2) any point in ( FP(S) )</td>
</tr>
<tr>
<td>ENF ⋊ CORF</td>
<td>(11) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) ) such that ( p_{i_2,j_2} + p_{i_2,j_2+1} = 0 ), or (C2) any point in ( FP(S) )</td>
</tr>
<tr>
<td>ENF ⋊ DORF</td>
<td>(12) : (C1) any point in ( \bigcup_{i=i_1}^{h_1} T P_i(S) \setminus \bigcup_{i=1}^{m} T P_i(S) ), or (C2) any point in ( FP(S) ) such that ( p_{i_2,j_2} + p_{i_2,j_2+1} = 0 )</td>
</tr>
<tr>
<td>Fault Class</td>
<td>(Expression No.) : Detection Condition</td>
</tr>
<tr>
<td>-------------</td>
<td>----------------------------------------</td>
</tr>
</tbody>
</table>
| TNF × TNF  | (13) : (C1) any point in \( (TP_{t_1}(S) \cap TP_{t_2}(S)) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \), or  
(C2) any point in \( FP(S) \) |
| TNF × TOF  | (14) : (C1) any point in \( (TP_{t_1}(S)) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \), or  
(C2) any point in \( FP(S) \) |
| TNF × DORF | (15) : (C1) any point in \( TP_{t_1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( (TP_{t_2}(S) \cap TP_{t_2+1}(S)) \right) \), or  
(C2) any point in \( FP(S) \) |
| TNF × CORF | (16) : (C1) any point in \( TP_{t_2}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \) |
| TOF × TOF  | (17) : (C1) any point in \( UTP_{t_1}(S) \) such that \( p_{t_2,1,j_2} + p_{t_2,2,j_2+1,k_2} = 0 \), or  
(C2) any point in \( FP(S) \) |
| TOF × DORF | (18) : (C1) any point in \( UTP_{t_2}(S) \), or  
(C2) any point in \( FP(S) \) such that \( p_{t_2,1,j_2} + p_{t_2,2,j_2+1,k_2} = 0 \) |
| TOF × CORF | (19) : (C1) any point in \( (TP_{t_1}(S) \cup TP_{t_2}(S)) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \) |
| DORF × DORF| (20) : (C1) any point in \( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( (TP_{t_2}(S) \cap TP_{t_2+1}(S)) \right) \cup \left( TP_{t_1+1}(S) \right) \) |
| DORF × CORF| (21) : (C1) any point in \( UTP_{t_1}(S) \) such that \( p_{t_2,1,j_2} + p_{t_2,2,j_2+1,k_2} = 0 \), or  
(C2) any point in \( FP(S) \) such that \( p_{t_2,1,j_2} + p_{t_2,2,j_2+1,k_2} = 1 \) |
| DORF × DORF| (22) : (C1) any point in \( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( (TP_{t_2}(S) \cap TP_{t_2+1}(S)) \right) \) |
| DORF × CORF| (23) : (C1) any point in \( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( (TP_{t_2}(S) \cap TP_{t_2+1}(S)) \right) \) |
| DORF × DORF| (24) : (C1) any point in \( UTP_{t_1}(S) \cup UTP_{t_1+1}(S) \) such that \( p_{t_2,1,j_2} + p_{t_2,2,j_2+1,k_2} = 0 \), or  
(C2) any point in \( FP(S) \) such that \( p_{t_2,1,j_2} + p_{t_2,2,j_2+1,k_2} = 1 \) |
| DORF × CORF| (25) : (C1) any point in \( UTP_{t_1+1}(S) \) such that \( p_{t_1,1,j_1} + p_{t_1,1,j_1+1,k_1} = 0 \), or  
(C2) any point in \( FP(S) \) such that \( p_{t_1,1,j_1} + p_{t_1,1,j_1+1,k_1} = 1 \) |
| DORF × CORF| (26) : (C1) any point in \( FP(S) \) such that \( p_{t_1,1,j_1} + p_{t_1,1,j_1+1,k_1} + p_{t_2,1,j_2} + p_{t_2,2,j_2+1,k_2} = 1 \) |
| DORF × CORF| (27) : (C1) any point in \( FP(S) \) such that \( p_{t_1,1,j_1} + p_{t_1,1,j_1+1,k_1} + p_{t_1,1,j_1+2} + p_{t_1,1,j_1+2} = 1 \) |
Table 2 (cont’d): Double fault, double-fault expression and detection condition

(b) Four double-fault expressions (53), (70), (73) and (76) due to double fault with ordering

<table>
<thead>
<tr>
<th>Fault Class</th>
<th>(Expression No.) : Detection Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>TOF ⊗ DORF</td>
<td>(53) : (C1) any point in ((TP_i(S) \cup TP_{i-1}(S)) \setminus \left( \bigcup_{j=1}^{m} TP_i(S) \right)) such that (p_{i+1,j} = 0), or (p_{i,j} = 0) when combined together. Most previous studies on fault coupling focused on the combination of two single faults.</td>
</tr>
<tr>
<td>CORF ⊗ ENF</td>
<td>(70) : (C1) any point in ((\bigcup_{t=i+1}^{m} TP_i(S)) \setminus \left( \bigcup_{i \neq i_1 \neq j_1}^{m} TP_i(S) \right)) such that (p_{i_1,j_1} = 0), or (p_{i,j} = 0) when combined together. Most previous studies on fault coupling focused on the combination of two single faults.</td>
</tr>
<tr>
<td>CORF ⊗ TNF</td>
<td>(73) : (C1) any point in (FP(S)) such that (p_{i+1,j} = 0), or (p_{i,j} = 0) when combined together. Most previous studies on fault coupling focused on the combination of two single faults.</td>
</tr>
<tr>
<td>CORF ⊗ TOF</td>
<td>(76) : (C1) any point in (FP(S)) such that (p_{i+1,j} = 0), or (p_{i,j} = 0) when combined together. Most previous studies on fault coupling focused on the combination of two single faults.</td>
</tr>
</tbody>
</table>

when combined together. Most previous studies on fault coupling focused on the combination of two single faults.

An empirical study on fault coupling via mutation analysis was done in [12]. A mutant is a program which differs from the original program by small syntactic changes. A 1-order (respectively, 2-order) mutant is a mutant that differs from the original program by 1 syntactic change (respectively, 2 syntactic changes). Three programs whose size ranges from 16 to 28 lines of code (LOC) were studied. Test sets that can kill all 1-order mutants were generated and used to kill 2-order mutants. As mentioned in [12], the experiment studies the mutation coupling effect, to be precise. It was found that test sets so generated can kill approximately 99.9% of 2-order mutants. It was then concluded that the effect of two faults being coupled together rarely occurs.

How Tai Wah [5] investigated fault coupling from a theoretical perspective. He also studied the behaviour of double faults. A program is modelled as a composition of several mathematical functions. For example, suppose that a program \(P\) is considered as a composition of three mathematical functions \(f, g, h\) in the order of \(f\) being computed first, followed by \(g\) and finally \(h\), then \(P\) is equivalent to the composite function \(h \circ g \circ f\). A single fault in a program is modelled as an incorrect use of one of the functions during the composition. For example, if a fault occurs in the program \(P\), it is possible that any one of the three functions \(f, g\) or \(h\) is implemented wrongly as \(f', g'\) or \(h'\),
respectively, to result in a faulty program which may be equivalent to $h \circ g \circ f'$, $h \circ g' \circ f$ or $h' \circ g \circ f$.

Moreover, a double fault in a program is modelled as the incorrect use of any two functions during the composition. Test sets that detect individual faults of a double fault are called proper test sets. Among the proper test sets, those that cannot detect the double fault are called coupled test sets. How Tai Wah [5] then calculates the coupling ratio, defined as the ratio of the number of coupled test sets to that of proper test sets. He shows analytically that the coupling ratio is approximately $1/|D|$ for test sets of size 1 and $1/|D|^2$ for test sets of size 2, where $|D|$ is the size of the input domain $D$. It should be noted that when a test set of size 1 detects a double fault, the only test case in the set must be able to detect each of the two individual faults in isolation. As $|D|$ is usually very large, the coupling ratio is very small, and he concludes that fault coupling rarely occurs.

In this study, we are more interested to know which double fault class can be detected by test cases that can detect the individual single fault classes. Hence, instead of performing empirical study via mutation analysis or calculating the coupling ratio via mathematical analysis, we analyse the relationship between single and double faults based on their detection conditions. More precisely, if $DC_A$, $DC_B$ and $DC_{A \bowtie B}$ are the detection conditions of the single fault classes $A$ and $B$ and the double fault class $A \bowtie B$, respectively, we would like to identify the relationship among $DC_A$, $DC_B$ and $DC_{A \bowtie B}$. Our aim is to find out which double fault class $A \bowtie B$ can be detected by test cases that can detect either of the two single fault classes $A$ and $B$.

For test cases that detect fault class $A$, there are three mutually exclusive possibilities:

(R1) All points that satisfy $DC_A$ will also satisfy $DC_{A \bowtie B}$; that is, any point that can detect $A$ will also detect $A \bowtie B$.

(R2) Some but not all points that satisfy $DC_A$ also satisfy $DC_{A \bowtie B}$; that is, some but not all points that can detect $A$ can also detect $A \bowtie B$.

(R3) None of the points that satisfy $DC_A$ also satisfies $DC_{A \bowtie B}$; that is, none of the points that can detect $A$ will also detect $A \bowtie B$. 
Similarly, for test cases that can detect $B$, there are three other mutually exclusive possibilities:

(R4) All points that satisfy $DC_B$ will also satisfy $DC_{A \times B}$; that is, any point that can detect $B$ will also detect $A \times B$.

(R5) Some but not all points that satisfy $DC_B$ also satisfy $DC_{A \times B}$; that is, some but not all points that can detect $B$ can also detect $A \times B$.

(R6) None of the points that satisfy $DC_B$ also satisfy $DC_{A \times B}$; that is, none of the points that can detect $B$ will also detect $A \times B$.

The analysis of the relationships between single faults and the 27 faulty implementations of double fault without ordering based on detection conditions are presented in Section 5.1 whereas those of the remaining 4 faulty implementations are introduced in Section 5.2.

### 5.1 Fault Coupling on 27 Faulty Implementations

#### 5.1.1 ENF and ENF

In this section, we analyse the relationship between the double fault involving two ENFs. Since there are two ENFs, we use ENF1 and ENF2 to identify the two different ENFs in double fault ENF1 $\times$ ENF2 for ease of understanding.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ ($i_1 < h_1$) in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{ENF_1}$, is “any point in $\bigcup_{i=i_1}^{h_1} TP_i(S) \setminus \bigcup_{i \neq i_1, \ldots, h_1}^{m} TP_i(S)$ or any point in $FP(S)$”. We use $TC_{ENF_1}$ to denote the set of all points that satisfy $DC_{ENF_1}$, that is $TC_{ENF_1} = \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \setminus \bigcup_{i \neq i_1, \ldots, h_1}^{m} TP_i(S) \right) \cup FP(S)$.

Similarly, if the subexpression $p_{i_2} + \cdots + p_{h_2}$ ($i_2 < h_2$) in $S$ is wrongly negated, the corresponding...
detection condition, denoted by $DC_{ENF2}$, is "any point in $(\bigcup_{i=i_2}^{h_2} TP_i(S)) \setminus \bigcup_{i=1}^{m} TP_i(S)$ or any point in $FP(S)$". We use $TC_{ENF2}$ to denote the set of all points that satisfy $DC_{ENF2}$, that is $TC_{ENF2} = \left((\bigcup_{i=i_2}^{h_2} TP_i(S)) \setminus \bigcup_{i=1}^{m} TP_i(S)\right) \cup FP(S)$.

For ENF1×ENF2, there are two subcases.

**Case 1**  
The double-fault expression is equivalent to Expression (1) in Table 1 where $i_1 < h_1 < i_2 < h_2$. The detection condition, denoted by $DC_{ENF1×ENF2−1}$, is "any point in \( ((\bigcup_{i=i_1}^{h_1} TP_i(S)) \cap (\bigcup_{i=i_2}^{h_2} TP_i(S))) \setminus \bigcup_{i=1}^{m} TP_i(S)\) or any point in $FP(S)$". We use $TC_{ENF1×ENF2−1}$ to denote the set of all points that satisfy $DC_{ENF1×ENF2−1}$, that is $TC_{ENF1×ENF2−1} = \left(((\bigcup_{i=i_1}^{h_1} TP_i(S)) \cap (\bigcup_{i=i_2}^{h_2} TP_i(S))) \setminus \bigcup_{i=1}^{m} TP_i(S)\right) \cup FP(S)$.

(A) Relationship between $DC_{ENF1}$ and $DC_{ENF1×ENF2−1}$: The relationship $R2$ holds because of the following reasons:

(a) Some points in $TC_{ENF1}$ can satisfy $DC_{ENF1×ENF2−1}$. For example, in $TC_{ENF1}$, false points of $S$ (that is, points in $FP(S)$) satisfy $DC_{ENF1×ENF2−1}$.

(b) Some points in $TC_{ENF1}$ cannot satisfy $DC_{ENF1×ENF2−1}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. For example, in $TC_{ENF1}$, we can find a point $\tilde{i}$ such that $\tilde{i} \in UTP_{i_1}(S) = \left(TP_{i_1}(S) \setminus \bigcup_{i=1}^{m} TP_i(S)\right)$. Note that, $\tilde{i}$ does not satisfy $DC_{ENF1×ENF2−1}$ because $\tilde{i} \notin FP(S)$, $\tilde{i} \notin \bigcup_{i=i_2}^{h_2} TP_i(S)$ and $i_1 < i_2 < h_2$.

Hence, only some, but not all, points satisfying $DC_{ENF1}$ can satisfy $DC_{ENF1×ENF2−1}$. 

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(B) Relationship between $DC_{ENF2}$ and $DC_{ENF1 \times ENF2-1}$: The relationship R5 holds because of the following reasons:

(a) Some points in $TC_{ENF2}$ can satisfy $DC_{ENF1 \times ENF2-1}$. For example, in $TC_{ENF2}$, false points of $S$ (that is, points in $FP(S)$) satisfy $DC_{ENF1 \times ENF2-1}$.

(b) Some points in $TC_{ENF2}$ cannot satisfy $DC_{ENF1 \times ENF2-1}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. For example, in $TC_{ENF2}$, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = \left( TP_{i_2}(S) \setminus \bigcup_{i \neq i_2} TP_i(S) \right)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF1 \times ENF2-1}$ because $\vec{t} \notin FP(S)$, $\vec{t} \notin \bigcup_{i = i_1} h_1$ and $i_1 < h_1 < i_2$.

Hence, only some, but not all, points satisfying $DC_{ENF2}$ can satisfy $DC_{ENF1 \times ENF2-1}$.

Case 2 The double-fault expression is equivalent to Expression (2) in Table 1 where $\{i_2, \ldots, h_2\} \subseteq \{i_1, \ldots, h_1\}$. The detection condition, denoted by $DC_{ENF1 \times ENF2-2}$, is “any point in $\left( \bigcup_{i = i_1} h_1 \right)$ \left( \bigcup_{i \neq i_1, \ldots, h_2} m \right) TP_i(S)$”. We use $TC_{ENF1 \times ENF2-2}$ to denote the set of all points that satisfy $DC_{ENF1 \times ENF2-2}$, that is $TC_{ENF1 \times ENF2-2} = \left( \bigcup_{i = i_1} h_1 \right)$ \left( \bigcup_{i \neq i_1, \ldots, h_2} m \right) TP_i(S)$.

(A) Relationship between $DC_{ENF1}$ and $DC_{ENF1 \times ENF2-2}$: The relationship R2 holds because of the following reasons:

(a) Some points in $TC_{ENF1}$ can satisfy $DC_{ENF1 \times ENF2-2}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. Since $\{i_2, \ldots, h_2\} \subseteq \{i_1, \ldots, h_1\}$, there exists $i_3 \in \{i_1, \ldots, h_1\}$ such that $i_3 \notin \{i_2, \ldots, h_2\}$.

Now, in $TC_{ENF1}$, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_3}(S) = \left( TP_{i_3}(S) \setminus \bigcup_{i \neq i_3} TP_i(S) \right)$ and $\vec{t}$ satisfies $DC_{ENF1 \times ENF2-2}$ because $\vec{t} \in \left( TP_{i_3}(S) \setminus \bigcup_{i \neq i_3} TP_i(S) \right) \subseteq \left( \bigcup_{i = i_1} h_1 \right)$ \left( \bigcup_{i \neq i_3} m \right) TP_i(S) \setminus \bigcup_{i \neq i_3} TP_i(S)$. 

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\[
\left( \bigcup_{i=1}^{m} TP_i(S) \right) = TC_{ENF1 \times ENF2-2} \text{ (by using } A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B).}
\]

(b) Some points in \(TC_{ENF1}\) cannot satisfy \(DC_{ENF1 \times ENF2-2}\). For example, in \(TC_{ENF1}\), false points of \(S\) (that is, points in \(FP(S)\)) do not satisfy \(DC_{ENF1 \times ENF2-2}\).

Hence, only some, but not all, points satisfying \(DC_{ENF1}\) can satisfy \(DC_{ENF1 \times ENF2-2}\).

(B) Relationship between \(DC_{ENF2}\) and \(DC_{ENF1 \times ENF2-2}\): The relationship R6 holds because:

\[
TC_{ENF2} \cap TC_{ENF1 \times ENF2-2}
= \left(\left(\bigcup_{i=1}^{h_2} TP_i(S) \setminus \bigcup_{i \neq i_1, \ldots, h_1}^{m} TP_i(S)\right) \cup FP(S)\right) \cap \left(\left(\bigcup_{i=1}^{h_1} TP_i(S) \setminus \bigcup_{i \neq i_2, \ldots, h_2}^{m} TP_i(S)\right)\right)
= \left(\bigcup_{i=1}^{h_2} \bigcup_{i \neq i_1}^{h_1} TP_i(S) \setminus \bigcup_{i \neq i_2}^{m} TP_i(S)\right) \cup \left(\bigcup_{i=1}^{h_1} \bigcup_{i \neq i_2}^{h_2} TP_i(S) \setminus \bigcup_{i \neq i_1}^{m} TP_i(S)\right) \cup \left(\bigcup_{i=1}^{h_2} \bigcup_{i \neq i_1}^{h_1} TP_i(S) \setminus \bigcup_{i \neq i_2}^{h_2} TP_i(S)\right) \cup \left(\bigcup_{i=1}^{h_1} \bigcup_{i \neq i_1}^{h_2} TP_i(S) \setminus \bigcup_{i \neq i_2}^{m} TP_i(S)\right)
\]

(by using \(A \setminus (B \cup C) \subseteq (A \setminus B)\))

\[= \emptyset \text{ (By using } (A \setminus B) \cap (B \setminus C) = \emptyset).\]

Hence, all points satisfying \(DC_{ENF2}\) do not satisfy \(DC_{ENF1 \times ENF2-2}\).

5.1.2 ENF and TNF

In this section, we analyse the relationship between the double fault ENF\(\times TNF\) and its corresponding single faults.
Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ $(i_1 < h_1)$ in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{ENF}$, is “any point in \((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{i=1}^{m} TP_i(S))\) or any point in $FP(S)$”. We use $TC_{ENF}$ to denote the set of all points that satisfy $DC_{ENF}$, that is $TC_{ENF} = \left(\left(\bigcup_{i=i_1}^{h_1} TP_i(S)\right) \setminus (\bigcup_{i=1}^{m} TP_i(S))\right) \cup FP(S)$.

Similarly, if the $i_2$-th term, $p_{i_2}$, in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{TNF}$, is “any point in $UTP_{i_2}(S)$ or any point in $FP(S)$”. We use $TC_{TNF}$ to denote the set of all points that satisfy $DC_{TNF}$, that is $TC_{TNF} = UTP_{i_2}(S) \cup FP(S)$.

For $ENF \not\iff TNF$, there are two subcases.

**Case 1** The double-fault expression is equivalent to Expression (3) in Table 1 where $i_1 < h_1 < i_2$.

The detection condition, denoted by $DC_{ENF \not\iff TNF-1}$, is “any point in \((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{i=1}^{m} TP_i(S))\) or any point in $FP(S)$”. We use $TC_{ENF \not\iff TNF-1}$ to denote the set of all points that satisfy $DC_{ENF \not\iff TNF-1}$, that is $TC_{ENF \not\iff TNF-1} = \left(\left(\bigcup_{i=i_1}^{h_1} TP_i(S)\right) \setminus (\bigcup_{i=1}^{m} TP_i(S))\right) \cup FP(S)$.

(A) **Relationship between $DC_{ENF}$ and $DC_{ENF \not\iff TNF-1}$**: The relationship $R2$ holds because of the following reasons:

(a) Some points in $TC_{ENF}$ can satisfy $DC_{ENF \not\iff TNF-1}$. For example, in $TC_{ENF}$, false points of $S$ (that is, points in $FP(S)$) satisfy $DC_{ENF \not\iff TNF-1}$.

(b) Some points in $TC_{ENF}$ cannot satisfy $DC_{ENF \not\iff TNF-1}$. Since $S$ is in IDNF, $UTP_{i_1}(S) \neq \emptyset$ for all $i$. For example, in $TC_{ENF}$, we can find a point $\vec{r}$ such that $\vec{r} \in UTP_{i_1}(S) = (TP_{i_1}(S) \setminus \bigcup_{i=1}^{m} TP_i(S))$. Note that, $\vec{r}$ does not satisfy $DC_{ENF \not\iff TNF-1}$ because $\vec{r} \notin FP(S), \vec{r} \notin TP_{i_2}(S)$.
and $i_1 < i_2$.

Hence, only some, but not all, points satisfying $DC_{ENF}$ can satisfy $DC_{ENF \times TNF - 1}$.

(B) Relationship between $DC_{TNF}$ and $DC_{ENF \times TNF - 1}$: The relationship R5 holds because of the following reasons:

(a) Some points in $TC_{TNF}$ can satisfy $DC_{ENF \times TNF - 1}$. For example, in $TC_{TNF}$, false points of $S$ (that is, points in $FP(S)$) satisfy $DC_{ENF \times TNF - 1}$.

(b) Some points in $TC_{TNF}$ cannot satisfy $DC_{ENF \times TNF - 1}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. For example, in $TC_{TNF}$, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = (TP_{i_2}(S) \setminus \bigcup_{i=1\atop i \neq i_2}^{m} TP_i(S))$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \times TNF - 1}$ because $\vec{t} \notin FP(S)$, $\vec{t} \notin \bigcup_{i=i_1}^{i_2} TP_i(S)$ and $i_1 < h_1 < i_2$.

Hence, only some, but not all, points satisfying $DC_{TNF}$ can satisfy $DC_{ENF \times TNF - 1}$.

Case 2 The double-fault expression is equivalent to Expression (4) in Table 1 where $i_1 \leq i_2 \leq h_1$ and $i_1 < h_1$. The detection condition, denoted by $DC_{ENF \times TNF - 2}$, is "any point in \Big( \bigcup_{i=i_1\atop i \neq i_2}^{h_1} TP_i(S) \big) \setminus \big( \bigcup_{i=i_1\atop i \neq i_2}^{m} TP_i(S) \big)". We use $TC_{ENF \times TNF - 2}$ to denote the set of all points that satisfy $DC_{ENF \times TNF - 2}$, that is $TC_{ENF \times TNF - 2} = \big( \bigcup_{i=i_1\atop i \neq i_2}^{h_1} TP_i(S) \big) \setminus \big( \bigcup_{i=i_1\atop i \neq i_3}^{m} TP_i(S) \big)$.

(A) Relationship between $DC_{ENF}$ and $DC_{ENF \times TNF - 2}$: The relationship R2 holds because of the following reasons:

(a) Some points in $TC_{ENF}$ can $DC_{ENF \times TNF - 2}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$.

Since $i_1 \leq i_2 \leq h_1$ and $i_1 \neq h_1$, there exists $i_3$ such that $i_1 \leq i_3 \leq h_1$ and $i_2 \neq i_3$. Now, in $TC_{ENF}$, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_3}(S) = (TP_{i_3}(S) \setminus \bigcup_{i=1\atop i \neq i_3}^{m} TP_i(S))$ and $\vec{t}$ satisfies
$DC_{ENF \times TNF - 2}$ because $\vec{t} \in (TP_{i_3}(S) \setminus \bigcup_{i \neq i_3} TP_i(S)) \subseteq (\bigcup_{i = 1}^{m} TP_i(S)) \setminus (\bigcup_{i = i_1, \ldots, h_1}^{m} TP_i(S)) = TC_{ENF \times TNF - 2}$ (by using $A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B$).

(b) Some points in $TC_{ENF}$ cannot $DC_{ENF \times TNF - 2}$. For example, in $TC_{ENF}$, false points of $S$ (that is, points in $FP(S)$) do not satisfy $DC_{ENF \times TNF - 2}$.

Hence, only some, but not all, points satisfying $DC_{ENF}$ can satisfy $DC_{ENF \times TNF - 2}$.

(B) Relationship between $DC_{TNF}$ and $DC_{ENF \times TNF - 2}$: The relationship R6 holds because:

$$TC_{TNF} \cap TC_{ENF \times TNF - 2} = \left( UTP_{i_2}(S) \cup FP(S) \right) \cap \left( \bigcup_{i = i_1}^{h_1} \left( \bigcup_{i \neq i_2}^{m} TP_i(S) \right) \right) \subseteq \left( UTP_{i_2}(S) \cap \left( \bigcup_{i = i_1}^{h_1} \left( \bigcup_{i \neq i_2}^{m} TP_i(S) \right) \right) \right) \cup \left( FP(S) \cap \left( \bigcup_{i = i_1}^{h_1} \left( \bigcup_{i \neq i_2}^{m} TP_i(S) \right) \right) \right) \subseteq \left( UTP_{i_2}(S) \cap \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \right) \cup \emptyset$$  

(By using $A \cap (B \setminus C) \subseteq A \cap B$)

$$= \emptyset$$  

(By definition of $UTP_{i_2}(S)$).

Hence, all points satisfying $DC_{TNF}$ do not satisfy $DC_{ENF \times TNF - 2}$.

5.1.3 ENF and TOF

In this section, we analyse the relationship between the double fault $ENF \times TOF$ and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ ($i_1 < h_1$) in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{ENF}$, is "any point in $\left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i = i_1, \ldots, h_1}^{m} TP_i(S) \right)$ or any point in $FP(S)$". We use $TC_{ENF}$ to denote the
set of all points that satisfy $D_{C_{ENF}}$, that is $T_{C_{ENF}} = \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup FP(S)$.

Similarly, if the $i_2$-th term, $p_{i_2}$, in $S$ is wrongly omitted, the corresponding detection condition, denoted by $D_{C_{TOF}}$, is “any point in $UTP_{i_2}(S)$”. We use $T_{C_{TOF}}$ to denote the set of all points that satisfy $D_{C_{TOF}}$, that is $T_{C_{TOF}} = UTP_{i_2}(S)$.

For $ENF \Join TOF$, there are two subcases.

**Case 1** The double-fault expression is equivalent to Expression (5) in Table 1 where $i_1 < h_1 < i_2$.

The detection condition, denoted by $D_{C_{ENF \Join TOF - 1}}$, is “any point in $\left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)$ or any point in $FP(S)$”. We use $T_{C_{ENF \Join TOF - 1}}$ to denote the set of all points that satisfy $D_{C_{ENF \Join TOF - 1}}$, that is $T_{C_{ENF \Join TOF - 1}} = \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup FP(S)$.

(A) Relationship between $D_{C_{ENF}}$ and $D_{C_{ENF \Join TOF - 1}}$: The relationship R1 holds because:

$$T_{C_{ENF}} = \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup FP(S)$$

$$= \left( \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup TP_{i_2}(S) \right) \cup FP(S)$$

$$\subseteq \left( \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup TP_{i_2}(S) \right) \cup FP(S)$$

(By using $A \setminus (B \cup C) \subseteq A \setminus B$)

$$= T_{C_{ENF \Join TOF - 1}}.$$  

Hence, any point satisfying $D_{C_{ENF}}$ satisfies $D_{C_{ENF \Join TOF - 1}}$.

(B) Relationship between $D_{C_{TOF}}$ and $D_{C_{ENF \Join TOF - 1}}$: The relationship R6 holds because:
\[
TC_{TOF} \cap TC_{ENF \times TOF-1} \\
= UTP_{i_2}(S) \cap \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1, \ldots, h_1} \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup FP(S) \\
= \left( UTP_{i_2}(S) \cap \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \right) \cup \left( UTP_{i_2}(S) \cap FP(S) \right) \\
\subseteq \left( UTP_{i_2}(S) \cap \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \right) \cup \emptyset \quad \text{(By using } A \setminus B \subseteq A) \\
= \emptyset \quad \text{(By definition of } UTP_{i_2}(S) \text{ and } i_1 < h_1 < i_2). 
\]

Hence, all points satisfying \( DC_{TOF} \) do not satisfy \( DC_{ENF \times TOF-1} \).

**Case 2** The double-fault expression is equivalent to Expression (6) in Table 1 where \( i_1 \leq i_2 \leq h_1 \) and \( i_1 < h_1 \). The detection condition, denoted by \( DC_{ENF \times TOF-2} \), is “any point in \( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1, \ldots, h_1} \bigcup_{i=1}^{m} TP_i(S) \right) \) or any point in \( FP(S) \)”. We use \( TC_{ENF \times TOF-2} \) to denote the set of points that satisfy \( DC_{ENF \times TOF-2} \), that is \( TC_{ENF \times TOF-2} = \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1, \ldots, h_1} \bigcup_{i=1}^{m} TP_i(S) \right) \cup FP(S) \).

(A) Relationship between \( DC_{ENF} \) and \( DC_{ENF \times TOF-2} \): The relationship R2 holds because of the following reasons:

(a) Some points in \( TC_{ENF} \) can satisfy \( DC_{ENF \times TOF-2} \). For example, in \( TC_{ENF} \), false points of \( S \) (that is, points in \( FP(S) \)) satisfy \( DC_{ENF \times TOF-2} \).

(b) Some points in \( TC_{ENF} \) cannot satisfy \( DC_{ENF \times TOF-2} \). Since \( S \) is in IDNF, \( UTP_i(S) \neq \emptyset \) for all \( i \). For example, in \( TC_{ENF} \), we can find a point \( \vec{t} \) such that \( \vec{t} \in UTP_{i_2}(S) = \left( TP_{i_2}(S) \setminus \bigcup_{i=1}^{m} TP_i(S) \right) \). Note that, \( \vec{t} \) does not satisfy \( DC_{ENF \times TOF-2} \) because \( \vec{t} \not\in FP(S) \).
and $\vec{t} \not\in \bigcup_{i=1 \atop i \neq 1,2}^{h_1} TP_i(S)$.

Hence, only some, but not all, points satisfying $DC_{ENF}$ can satisfy $DC_{ENF \times TOF - 2}$.

(B) Relationship between $DC_{TOF}$ and $DC_{ENF \times TOF - 2}$: The relationship $R6$ holds because:

$$TC_{TOF} \cap TC_{ENF \times TOF - 2}$$

$$= \bigcup_{i=1 \atop i \neq 1,2}^{h_1} TP_i(S) \cap \left( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1 \atop i \neq 1, \ldots, h_1}^{m} TP_i(S) \right) \right) \cup FP(S)$$

$$= \left( \bigcup_{i=1 \atop i \neq 1,2}^{h_1} TP_i(S) \cap \left( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1 \atop i \neq 1, \ldots, h_1}^{m} TP_i(S) \right) \right) \right) \cup \left( UTP_{i_2}(S) \cap FP(S) \right)$$

$$\subseteq \left( \bigcup_{i=1 \atop i \neq 1,2}^{h_1} TP_i(S) \right) \cup \emptyset \quad \text{(By using $A \setminus B \subseteq A$)}$$

$$= \emptyset \quad \text{(By definition of $UTP_{i_2}(S)$).}$$

Hence, all points satisfying $DC_{TOF}$ do not satisfy $DC_{ENF \times TOF - 2}$.

5.1.4 ENF and DORF

In this section, we analyse the relationship between the double fault $ENF \times DORF$ and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ ($i_1 < h_1$) in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{ENF}$, is “any point in $\left( \bigcup_{i=1 \atop i \neq 1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1 \atop i \neq 1, \ldots, h_1}^{m} TP_i(S) \right)$ or any point in $FP(S)$”. We use $TC_{ENF}$ to denote the set of all points that satisfy $DC_{ENF}$, that is $TC_{ENF} = \left( \bigcup_{i=1 \atop i \neq 1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1 \atop i \neq 1, \ldots, h_1}^{m} TP_i(S) \right) \cup FP(S)$.

Similarly, if the subexpression $p_{i_2} + p_{i_2+1}$ in $S$ is wrongly implemented as $p_{i_2} \cdot p_{i_2+1}$, the corresponding detection condition, denoted by $DC_{DORF}$, is “any point in $UTP_{i_2}(S)$ or any point in
We use $TC_{DORF}$ to denote the set of all points that satisfy $DC_{DORF}$, that is $TC_{DORF} = UTP_{i_2 + 1}(S)$. For $ENF \times DORF$, there are four subcases.

**Case 1** The double-fault expression is equivalent to Expression (7) in Table 1 where $i_1 < h_1 < i_2$.

The detection condition, denoted by $DC_{ENF \times DORF - 1}$, is “any point in $(\bigcup_{i = i_1}^{h_1} TP_i(S)) \setminus \left( \bigcup_{i \neq i_1, h_1, i_2, i_2 + 1} TP_i(S) \cup \left( TP_{i_2}(S) \cap TP_{i_2 + 1}(S) \right) \right)$ or any point in $FP(S)$”. We use $TC_{ENF \times DORF - 1}$ to denote the set of all points that satisfy $DC_{ENF \times DORF - 1}$, that is $TC_{ENF \times DORF - 1} = \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \setminus \left( \bigcup_{i \neq i_1, h_1, i_2, i_2 + 1} TP_i(S) \cup \left( TP_{i_2}(S) \cap TP_{i_2 + 1}(S) \right) \right) \right) \cup FP(S)$.

**A** Relationship between $DC_{ENF}$ and $DC_{ENF \times DORF - 1}$: The relationship R1 holds because

$$
TC_{ENF} = \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \cup \left( TP_{i_2}(S) \cup TP_{i_2 + 1}(S) \right) \right) \cup FP(S)
$$

$$
= \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1, h_1, i_2, i_2 + 1} TP_i(S) \cup \left( TP_{i_2}(S) \cup TP_{i_2 + 1}(S) \right) \right) \cup FP(S)
$$

$$
\subseteq \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1, h_1, i_2, i_2 + 1} TP_i(S) \cup \left( TP_{i_2}(S) \cup TP_{i_2 + 1}(S) \right) \right) \cup FP(S)
$$

(By using $A \setminus (B \cup (C \cup D)) \subseteq A \setminus (B \cup (C \cap D))$)

$$
= TC_{ENF \times DORF - 1}.
$$

Hence, any point satisfying $DC_{ENF}$ satisfies $DC_{ENF \times DORF - 1}$.

**B** Relationship between $DC_{DORF}$ and $DC_{ENF \times DORF - 1}$: The relationship R6 holds because:
\[ TC_{DORF} \cap TC_{ENF \times DORF - 1} \]
\[ = (UTP_{i_2}(S) \cup UTP_{i_2+1}(S)) \cap \left( \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i = i_1}^{m} TP_i(S) \right) \right) \]
\[ \cup (TP_{i_2}(S) \cap TP_{i_2+1}(S)) \cup FP(S) \]
\[ = \left( (UTP_{i_2}(S) \cup UTP_{i_2+1}(S)) \cap \left( \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i = i_1}^{m} TP_i(S) \right) \right) \right) \cup (UTP_{i_2}(S) \cup UTP_{i_2+1}(S)) \cap FP(S) \]
\[ \subseteq \left( (UTP_{i_2}(S) \cup UTP_{i_2+1}(S)) \cap \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \right) \cup \emptyset \quad \text{(By using } A \setminus B \subseteq A) \]
\[ = (UTP_{i_2}(S) \cap \bigcup_{i = i_1}^{h_1} TP_i(S)) \cup (UTP_{i_2+1}(S) \cap \bigcup_{i = i_1}^{h_1} TP_i(S)) \]
\[ = \emptyset \quad \text{(By definitions of } UTP_{i_2}(S) \text{ and } UTP_{i_2+1}(S), \text{ and } i_1 < h_1 < i_2) . \]

Hence, all points satisfying \( DC_{DORF} \) do not satisfy \( DC_{ENF \times DORF - 1} \).

**Case 2** The double-fault expression is equivalent to Expression (8) in Table 1 where \( i_1 < h_1 = i_2 \).

The detection condition, denoted by \( DC_{ENF \times DORF - 2} \), is “any point in \( \left( \bigcup_{i = i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i = i_1}^{m} TP_i(S) \right) \).”

We use \( TC_{ENF \times DORF - 2} \) to denote the set of points that satisfy \( DC_{ENF \times DORF - 2} \), that is \( TC_{ENF \times DORF - 2} = (\bigcup_{i = i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{i = i_1}^{m} TP_i(S)) \).

(A) **Relationship between \( DC_{ENF} \) and \( DC_{ENF \times DORF - 2} \)**: The relationship R2 holds because of the following reasons:

(a) Some points in \( TC_{ENF} \) can satisfy \( DC_{ENF \times DORF - 2} \). Since \( S \) is in IDNF, \( UTP_i(S) \neq \emptyset \) for all \( i \). For example, in \( TC_{ENF} \), we can find a point \( \vec{r} \) such that \( \vec{r} \in UTP_{i_1}(S) = (TP_{i_1}(S) \setminus \bigcup_{i = i_1}^{m} TP_i(S)) \). Note that, \( \vec{r} \) satisfies \( DC_{ENF \times DORF - 2} \) because \( \vec{r} \in (TP_{i_1}(S) \setminus \bigcup_{i = i_1}^{m} TP_i(S)) \subseteq (\bigcup_{i = i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{i = i_1}^{m} TP_i(S)) \).
\[
\left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) = TC_{ENF} \times DORF - 2 \quad \text{(by using } A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B).}
\]

(b) Some points in \( TC_{ENF} \) cannot satisfy \( DC_{ENF} \times DORF - 2 \). For example, in \( TC_{ENF} \), false points of \( S \) (that is, points in \( FP(S) \)) do not satisfy \( DC_{ENF} \times DORF - 2 \).

Hence, only some, but not all, points satisfying \( DC_{ENF} \) can satisfy \( DC_{ENF} \times DORF - 2 \).

(B) Relationship between \( DC_{DORF} \) and \( DC_{ENF} \times DORF - 2 \): The relationship R5 holds because of the following reasons:

(a) Some points in \( TC_{DORF} \) can satisfy \( DC_{ENF} \times DORF - 2 \). Since \( S \) is in IDNF, \( UTP_i(S) \neq \emptyset \) for all \( i \). For example, in \( TC_{DORF} \), we can find a point \( \vec{i} \) such that \( \vec{i} \in UTP_{i_2}(S) = (TP_{i_2}(S) \setminus \bigcup_{i=1}^{m} TP_i(S)) \). Note that, \( \vec{i} \) satisfies \( DC_{ENF} \times DORF - 2 \) because \( \vec{i} \in (TP_{i_2}(S) \setminus \bigcup_{i=1}^{m} TP_i(S)) \subseteq (A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B \) and \( i_2 = h_1 \).

(b) Some points in \( TC_{DORF} \) cannot satisfy \( DC_{ENF} \times DORF - 2 \). For example, in \( TC_{DORF} \), we can find a point \( \vec{i} \in UTP_{i_2+1}(S) = (TP_{i_2+1}(S) \setminus \bigcup_{i=1}^{m} TP_i(S)) \). Note that, \( \vec{i} \) does not satisfy \( DC_{ENF} \times DORF - 2 \) because \( \vec{i} \notin \bigcup_{i=i_1}^{h_1} TP_i(S) \) and \( i_2 = h_1 \).

Hence, only some, but not all, points satisfying \( DC_{DORF} \) can satisfy \( DC_{ENF} \times DORF - 2 \).

Case 3 The double-fault expression is equivalent to Expression (9) in Table 1 where \( i_1 = i_2 \) and \( h_1 = i_2 + 1 \). The detection condition, denoted by \( DC_{ENF} \times DORF - 3 \), is “any point in \( (TP_{i_1}(S) \cap TP_{i_1+1}(S)) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \) or any point in \( FP(S) \)”. We use \( TC_{ENF} \times DORF - 3 \) to denote the
set of all points that satisfy $DC_{ENF \times DORF-3}$, that is $TC_{ENF \times DORF-3} = \left( \left( TP_{i_1}(S) \cap TP_{i_1+1}(S) \right) \setminus \bigcup_{i=1}^{m} TP_{i}(S) \right) \cup FP(S)$.

(A) Relationship between $DC_{ENF}$ and $DC_{ENF \times DORF-3}$: The relationship R2 holds because of the following reasons:

(a) Some points in $TC_{ENF}$ can satisfy $DC_{ENF \times DORF-3}$. For example, in $TC_{ENF}$, false points of $S$ (that is, points in $FP(S)$) satisfy $DC_{ENF \times DORF-3}$.

(b) Some points in $TC_{ENF}$ cannot satisfy $DC_{ENF \times DORF-3}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. For example, in $TC_{ENF}$, we can find a point $\bar{t}$ such that $\bar{t} \in UTP_{i_1}(S) = \left( TP_{i_1}(S) \setminus \bigcup_{i=1}^{m} TP_{i}(S) \right)$. Note that, $\bar{t}$ does not satisfy $DC_{ENF \times DORF-3}$ because $\bar{t} \not\in TP_{i_1+1}(S)$.

Hence, only some, but not all, points satisfying $DC_{ENF}$ can satisfy $DC_{ENF \times DORF-3}$.

(B) Relationship between $DC_{DORF}$ and $DC_{ENF \times DORF-3}$: The relationship R6 holds because:
The double-fault expression is equivalent to Expression (10) in Table 1 where

\[ TC_{DORF} \cap TC_{ENF \times DORF - 3} \]

\[ = (UTP_{i_2}(S) \cup UTP_{i_2+1}(S)) \cap \left( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup FP(S) \right) \]

\[ = \left( (UTP_{i_2}(S) \cup UTP_{i_2+1}(S)) \cap \left( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup FP(S) \right) \right) \]

\[ \cup \left( (UTP_{i_2}(S) \cup UTP_{i_2+1}(S)) \cap FP(S) \right) \]

\[ = \left( UTP_{i_2}(S) \cap \left( \left( TP_{i_1}(S) \cap TP_{i_1+1}(S) \right) \cup \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \right) \]

\[ \cup \left( UTP_{i_2+1}(S) \cap \left( TP_{i_1}(S) \cap TP_{i_1+1}(S) \right) \cup \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup \emptyset \]

\[ \subseteq \left( UTP_{i_2}(S) \cap \left( TP_{i_1}(S) \cap TP_{i_1+1}(S) \right) \right) \cup \left( UTP_{i_2+1}(S) \cap \left( TP_{i_1}(S) \cap TP_{i_1+1}(S) \right) \right) \]

(By using \( A \setminus B \subseteq A \))

\[ = \left( UTP_{i_2}(S) \cap TP_{i_1+1}(S) \right) \cup \left( UTP_{i_2+1}(S) \cap TP_{i_1}(S) \right) \]

(When \( i_1 = i_2 \), \( UTP_{i_2}(S) \subseteq TP_{i_1}(S) \) and \( UTP_{i_2+1}(S) \subseteq TP_{i_1+1}(S) \))

\[ = \emptyset \] (By definitions of \( UTP_{i_2}(S) \) and \( UTP_{i_2+1}(S) \), and \( i_1 = i_2 \)).

Hence, all points satisfying \( DC_{DORF} \) do not satisfy \( DC_{ENF \times DORF - 3} \).

**Case 4** The double-fault expression is equivalent to Expression (10) in Table 1 where \( i_1 \leq i_2 < h_1 \) and \( h_1 \neq i_1 + 1 \). The detection condition, denoted by \( DC_{ENF \times DORF - 4} \), is “any point in

\[ \left( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \] or any point in \( FP(S) \)”.

We use \( TC_{ENF \times DORF - 4} \) to denote the set of all points that satisfy \( DC_{ENF \times DORF - 4} \), that is

\[ TC_{ENF \times DORF - 4} = \left( \left( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup FP(S). \]

(A) Relationship between \( DC_{ENF} \) and \( DC_{ENF \times DORF - 4} \): The relationship R2 holds because of the following reasons:

(a) Some points in \( TC_{ENF} \) can satisfy \( DC_{ENF \times DORF - 4} \). For example, in \( TC_{ENF} \), false points of
S (that is, points in $FP(S)$) satisfy $DC_{ENF \times DORF-4}$.

(b) Some points in $TC_{ENF}$ cannot satisfy $DC_{ENF \times DORF-4}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. For example, in $TC_{ENF}$, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = \left( TP_{i_2}(S) \setminus \bigcup_{i=1}^{m} TP_i(S) \right)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \times DORF-4}$ because $\vec{t} \notin TP_{i_2+1}(S)$.

Hence, only some, but not all, points satisfying $DC_{ENF}$ can satisfy $DC_{ENF \times DORF-4}$.

**B** Relationship between $DC_{DORF}$ and $DC_{ENF \times DORF-4}$: The relationship R6 holds because:

\[
TC_{DORF} \cap TC_{ENF \times DORF-4} = \left( UTP_{i_2}(S) \cup UTP_{i_2+1}(S) \right) \cap \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \right) \cup \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \right) \cup \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \right) \cup \emptyset \quad (\text{By using } A \setminus B \subseteq A)
\]

\[
= \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \right) \cup \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \right) \cup \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \cup \left( TP_{i_2}(S) \cap TP_{i_2+1}(S) \right) \right) \cup \emptyset \quad (\text{By definitions of } UTP_{i_2}(S) \text{ and } UTP(S) \cap FP(S) = \emptyset)
\]

\[
= \left( UTP_{i_2}(S) \cap \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \right) \cup \left( UTP_{i_2+1}(S) \cap \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \right) \cup \emptyset \quad (\text{Because } UTP_{i_2}(S) \subseteq UTP(S) \text{ and } UTP(S) \cap OTP(S) = \emptyset)
\]

\[
= \emptyset \quad (\text{By definitions of } UTP_{i_2}(S) \text{ and } UTP_{i_2+1}(S)).
\]

Hence, all points satisfying $DC_{DORF}$ do not satisfy $DC_{ENF \times DORF-4}$.
5.1.5 ENF and CORF

In this section, we analyse the relationship between the double fault ENF×CORF and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{i_h}$ ($i_1 < h_1$) in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{ENF}$, is “any point in $\left( \bigcup_{i=h_1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i\neq i_1,\ldots,h_1}^{m} TP_i(S) \right)$ or any point in $FP(S)$”. We use $TC_{ENF}$ to denote the set of all points that satisfy $DC_{ENF}$, that is $TC_{ENF} = \left( \left( \bigcup_{i=h_1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i\neq i_1,\ldots,h_1}^{m} TP_i(S) \right) \right) \cup FP(S)$.

Similarly, if the $i_2$-th term, $p_{i_2}$, in $S$ is wrongly implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2}$ where $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_2}$, the corresponding detection condition, denoted by $DC_{CORF}$, is “any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1$”. We use $TC_{CORF}$ to denote the set of all points that satisfy $DC_{CORF}$, that is $TC_{CORF} = \{ i \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \}$.

For ENF×CORF, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (11) in Table 1 where $i_1 < h_1 < i_2$.

The detection condition, denoted by $DC_{ENF\times CORF^{−1}}$, is “any point in $\left( \bigcup_{i=h_1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i\neq i_1,\ldots,h_1}^{m} TP_i(S) \right)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0$ or any point in $FP(S)$”. We use $TC_{ENF\times CORF^{−1}}$ to denote the set of all points that satisfy $DC_{ENF\times CORF^{−1}}$, that is $TC_{ENF\times CORF^{−1}} = \{ i \in \left( \bigcup_{i=h_1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i\neq i_1,\ldots,h_1}^{m} TP_i(S) \right) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup FP(S)$.

(A) Relationship between $DC_{ENF}$ and $DC_{ENF\times CORF^{−1}}$:

First, $TC_{ENF\times CORF^{−1}} = \{ i \in \left( \bigcup_{i=h_1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i\neq i_1,\ldots,h_1}^{m} TP_i(S) \right) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup$
\( \text{FP}(S) \subseteq \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup \text{FP}(S) = \text{TC}_{\text{ENF}}. \) Hence the relationship between \( \text{DC}_{\text{ENF}} \) and \( \text{DC}_{\text{ENF} \times \text{CORF} - 1} \) depends on whether \( \text{TC}_{\text{ENF} \times \text{CORF} - 1} = \text{TC}_{\text{ENF}}. \)

If \( \text{TC}_{\text{ENF} \times \text{CORF} - 1} \) and \( \text{TC}_{\text{ENF}} \) are not equal, R2 holds. That means, some points in \( \text{TC}_{\text{ENF}} \) do not satisfy \( \text{DC}_{\text{ENF} \times \text{CORF} - 1} \). Here is an example. Let \( S = ab + cd + ef \). If the subexpression \( ab + cd \) is negated and the third term \( ef \) is wrongly split as \( e + f \), \( \text{I}_{\text{ENF}} = ab + cd + ef \) and \( \text{I}_{\text{ENF} \times \text{CORF} - 1} = \overline{ab + cd} + e + f \). Now, the point \( 110001 \in \text{UTP}_1(S) = TP_1(S) \setminus \left( \bigcup_{i=2}^{6} TP_i(S) \right) \subseteq \text{TC}_{\text{ENF}} \) can distinguish \( S \) from \( \text{I}_{\text{ENF}} \) because \( S \) and \( \text{I}_{\text{ENF}} \) evaluate to 1 and 0 on this point, respectively. However, it cannot be used to distinguish \( S \) from \( \text{I}_{\text{ENF} \times \text{CORF} - 1} \) because both \( S \) and \( \text{I}_{\text{ENF} \times \text{CORF} - 1} \) evaluate to 1 on \( 110001 \).

On the other hand, if \( \text{TC}_{\text{ENF} \times \text{CORF} - 1} = \text{TC}_{\text{ENF}}, \) R1 holds. That means, \( p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2} \) evaluates to 0 on all points in \( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \). Under such circumstances, all test cases that can detect \( \text{ENF} \times \text{CORF} - 1 \) can also be used to detect \( \text{ENF} \). Here is an example. Let \( S = abc + de + \overline{a}bd + cd + \overline{a}e + \overline{b}e \). Note that \( \left( TP_1(S) \cup TP_2(S) \right) \setminus \left( \bigcup_{i=3}^{6} TP_i(S) \right) = \{11110,11011,11111\} \). If the subexpression \( abc + de \) is negated and the third term is wrongly split as \( \overline{a} + bd \), \( \text{I}_{\text{ENF}} = \overline{abc} + \overline{de} + \overline{a}bd + cd + \overline{a}e + \overline{b}e \) and \( \text{I}_{\text{ENF} \times \text{CORF} - 1} = abc + de + a + bd + cd + \overline{ae} + \overline{be} \). For ease of reference, let \( X = \left( \left( TP_1(S) \cup TP_2(S) \right) \setminus \left( \bigcup_{i=3}^{6} TP_i(S) \right) \right) \). Note that \( X = \{11110,11011,11111\} \), and \( \text{TC}_{\text{ENF}} = X \cup \text{FP}(S) = \{11110,11011,11111\} \cup \text{FP}(S) \), and \( \text{TC}_{\text{ENF} \times \text{CORF} - 1} = \{i \in X : \overline{a} + \overline{b}d = 0\} \cup \text{FP}(S) = \{11110,11011,11111\} \cup \text{FP}(S) = \text{TC}_{\text{ENF}} \) because \( \overline{a} + \overline{b}d = 0 \) on all points in \( X \).

(B) Relationship between \( \text{DC}_{\text{CORF}} \) and \( \text{DC}_{\text{ENF} \times \text{CORF} - 1} \): The relationship R4 holds because:

\[
\text{TC}_{\text{CORF}} = \{i \in \text{FP}(S) : p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2} = 1\} \subseteq \text{FP}(S) \subseteq \{i \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) : p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_2} = 0\} \cup \text{FP}(S) = \text{TC}_{\text{ENF} \times \text{CORF} - 1}.
\]
Hence, any point satisfying $DC_{CORF}$ satisfies $DC_{ENF \times CORF - 1}$.

Case 2  The double-fault expression is equivalent to Expression (12) in Table 1 where $i_1 \leq i_2 \leq h_1$ and $i_1 < h_1$. The detection condition, denoted by $DC_{ENF \times CORF - 2}$, is “any point in $\left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1}^{h_1} TP_i(S) \right)$ or any point in $FP(S)$ such that $p_{i_1,1,j_2} + p_{i_2,j_2 + 1,k_2} = 0$”. We use $TC_{ENF \times CORF - 2}$

to denote the set of all points that satisfy $DC_{ENF \times CORF - 2}$, that is $TC_{ENF \times CORF - 2} = \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1}^{h_1} TP_i(S) \right) \right) \cup \{ \vec{t} \in FP(S) : p_{i_1,1,j_2} + p_{i_2,j_2 + 1,k_2} = 0 \}$. 

(A) Relationship between $DC_{ENF}$ and $DC_{ENF \times CORF - 2}$: The relationship R2 holds because of the following reasons:

(a) Some points in $TC_{ENF}$ can satisfy $DC_{ENF \times CORF - 2}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. For example, in $TC_{ENF}$, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_i(S) = \left( TP_i(S) \setminus \bigcup_{i \neq i_1}^{h_1} TP_i(S) \right)$. Note that, $\vec{t}$ satisfies $DC_{ENF \times CORF - 2}$ because $\vec{t} \in \left( TP_i(S) \setminus \bigcup_{i \neq i_1}^{h_1} TP_i(S) \right) \subseteq \left( \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i \neq i_1}^{h_1} TP_i(S) \right) \right) \subseteq TC_{ENF \times CORF - 2}$ (by using $A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B$).

(b) Some points in $TC_{ENF}$ cannot satisfy $DC_{ENF \times CORF - 2}$. Since $S$ is in IDNF, $NFP_{i,j}(S) \neq \emptyset$ for all $i$ and $j$. For example, in $TC_{ENF}$, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i,j}(S) \subseteq FP(S)$.

By definition of $NFP_{i,j}(S)$, $p_{i,j} = 1$ on $\vec{t}$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \times CORF - 2}$ because $\vec{t}$ is a false point of $S$ and $p_{i_1,1,j_2} + p_{i_2,j_2 + 1,k_2} = 0 + 1 = 1 \neq 0$ on $\vec{t}$.

Hence, only some, but not all, points satisfying $DC_{ENF}$ can satisfy $DC_{ENF \times CORF - 2}$.

(B) Relationship between $DC_{CORF}$ and $DC_{ENF \times CORF - 2}$: The relationship R6 holds because:
\[ DC_{CORF} \cap DC_{ENF \times CORF^{-2}} \]
\[ = \left\{ \vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \right\} \cap \left( \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=i_1}^{m} TP_i(S) \right) \right) \]
\[ \cup \{ \vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \]
\[ = \left( \left\{ \vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \right\} \cap \left( \left( \bigcup_{i=i_1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=i_1}^{m} TP_i(S) \right) \right) \right) \]
\[ \cup \{ \vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \]
\[ = \emptyset. \]

Hence, all points satisfying \( DC_{CORF} \) do not satisfy \( DC_{ENF \times CORF^{-2}} \).

5.1.6 TNF and TNF

In this section, we analyse the relationship between the double fault involving two TNFs. Since there are two TNFs, we use TNF1 and TNF2 to identify the two different TNFs in double fault \( TNF_1 \times TNF_2 \) for ease of understanding.

Let \( S \) be a Boolean expression in IDNF. As discussed in Section 2.2, if the \( i_1 \)-th term, \( p_{i_1} \), in \( S \) is wrongly negated, the corresponding detection condition, denoted by \( DC_{TNF_1} \), is “any point in \( UTP_{i_1}(S) \) or any point in \( FP(S) \)”. We use \( TC_{TNF_1} \) to denote the set of all points that satisfy \( DC_{TNF_1} \), that is \( TC_{TNF_1} = UTP_{i_1}(S) \cup FP(S) \).

Similarly, if the \( i_2 \)-th term, \( p_{i_2} \), in \( S \) is wrongly negated, the corresponding detection condition, denoted by \( DC_{TNF_2} \), is “any point in \( UTP_{i_2}(S) \) or any point in \( FP(S) \)”. We use \( TC_{TNF_2} \) to denote the set of all points that satisfy \( DC_{TNF_2} \), that is \( TC_{TNF_2} = UTP_{i_2}(S) \cup FP(S) \).

The double-fault expression is equivalent to Expression (13) in Table 1 where \( i_1 < i_2 \). The detection condition, denoted by \( DC_{TNF_1 \times TNF_2} \), is “any point in \( (TP_{i_1}(S) \cap TP_{i_2}(S)) \setminus \bigcup_{i=i_1, j_2}^{m} TP_i(S) \) or any point in \( FP(S) \)”. We use \( TC_{TNF_1 \times TNF_2} \) to denote the set of all points that satisfy \( DC_{TNF_1 \times TNF_2} \), that
is \( TC_{TNF_1 \times TNF_2} = \left( (TP_{i_1}(S) \cap TP_{i_2}(S)) \setminus \left( \bigcup_{i=1 \atop i \neq i_1, i_2}^{m} TP_i(S) \right) \right) \cup FP(S). \)

(A) Relationship between \( DC_{TNF_1} \) and \( DC_{TNF_1 \times TNF_2} \): The relationship R2 holds because of the following reasons:

(a) Some points in \( TC_{TNF_1} \) can satisfy \( DC_{TNF_1 \times TNF_2} \). For example, in \( TC_{TNF_1} \), false points of \( S \) (that is, points in \( FP(S) \)) satisfy \( DC_{TNF_1 \times TNF_2} \).

(b) Some points in \( TC_{TNF_1} \) cannot satisfy \( DC_{TNF_1 \times TNF_2} \). Since \( S \) is in IDNF, \( UTP_i(S) \neq \emptyset \) for all \( i \). For example, in \( TC_{TNF_1} \), we can find a point \( \bar{r} \) such that \( \bar{r} \in UTP_{i_1}(S) = (TP_{i_1}(S) \setminus \bigcup_{i=1 \atop i \neq i_1}^{m} TP_i(S)) \). Note that, \( \bar{r} \) does not satisfy \( DC_{TNF_1 \times TNF_2} \) because \( \bar{r} \notin FP(S) \), \( \bar{r} \notin TP_{i_2}(S) \) and \( i_1 \neq i_2 \).

Hence, only some, but not all, points satisfying \( DC_{TNF_1} \) can satisfy \( DC_{TNF_1 \times TNF_2} \).

(B) Relationship between \( DC_{TNF_2} \) and \( DC_{TNF_1 \times TNF_2} \): The relationship R5 holds because of the following reasons:

(a) Some points in \( TC_{TNF_2} \) can satisfy \( DC_{TNF_1 \times TNF_2} \). For example, in \( TC_{TNF_2} \), false points of \( S \) (that is, points in \( FP(S) \)) satisfy \( DC_{TNF_1 \times TNF_2} \).

(b) Some points in \( TC_{TNF_2} \) cannot satisfy \( DC_{TNF_1 \times TNF_2} \). Since \( S \) is in IDNF, \( UTP_i(S) \neq \emptyset \) for all \( i \). For example, in \( TC_{TNF_2} \), we can find a point \( \bar{r} \) such that \( \bar{r} \in UTP_{i_2}(S) = (TP_{i_2}(S) \setminus \bigcup_{i=1 \atop i \neq i_2}^{m} TP_i(S)) \). Note that, \( \bar{r} \) does not satisfy \( DC_{TNF_1 \times TNF_2} \) because \( \bar{r} \notin FP(S) \), \( \bar{r} \notin TP_{i_1}(S) \) and \( i_1 \neq i_2 \).

Hence, only some, but not all, points satisfying \( DC_{TNF_2} \) can satisfy \( DC_{TNF_1 \times TNF_2} \).
5.1.7 TNF and TOF

In this section, we analyse the relationship between the double fault TNF $\bowtie$ TOF and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the $i_1$-th term, $p_{i_1}$, in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{TNF}$, is “any point in $UTP_{i_1}(S)$ or any point in $FP(S)$”. We use $TC_{TNF}$ to denote the set of all points that satisfy $DC_{TNF}$, that is $TC_{TNF} = UTP_{i_1}(S) \cup FP(S)$.

Similarly, if the $i_2$-th term, $p_{i_2}$, in $S$ is wrongly omitted, the corresponding detection condition, denoted by $DC_{TOF}$, is “any point in $UTP_{i_2}(S)$”. We use $TC_{TOF}$ to denote the set of all points that satisfy $DC_{TOF}$, that is $TC_{TOF} = UTP_{i_2}(S)$.

The double-fault expression is equivalent to Expression (14) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TNF \bowtie TOF}$, is “any point in $(TP_{i_1}(S) \setminus \bigcup_{i \neq i_1,i_2} TP_i(S))$ or any point in $FP(S)$”. We use $TC_{TNF \bowtie TOF}$ to denote the set of all points that satisfy $DC_{TNF \bowtie TOF}$, that is $DC_{TNF \bowtie TOF} = (TP_{i_1}(S) \setminus \bigcup_{i \neq i_1,i_2} TP_i(S)) \cup FP(S)$.

(A) Relationship between $DC_{TNF}$ and $DC_{TNF \bowtie TOF}$: The relationship R1 holds because:

\[
TC_{TNF} = UTP_{i_1}(S) \cup FP(S) = (TP_{i_1}(S) \setminus \bigcup_{i \neq i_1} TP_i(S)) \cup FP(S) = (TP_{i_1}(S) \setminus (\bigcup_{i \neq i_1} TP_i(S)) \cup TP_{i_2}(S)) \cup FP(S) \subseteq (TP_{i_1}(S) \setminus (\bigcup_{i \neq i_1,i_2} TP_i(S)) \cup FP(S) \quad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B) \]

\[
= TC_{TNF \bowtie TOF}.
\]
Hence, any point satisfying $DC_{TNF}$ satisfies $DC_{TNF \ltimes TOF}$.

(B) Relationship between $DC_{TOF}$ and $DC_{TNF \ltimes TOF}$: The relationship R6 holds because:

$$TC_{TOF} \cap TC_{TNF \ltimes TOF}$$

$$= UTP_{i_2}(S) \cap \left( \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup FP(S) \right)$$

$$= \left( UTP_{i_2}(S) \cap \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup \left( UTP_{i_2}(S) \cap FP(S) \right)$$

$$\subseteq \left( UTP_{i_2}(S) \cap TP_{i_1}(S) \right) \cup \emptyset \quad \text{(By using } A \setminus B \subseteq A)$$

$$= \emptyset \quad \text{(By definition of } UTP_{i_2}(S) \text{ and } i_1 \neq i_2).$$

Hence, all points satisfying $DC_{TOF}$ do not satisfy $DC_{TNF \ltimes TOF}$.

5.1.8 TNF and DORF

In this section, we analyse the relationship between the double fault $TNF \ltimes DORF$ and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the $i_1$-th term, $p_{i_1}$ in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{TNF}$, is “any point in $UTP_{i_1}(S)$ or any point in $FP(S)$”. We use $TC_{TNF}$ to denote the set of all points that satisfy $DC_{TNF}$, that is $TC_{TNF} = UTP_{i_1}(S) \cup FP(S)$.

Similarly, if the subexpression $p_{i_2} + p_{i_2+1}$ in $S$ is wrongly implemented as $p_{i_2} \cdot p_{i_2+1}$, the corresponding detection condition, denoted by $DC_{DORF}$, is “any point in $UTP_{i_2}(S)$ or any point in $UTP_{i_2+1}(S)$”. We use $TC_{DORF}$ to denote the set of all points that satisfy $DC_{DORF}$, that is $TC_{DORF} = UTP_{i_2}(S) \cup UTP_{i_2+1}(S)$.

For $TNF \ltimes DORF$, there are two subcases.
Case 1  The double-fault expression is equivalent to Expression (15) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TNF \times DORF - 1}$, is "any point in $TP_{i_1}(S) \setminus \left( \bigcup_{i \neq i_1, i_2, i_2 + 1}^{m} TP_i(S) \right) \cup (TP_{i_2}(S) \cap TP_{i_2 + 1}(S))" or any point in $FP(S)". We use $TC_{TNF \times DORF - 1}$ to denote the set of all points that satisfy $DC_{TNF \times DORF - 1}$, that is $TC_{TNF \times DORF - 1} = \left( TP_{i_1}(S) \setminus \left( \bigcup_{i \neq i_1, i_2, i_2 + 1}^{m} TP_i(S) \right) \cup (TP_{i_2}(S) \cap TP_{i_2 + 1}(S)) \right) \cup FP(S)$.

(A) Relationship between $DC_{TNF}$ and $DC_{TNF \times DORF - 1}$: The relationship $R_1$ holds because:

$$TC_{TNF}$$

$$= UTP_{i_1}(S) \cup FP(S)$$

$$= \left( TP_{i_1}(S) \setminus \left( \bigcup_{i \neq i_1}^{m} TP_i(S) \right) \right) \cup FP(S)$$

$$= \left( TP_{i_1}(S) \setminus \left( \bigcup_{i \neq i_1, i_2, i_2 + 1}^{m} TP_i(S) \cup TP_{i_2}(S) \cup TP_{i_2 + 1}(S) \right) \right) \cup FP(S)$$

$$\subseteq \left( TP_{i_1}(S) \setminus \left( \bigcup_{i \neq i_1, i_2, i_2 + 1}^{m} TP_i(S) \cup (TP_{i_2}(S) \cap TP_{i_2 + 1}(S)) \right) \right) \cup FP(S) \quad (By \ using \ A \setminus (B \cup C \cup D) \subseteq A \setminus (B \cup (C \cap D)))$$

$$= TC_{TNF \times DORF - 1}.$$  

Hence, any point satisfying $DC_{TNF}$ satisfies $DC_{TNF \times DORF - 1}$.

(B) Relationship between $DC_{DORF}$ and $DC_{TNF \times DORF - 1}$: The relationship $R_6$ holds because:
Case 2  The double-fault expression is equivalent to Expression (16) in Table 1 where \( i_1 = i_2 \). The detection condition, denoted by \( DC_{TNF \times DORF - 2} \), is “any point in \( TP_{i_1} (S) \setminus \left( \bigcup_{i \neq i_1, i_1 + 1}^{m} TP_i (S) \right) \). We use \( TC_{TNF \times DORF - 2} \) to denote the set of points that satisfy \( DC_{TNF \times DORF - 2} \), that is \( TC_{TNF \times DORF - 2} = TP_{i_1} (S) \setminus \left( \bigcup_{i \neq i_1, i_1 + 1}^{m} TP_i (S) \right) \).

(A) Relationship between \( DC_{TNF} \) and \( DC_{TNF \times DORF - 2} \): The relationship R2 holds because of the following reasons:

(a) Some points in \( TC_{TNF} \) can satisfy \( DC_{TNF \times DORF - 2} \). Since \( S \) is in IDNF, \( UTP_i (S) \neq \emptyset \) for all \( i \). For example, in \( TC_{TNF} \), we can find a point \( \bar{t} \) such that \( \bar{t} \in UTP_{i_1} (S) = \left( TP_{i_1} (S) \setminus \bigcup_{i = 1}^{m} TP_i (S) \right) \setminus \bigcup_{i \neq i_1, i_1 + 1}^{m} TP_i (S) \) and \( \bar{t} \) satisfies \( DC_{TNF \times DORF - 2} \) because \( \bar{t} \in \left( TP_{i_1} (S) \setminus \bigcup_{i = 1}^{m} TP_i (S) \right) \subseteq TP_{i_1} (S) \setminus \bigcup_{i \neq i_1, i_1 + 1}^{m} TP_i (S) \) = \( TC_{TNF \times DORF - 2} \) (by using \( A \setminus (B \cup C) \subseteq (A \setminus B) \)).

(b) Some points in \( TC_{TNF} \) cannot satisfy \( DC_{TNF \times DORF - 2} \). For example, in \( TC_{TNF} \), false points of \( S \) (that is, points in \( FP(S) \)) do not satisfy \( DC_{TNF \times DORF - 2} \).
Hence, only some, but not all, points satisfying $DC_{TNF}$ can satisfy $DC_{TNF \times DORF - 2}$.

(B) Relationship between $DC_{DORF}$ and $DC_{TNF \times DORF - 2}$: The relationship R5 holds because of the following reasons:

(a) Some points in $TC_{DORF}$ can satisfy $DC_{TNF \times DORF - 2}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. Given that $i_1 = i_2$, we can find a point $\bar{r}$ in $TC_{DORF}$ such that $\bar{r} \in UTP_{i_2}(S) = UTP_{i_1}(S) = \left( TP_{i_1}(S) \setminus \bigcup_{i \neq i_1} TP_i(S) \right)$ and $\bar{r}$ satisfies $DC_{TNF \times DORF - 2}$.

(b) Some points in $TC_{DORF}$ cannot satisfy $DC_{TNF \times DORF - 2}$. Since $S$ is in IDNF, $UTP_i(S) \neq \emptyset$ for all $i$. Given that $i_1 = i_2$, we can find a point $\bar{r}$ in $TC_{DORF}$ such that $\bar{r} \in UTP_{i_2+1}(S) = UTP_{i_1+1}(S) = \left( TP_{i_1+1}(S) \setminus \bigcup_{i \neq i_1+1} TP_i(S) \right)$ and $\bar{r}$ does not satisfy $DC_{TNF \times DORF - 2}$ because $\bar{r} \notin TP_{i_1}(S)$.

Hence, only some, but not all, points satisfying $DC_{DORF}$ can satisfy $DC_{TNF \times DORF - 2}$.

5.1.9 TNF and CORF

In this section, we analyse the relationship between the double fault $TNF \times CORF$ and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the $i_1$-th term, $p_{i_1}$ in $S$ is wrongly negated, the corresponding detection condition, denoted by $DC_{TNF}$, is “any point in $UTP_{i_1}(S)$ or any point in $FP(S)$”. We use $TC_{TNF}$ to denote the set of all points that satisfy $DC_{TNF}$, that is $TC_{TNF} = UTP_{i_1}(S) \cup FP(S)$.

Similarly, if the $i_2$-th term, $p_{i_2}$, in $S$ is wrongly implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$, the corresponding detection condition, denoted by $DC_{CORF}$, is “any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$”. We use $TC_{CORF}$ to denote the set of all points that satisfy $DC_{CORF}$, that is $TC_{CORF} = \{ \bar{r} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1 \}$. 
The detection condition, denoted by $DC_{\text{TNF} \times \text{CORF}}$, is “any point in $UTP_i(S)$ such that $p_{i_1,j_2} + p_{i_2,j_2+1,k_2} = 0$ or any point in $FP(S)$”. We use $TC_{\text{TNF} \times \text{CORF}}$ to denote the set of all points that satisfy $DC_{\text{TNF} \times \text{CORF}}$, that is $TC_{\text{TNF} \times \text{CORF}} = \{ \bar{t} \in UTP_i(S) : p_{i_1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup FP(S)$.

(A) Relationship between $DC_{\text{TNF}}$ and $DC_{\text{TNF} \times \text{CORF}}$

First, we observe that $TC_{\text{TNF} \times \text{CORF}} = \{ \bar{t} \in UTP_i(S) : p_{i_1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup FP(S) \subseteq (UTP_i(S) \cup FP(S)) = TC_{\text{TNF}}$. Hence the relationship between $DC_{\text{TNF}}$ and $DC_{\text{TNF} \times \text{CORF}}$ depends on whether $TC_{\text{TNF} \times \text{CORF}} = TC_{\text{TNF}}$.

If $TC_{\text{TNF} \times \text{CORF}}$ and $TC_{\text{TNF}}$ are not equal, R2 holds. That means, some points in $TC_{\text{TNF}}$ do not satisfy $DC_{\text{TNF} \times \text{CORF}}$. Here is an example. Let $S = abc + \bar{a}bd + cd\bar{a}$. If the first term $abc$ is negated and the second term is wrongly split as $\bar{a}b + d$, $I_{\text{TNF}} = \bar{a}bc + \bar{a}bd + cd\bar{a}$ and $I_{\text{TNF} \times \text{CORF}} = \bar{a}bc + \bar{a}b + d + cd\bar{a}$. Now, the point 1111 $\in UTP_1(S) \subseteq TC_{\text{TNF}}$ can distinguish $S$ from $I_{\text{TNF}}$ because $S$ and $I_{\text{TNF}}$ evaluate to 1 and 0 on this point, respectively. However, it cannot be used to distinguish $S$ from $I_{\text{TNF} \times \text{CORF}}$ because both $S$ and $I_{\text{TNF} \times \text{CORF}}$ evaluate to 1 on 1111.

On the other hand, if $TC_{\text{TNF} \times \text{CORF}} = TC_{\text{TNF}}$, R1 holds. That means, $p_{i_1,j_2} + p_{i_2,j_2+1,k_2}$ evaluate to 0 on all points in $UTP_i(S)$. Under such special circumstances, all test cases that can detect $\text{TNF} \times \text{CORF} - 1$ can also be used to detect $\text{TNF}$. Here is an example. Let $S = ab + cd + ac + bd$. Note that $UTP_1(S) = \{1100\}$. If the first term $ab$ is negated and the second term is wrongly split as $c + d$, $I_{\text{TNF}} = \bar{a}b + cd + ac + bd$ and $I_{\text{TNF} \times \text{CORF}} = \bar{a}b + c + d + ac + bd$. Now, $TC_{\text{TNF}} = UTP_1(S) \cup FP(S) = \{1100\} \cup FP(S)$, and $TC_{\text{TNF} \times \text{CORF}} = \{ \bar{t} \in UTP_1(S) : c + d = 0 \} \cup FP(S) = \{1100\} \cup FP(S) = TC_{\text{TNF}}$ because $c + d = 0$ on all points of $UTP_1(S) = \{1100\}$.

(B) Relationship between $DC_{\text{CORF}}$ and $DC_{\text{TNF} \times \text{CORF}}$

The relationship R4 holds because:
\[ TC_{\text{CORF}} = \{ t \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \} \]
\[ \subseteq \{ t \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup FP(S) \]
\[ \subseteq \{ t \in UTP_{i_2}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \]
\[ = TC_{\text{TNF} \ltimes \text{CORF} - 1}. \]

Hence, any point satisfying \( DC_{\text{CORF}} \) satisfies \( DC_{\text{TNF} \ltimes \text{CORF} - 1}. \)

**Case 2** The double-fault expression is equivalent to Expression (18) in Table 1 where \( i_1 = i_2 \). The detection condition, denoted by \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \), is “any point in \( UTP_{i_2}(S) \) or any point in \( FP(S) \) such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \)”. We use \( TC_{\text{TNF} \ltimes \text{CORF} - 2} \) to denote the set of all points that satisfy \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \), that is \( TC_{\text{TNF} \ltimes \text{CORF} - 2} = UTP_{i_2}(S) \cup \{ t \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \}. \)

(A) Relationship between \( DC_{\text{TNF}} \) and \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \): The relationship R2 holds because of the following reasons:

(a) Some points in \( TC_{\text{TNF}} \) can satisfy \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \). Since \( S \) is in \( \text{IDNF} \), \( UTP_i(S) \neq \emptyset \) for all \( i \). Given that \( i_1 = i_2 \), we can find a point \( t \) in \( TC_{\text{TNF}} \) such that \( t \in UTP_{i_1}(S) = UTP_{i_2}(S) \) and \( t \) satisfies \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \).

(b) Some points in \( TC_{\text{TNF}} \) cannot satisfy \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \). Since \( S \) is in \( \text{IDNF} \), \( \text{NFP}_{i,j}(S) \neq \emptyset \) for all \( i \) and \( j \). For example, in \( TC_{\text{TNF}} \), we can find a point \( t \) such that \( t \in \text{NFP}_{i_2,j_2}(S) \subseteq FP(S) \).

By definition of \( \text{NFP}_{i_2,j_2}(S) \), \( p_{i_2,j_2} = 1 \) on \( t \). Note that, \( t \) does not satisfy \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \) because \( t \) is a false point of \( S \) and \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 + 1 = 1 \neq 0 \) on \( t \).

Hence, only some, but not all, points satisfying \( DC_{\text{TNF}} \) can satisfy \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \).

(B) Relationship between \( DC_{\text{CORF}} \) and \( DC_{\text{TNF} \ltimes \text{CORF} - 2} \): The relationship R6 holds because:
\[ \text{TC}_{\text{CORF}} \cap \text{TC}_{\text{TNF} \times \text{CORF}} = 2 \]
\[ = \{ \vec{t} \in \text{FP}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \} \cap \{ \vec{t} \in \text{FP}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \]
\[ = (\{ \vec{t} \in \text{FP}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \} \cap \text{UTP}_{i_1}(S)) \]
\[ \cup (\{ \vec{t} \in \text{FP}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \} \cap \{ \vec{t} \in \text{FP}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \}) \]
\[ = \emptyset \quad \text{(because } i_1 = i_2). \]

Hence, all points satisfying \( DC_{\text{CORF}} \) do not satisfy \( DC_{\text{TNF} \times \text{CORF}} = 2 \).

5.1.10 TOF and TOF

In this section, we analyse the relationship between the double fault involving two TOFs. Since there are two TOFs, we use TOF1 and TOF2 to identify the two different TOFs in double fault \( \text{TOF1} \times \text{TOF2} \) for ease of understanding.

Let \( S \) be a Boolean expression in IDNF. As discussed in Section 2.2, if the \( i_1 \)-th term, \( p_{i_1} \), in \( S \) is wrongly omitted, the corresponding detection condition, denoted by \( DC_{\text{TOF1}} \), is “any point in \( \text{UTP}_{i_1}(S) \)”. We use \( \text{TC}_{\text{TOF1}} \) to denote the set of all points that satisfy \( DC_{\text{TOF1}} \), that is \( \text{TC}_{\text{TOF1}} = \text{UTP}_{i_1}(S) \).

Similarly, if the \( i_2 \)-th term, \( p_{i_2} \), in \( S \) is wrongly omitted, the corresponding detection condition, denoted by \( DC_{\text{TOF2}} \), is “any point in \( \text{UTP}_{i_2}(S) \)”. We use \( \text{TC}_{\text{TOF2}} \) to denote the set of all points that satisfy \( DC_{\text{TOF2}} \), that is \( \text{TC}_{\text{TOF2}} = \text{UTP}_{i_2}(S) \).

The double-fault expression is equivalent to Expression (19) in Table 1 where \( i_1 < i_2 \). The detection condition, denoted by \( DC_{\text{TOF1} \times \text{TOF2}} \), is “any point in \( (\text{TP}_{i_1}(S) \cup \text{TP}_{i_2}(S)) \) \( \setminus \left( \bigcup_{i \neq i_1,i_2}^{m} \text{TP}_{i}(S) \right) \)”. We use \( \text{TC}_{\text{TOF1} \times \text{TOF2}} \) to denote the set of all points that satisfy \( DC_{\text{TOF1} \times \text{TOF2}} \), that is \( \text{TC}_{\text{TOF1} \times \text{TOF2}} = (\text{TP}_{i_1}(S) \cup \text{TP}_{i_2}(S)) \setminus \left( \bigcup_{i \neq i_1,i_2}^{m} \text{TP}_{i}(S) \right) \).

(A) Relationship between \( DC_{\text{TOF1}} \) and \( DC_{\text{TOF1} \times \text{TOF2}} \): The relationship R1 holds because:
\[
DC_{TOF1} = UTP_{i_1}(S) \\
= TP_{i_1}(S) \setminus \left( \bigcup_{i \neq i_1}^{m} TP_i(S) \right) \\
= TP_{i_1}(S) \setminus \left( \left( \bigcup_{i = 1}^{m} TP_i(S) \right) \cup TP_{i_2}(S) \right) \\
\subseteq TP_{i_1}(S) \setminus \left( \bigcup_{i = 1}^{m} TP_i(S) \right) \quad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B) \\
\subseteq \left( TP_{i_1}(S) \cup TP_{i_2}(S) \right) \setminus \bigcup_{i = 1}^{m} TP_i(S) \quad \text{(By using } A \setminus B \subseteq (A \cup C) \setminus B) \\
= TC_{TOF1 \times TOF2}.
\]

Hence, any point satisfying \( DC_{TOF1} \) satisfies \( DC_{TOF1 \times TOF2} \).

(B) Relationship between \( DC_{TOF2} \) and \( DC_{TOF1 \times TOF2} \): The relationship R4 holds because:

\[
DC_{TOF2} = UTP_{i_2}(S) \\
= TP_{i_2}(S) \setminus \left( \bigcup_{i \neq i_2}^{m} TP_i(S) \right) \\
= TP_{i_2}(S) \setminus \left( \left( \bigcup_{i = 1}^{m} TP_i(S) \right) \cup TP_{i_1}(S) \right) \\
\subseteq TP_{i_2}(S) \setminus \left( \bigcup_{i = 1}^{m} TP_i(S) \right) \quad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B) \\
\subseteq \left( TP_{i_1}(S) \cup TP_{i_2}(S) \right) \setminus \bigcup_{i = 1}^{m} TP_i(S) \quad \text{(By using } A \setminus B \subseteq (A \cup C) \setminus B) \\
= TC_{TOF1 \times TOF2}.
\]

Hence, any point satisfying \( DC_{TOF2} \) satisfies \( DC_{TOF1 \times TOF2} \).
5.1.11 TOF and DORF

In this section, we analyse the relationship between the double fault TOF×DORF and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the $i_1$-th term, $p_{i_1}$ in $S$ is wrongly omitted, the corresponding detection condition, denoted by $DC_{TOF}$, is “any point in $UTP_{i_1}(S)$”. We use $TC_{TOF}$ to denote the set of all points that satisfy $DC_{TOF}$, that is $TC_{TOF} = UTP_{i_1}(S)$.

Similarly, if the subexpression $p_{i_2} + p_{i_2+1}$ in $S$ is wrongly implemented as $p_{i_2} \cdot p_{i_2+1}$, the corresponding detection condition, denoted by $DC_{DORF}$, is “any point in $UTP_{i_2}(S)$ or any point in $UTP_{i_2+1}(S)$”. We use $TC_{DORF}$ to denote the set of all points that satisfy $DC_{DORF}$, that is $TC_{DORF} = UTP_{i_2}(S) \cup UTP_{i_2+1}(S)$.

The double-fault expression is equivalent to Expression (20) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TOF \times DORF}$, is “any point in $\left( (TP_{i_{1}}(S) \cup TP_{i_{2}}(S)) \setminus \left( \bigcup_{i=1}^{m} TP_{i}(S) \right) \bigg) \right) \setminus \left( \bigcup_{i=1}^{m} TP_{i}(S) \right)$”.

(A) Relationship between $DC_{TOF}$ and $DC_{TOF \times DORF}$: The relationship R1 holds because:
$$TC_{TOF}$$

$$= UTP_{i_1}(S)$$

$$= TP_{i_1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)_{i \neq i_1}$$

$$= TP_{i_1}(S) \setminus \left( \bigcup_{i \neq i_1,i_2}^{m} TP_i(S) \cap TP_{i_2}(S) \right)$$

$$\subseteq TP_{i_1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)$$  (By using $A \setminus (B \cup C) \subseteq A \setminus B$)

$$\subseteq \left( TP_{i_1}(S) \cup TP_{i_2}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right)$$  (By using $(A \setminus B) \subseteq (A \cup C) \setminus B$)

$$\subseteq \left( \left( TP_{i_1}(S) \cup TP_{i_2}(S) \right) \setminus \bigcup_{i \neq i_1,i_2}^{m} TP_i(S) \right) \cup \left( \left( TP_{i_1}(S) \cup TP_{i_2+1}(S) \right) \setminus \bigcup_{i=1}^{m} TP_i(S) \right)$$

$$= TC_{TOF \times DORF}.$$  

Hence, any point satisfying $DC_{TOF}$ satisfies $DC_{TOF \times DORF}.$

(B) Relationship between $DC_{DORF}$ and $DC_{TOF \times DORF}$: The relationship R4 holds because:


\[ DC_{DORF} \]

\[ = \ UTP_{i_2}(S) \cup UTP_{i_2+1}(S) \]

\[ = \left( TP_{i_2}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup \left( TP_{i_2+1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \]

\[ = \left( TP_{i_2}(S) \setminus \left( \bigcup_{i \neq i_1,i_2}^{m} TP_i(S) \right) \right) \cup \left( TP_{i_2+1}(S) \setminus \left( \bigcup_{i \neq i_1,i_2+1}^{m} TP_i(S) \right) \right) \]

\[ \subseteq \left( TP_{i_2}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup \left( TP_{i_2+1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \]

\[ \subseteq \left( TP_{i_1}(S) \cup TP_{i_2}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \]

\[ \subseteq \left( TP_{i_1}(S) \cup TP_{i_2}(S) \right) \setminus \left( \bigcup_{i \neq i_1,i_2}^{m} TP_i(S) \right) \]

\[ = TC_{TOF \times DORF}. \]

Hence, any point satisfying \( DC_{DORF} \) satisfies \( DC_{TOF \times DORF} \).

5.1.12 TOF and CORF

In this section, we analyse the relationship between the double fault \( TOF \times CORF \) and its corresponding single faults.

Let \( S \) be a Boolean expression in IDNF. As discussed in Section 2.2, if the \( i_1 \)-th term, \( p_{i_1} \) in \( S \) is wrongly omitted, the corresponding detection condition, denoted by \( DC_{TOF} \), is “any point in \( UTP_{i_1}(S) \)”. We use \( TC_{TOF} \) to denote the set of all points that satisfy \( DC_{TOF} \), that is \( TC_{TOF} = UTP_{i_1}(S) \).

Similarly, if the \( i_2 \)-th term, \( p_{i_2} \), in \( S \) is wrongly implemented as \( p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_2} \) where \( p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,1,j_2+1,k_2} \), the corresponding detection condition, denoted by \( DC_{CORF} \), is “any point in \( FP(S) \) such that \( p_{i_2,1,j_2} + p_{i_2,1,j_2+1,k_2} = 1 \)”. We use \( TC_{CORF} \) to denote the set of all points that satisfy...
\( DC_{\text{CORF}} \), that is \( TC_{\text{CORF}} = \{ \vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \} \).

The double-fault expression is equivalent to Expression (21) in Table 1 where \( i_1 < i_2 \). The detection condition, denoted by \( DC_{\text{TOF} \times \text{CORF}} \), is “any point in \( UTP_{i_1}(S) \) such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \) or any point in \( FP(S) \) such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \)”. We use \( TC_{\text{TOF} \times \text{CORF}} \) to denote the set of all points that satisfy \( DC_{\text{TOF} \times \text{CORF}} \), that is \( TC_{\text{TOF} \times \text{CORF}} = \{ \vec{t} \in UTP_{i_1}(S) : p_{i_1,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup \{ \vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \} \).

(A) Relationship between \( DC_{\text{TOF}} \) and \( DC_{\text{TOF} \times \text{CORF}-1} \):

Note that, \( TC_{\text{TOF}} = UTP_{i_1}(S) \) and \( TC_{\text{TOF} \times \text{CORF}} = \{ \vec{t} \in UTP_{i_1}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup \{ \vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \} \).

Hence, the relationship \( DC_{\text{TOF}} \) and \( DC_{\text{TOF} \times \text{CORF}-1} \) depends on whether there are points in \( UTP_{i_1}(S) \) such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \). There are three subcases.

(a) Some points in \( UTP_{i_1}(S) \) are such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \) and some points in \( UTP_{i_1}(S) \) are such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \). Then, R2 holds. Here is an example. Let \( S = ab + cd + ef \). If the first term \( ab \) is omitted and the second term is wrongly split as \( c + d \), \( I_{\text{TOF}} = cd + ef \) and \( I_{\text{TOF} \times \text{CORF}-1} = c + d + ef \). Note that, the unique true point \( 110010 \) in \( UTP_1(S) = TC_{\text{TOF}} \) can detect \( I_{\text{TOF} \times \text{CORF}-1} \) because \( c + d = 0 \) on this point. However, another unique true point \( 110110 \) in \( UTP_1(S) = TC_{\text{TOF}} \) does not satisfy \( DC_{\text{TOF} \times \text{CORF}-1} \) because \( c + d = 1 \) on this point.

(b) All points in \( UTP_{i_1}(S) \) are such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \). Then, R1 holds. Here is an example. Let \( S = ab + cd + ac + bd \). If the first term \( ab \) is omitted and the second term is wrongly split as \( c + d \), \( I_{\text{TOF}} = cd + ac + bd \) and \( I_{\text{TOF} \times \text{CORF}-1} = c + d + ac + bd \). Note that \( TC_{\text{TOF}} = UTP_1(S) = \{ 1100 \} \), and \( TC_{\text{TOF} \times \text{CORF}-1} = \{ \vec{t} \in UTP_1(S) : c + d = 0 \} = \{ 1100 \} \) because \( c + d = 0 \) on \( 1100 \). Hence, any point that can detect \( I_{\text{TOF}} \) can also detect \( I_{\text{TOF} \times \text{CORF}-1} \).

(c) All points in \( UTP_{i_1}(S) \) are such that \( p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \). Then, R3 holds. Let \( S = abc + \bar{a}bd + cd \). If the first term \( abc \) is omitted and the second term is wrongly split as \( \bar{a}b + d \),
\[ I_{TOF} = \bar{a}bd + c\bar{d} \text{ and } I_{TOF \times CORF -1} = \bar{a}b + d + c\bar{d}. \] Note that, \( TC_{TOF} = UTP_1(S) = \{1111\}. \)

The point 1111 can distinguish \( S \) from \( I_{TOF} \) because \( S \) and \( I_{TOF} \) evaluate to 1 and 0 on this point, respectively. However, it cannot be used to distinguish \( S \) from \( I_{TOF \times CORF -1} \) because both \( S \) and \( I_{TOF \times CORF -1} \) evaluate to 1 on 1111.

(B) Relationship between \( DC_{CORF} \) and \( DC_{TOF \times CORF} \): The relationship R4 holds because:

\[
TC_{CORF} = \{ \bar{t} \in FP(S) : p_{i_1, j_2} + p_{i_2, j_2+1, k_2} = 1 \} \subseteq \{ \bar{t} \in UTP_{i_1}(S) : p_{i_1, j_2} + p_{i_2, j_2+1, k_2} = 0 \} \cup \{ \bar{t} \in FP(S) : p_{i_1, j_2} + p_{i_2, j_2+1, k_2} = 1 \} = TC_{TOF \times CORF}.
\]

Hence, any point satisfying \( DC_{CORF} \) satisfies \( DC_{TOF \times CORF -1} \).

5.1.13 DORF and DORF

In this section, we analyse the relationship between the double fault involving two DORFs. Since there are two DORFs, we use DORF1 and DORF2 to identify the two different DORFs in double fault DORF1 \( \times \) DORF2 for ease of understanding.

Let \( S \) be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression \( p_{i_2} + p_{i_2+1} \) in \( S \) is wrongly implemented as \( p_{i_2} \cdot p_{i_2+1} \), the corresponding detection condition, denoted by \( DC_{DORF1} \), is “any point in \( UTP_{i_1}(S) \) or any point in \( UTP_{i_1+1}(S) \)”. We use \( TC_{DORF1} \) to denote the set of all points that satisfy \( DC_{DORF1} \), that is \( TC_{DORF1} = UTP_{i_1}(S) \cup UTP_{i_1+1}(S) \).

Similarly, if the subexpression \( p_{i_2} + p_{i_2+1} \) in \( S \) is wrongly implemented as \( p_{i_2} \cdot p_{i_2+1} \), the corresponding detection condition, denoted by \( DC_{DORF2} \), is “any point in \( UTP_{i_2}(S) \) or any point in \( UTP_{i_2+1}(S) \)”. We use \( TC_{DORF2} \) to denote the set of all points that satisfy \( DC_{DORF2} \), that is \( TC_{DORF2} = UTP_{i_2}(S) \cup UTP_{i_2+1}(S) \).

For DORF1 \( \times \) DORF2, there are two subcases.
Case 1  The double-fault expression is equivalent to Expression (22) in Table 1 where $i_1 + 1 < i_2$.

The detection condition, denoted by $DC_{DORF1 \times DORF2^{1-1}}$, is "any point in $\bigcup_{i=i_1, i_1+1, i_2, i_2+1} TP_i(S)$ \left(\bigcup_{i=1}^{m} TP_i(S) \cup (TP_{i_1}(S) \cap TP_{i_1+1}(S)) \cup (TP_{i_2}(S) \cap TP_{i_2+1}(S))\right)^\bigcup$. We use $TC_{DORF1 \times DORF2^{1-1}}$ to denote the set of all points that satisfy $DC_{DORF1 \times DORF2^{1-1}}$, that is $TC_{DORF1 \times DORF2^{1-1}} = (\bigcup_{i=i_1, i_1+1, i_2, i_2+1} TP_i(S)) \setminus \left( (\bigcup_{i=1}^{m} TP_i(S) \cup (TP_{i_1}(S) \cap TP_{i_1+1}(S)) \cup (TP_{i_2}(S) \cap TP_{i_2+1}(S))\right)^\bigcup$.

(A) Relationship between $DC_{DORF1}$ and $DC_{DORF1 \times DORF2^{1-1}}$: The relationship R1 holds because:
\[ TC_{DORF1} \]
\[ = UTP_{i_1} (S) \cup UTP_{i_1+1} (S) \]
\[ = \left( \bigcup_{i=1}^{m} TP_{i_1} (S) \right) \setminus \left( \bigcup_{i \neq i_1}^{m} TP_{i_1} (S) \right) \bigcup \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1} (S) \right) \]
\[ \cup \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1+1} (S) \right) \setminus \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1} (S) \right) \]
\[ \cup \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1+1} (S) \right) \setminus \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1} (S) \right) \]
\[ \subseteq \left( \bigcup_{i=1}^{m} TP_{i_1} (S) \right) \setminus \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1} (S) \right) \]
\[ \cup \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1+1} (S) \right) \setminus \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1} (S) \right) \]
\[ \cup \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1+1} (S) \right) \setminus \left( \bigcup_{i \neq i_1, i_1+1, i_2, i_2+1}^{m} TP_{i_1} (S) \right) \]
\[ = TC_{DORF1 \& DORF2-1} \]

Hence, any point satisfying \( DC_{DORF1} \) satisfies \( DC_{DORF1 \& DORF2-1} \).

(B) Relationship between \( DC_{DORF2} \) and \( DC_{DORF2 \& DORF2-1} \): The relationship R4 holds because:
\[TC_{DORF2}\]
\[=\ UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\]
\[\subseteq (\bigcup_{i=i_1,i_1+1,i_2,i_2+1}^m TP_i(S)) \setminus \left(\bigcup_{i=i_1}^m TP_i(S) \cup (TP_{i_1}(S) \cap TP_{i_1+1}(S)) \cup (TP_{i_2}(S) \cap TP_{i_2+1}(S))\right)\]
\[= TC_{DORF1 \times DORF2 - 1}\]
(By using a similar argument as in (A) above)

Hence, any point satisfying \(DC_{DORF2}\) satisfies \(DC_{DORF1 \times DORF2 - 1}\).

Case 2  The double-fault expression is equivalent to Expression (23) in Table 1 where \(i_1 + 1 = i_2\).

The detection condition, denoted by \(DC_{DORF1 \times DORF2 - 2}\), is “any point in \(\bigcup_{i=i_1,i_1+1,i_1+2}^m TP_i(S)\) \setminus \left(\bigcup_{i=i_1,i_1+1,i_1+2}^m TP_i(S) \cup (TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S))\right)\)”.

We use \(TC_{DORF1 \times DORF2 - 2}\) to denote the set of all points that satisfy \(DC_{DORF1 \times DORF2 - 2}\), that is
\[TC_{DORF1 \times DORF2 - 2} = \left(\bigcup_{i=i_1,i_1+1,i_1+2}^m TP_i(S)\right) \setminus \left(\bigcup_{i=i_1,i_1+1,i_1+2}^m TP_i(S) \cup (TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S))\right)\).

(A) Relationship between \(DC_{DORF1}\) and \(DC_{DORF1 \times DORF2 - 2}\): The relationship R1 holds because:
\( TC_{DORF1} \)
\[
= UTP_{i_1}(S) \cup UTP_{i_1+1}(S)
\]
\[
= \left( TP_{i_1}(S) \setminus \bigcup_{i=1}^{m} TP_i(S) \right) \cup \left( TP_{i_1+1}(S) \setminus \bigcup_{i=1}^{m} TP_i(S) \right)
\]
\[
= \left( TP_{i_1}(S) \setminus \bigcup_{i \neq i_1, i_1+1, i_1+2} TP_i(S) \right) \cup \left( TP_{i_1+1}(S) \setminus \bigcup_{i \neq i_1, i_1+1, i_1+2} TP_i(S) \right)
\]
\[
\subseteq \left( TP_{i_1}(S) \setminus \bigcup_{i = 1}^{m} TP_i(S) \right) \cup \left( TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S) \right)
\]
\[
= TC_{DORF1 \times DORF2-2}
\]
Hence, any points satisfying \( DC_{DORF1} \) can satisfy \( DC_{DORF1 \times DORF2-2} \).

(B) Relationship between \( DC_{DORF2} \) and \( DC_{DORF1 \times DORF2-2} \): The relationship R4 holds because:

\( TC_{DORF2} \)
\[
= UTP_{i_1+1}(S) \cup UTP_{i_1+2}(S)
\]
\[
\subseteq \left( \bigcup_{i = i_1, i_1+2, i_1+2} TP_i(S) \right) \cup \left( \bigcup_{i = 1}^{m} TP_i(S) \right) \cup \left( TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S) \right)
\]
\[
= TC_{DORF1 \times DORF2-2} \quad \text{(By using a similar argument as in (A) above)}
\]

Hence, any points satisfying \( DC_{DORF2} \) can satisfy \( DC_{DORF1 \times DORF2-2} \).
5.1.14 DORF and CORF

In this section, we analyse the relationship between the double fault DORF×CORF and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + p_{i_1+1}$ in $S$ is wrongly implemented as $p_{i_1} \cdot p_{i_1+1}$, the corresponding detection condition, denoted by $DC_{DORF}$, is “any point in $UTP_{j_1}(S)$ or any point in $UTP_{i_1+1}(S)$”. We use $TC_{DORF}$ to denote the set of all points that satisfy $DC_{DORF}$, that is $TC_{DORF} = UTP_{j_1}(S) \cup UTP_{i_1+1}(S)$.

Similarly, if the $i_2$-th term, $p_{i_2}$, in $S$ is wrongly implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2}$, the corresponding detection condition, denoted by $DC_{CORF}$, is “any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1$”. We use $TC_{CORF}$ to denote the set of all points that satisfy $DC_{CORF}$, that is $TC_{CORF} = \{ \mathbf{\bar{t}} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \}$.

For DORF×CORF, there are two subcases.

**Case 1** The double-fault expression is equivalent to Expression (24) in Table 1 where $i_1 < i_2 - 1$.

The detection condition, denoted by $DC_{DORF\times CORF-1}$, is “any point in $UTP_{j_1}(S) \cup UTP_{i_1+1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0$ or any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1$”. We use $TC_{DORF\times CORF-1}$ to denote the set of all points that satisfy $DC_{DORF\times CORF-1}$, that is $TC_{DORF\times CORF-1} = \{ \mathbf{\bar{t}} \in (UTP_{j_1}(S) \cup UTP_{i_1+1}(S)) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup \{ \mathbf{\bar{t}} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \}$.

(A) Relationship between $DC_{DORF}$ and $DC_{DORF\times CORF-1}$: We observe that $TC_{DORF} = UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ and $TC_{DORF\times CORF-1} = \{ \mathbf{\bar{t}} \in (UTP_{j_1}(S) \cup UTP_{i_1+1}(S)) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0 \} \cup \{ \mathbf{\bar{t}} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1 \}$. Hence, the relationship between $DC_{DORF}$ and $DC_{DORF\times CORF-1}$ depends on whether there are points in $UTP_{j_1}(S) \cup UTP_{i_1+1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0$. There are three cases:

(a) Some points in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0$ and some

...
points in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1$. Then, R2 holds. Here is an example. Let $S = ab + cd + ef$. If the subexpression $ab + cd$ is implemented as $abcd$ and the third term $ef$ is wrongly split as $e + f$, $I_{DORF} = abcd + ef$ and $I_{DORF \ast CORF - 1} = abcd + e + f$. Note that, the unique true point 110000 in $UTP_1(S) \subseteq TC_{DORF}$ can detect $I_{DORF \ast CORF - 1}$ because $e + f$ evaluates to 0 on this point in $UTP_1(S)$. However, another unique true point 110110 in $UTP_1(S) \subseteq TC_{DORF}$ is such that $I_{DORF \ast CORF - 1}(110110) = 1$, and hence cannot be used to detect $I_{DORF \ast CORF - 1}$.

(b) All points in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0$. Then, R1 holds. Here is an example. Let $S = ab + cd + ef + ae + bf + ce + df$. If the subexpression $ab + cd$ is wrongly implemented as $abcd$ and the third term is wrongly split as $e + f$, $I_{DORF} = abcd + ef + ae + bf + ce + df$ and $I_{DORF \ast CORF - 1} = abcd + e + f + ae + bf + ce + df$. Note that $TC_{DORF} = UTP_1(S) \cup UTP_2(S) = \{110000, 110100, 111000, 001100, 011100, 101100\}$, and $TC_{DORF \ast CORF - 1} = \{i \in UTP_1(S) \cup UTP_2(S) : e + f = 0\} = UTP_1(S) \cup UTP_2$ because $e + f = 0$ on all points in $TC_{DORF}$. Hence, $TC_{DORF} = TC_{DORF \ast CORF - 1}$.

(c) All points $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1$. Then, R3 holds. Here is an example. Let $S = ab + cd + ac$. If the subexpression $ab + cd$ is wrongly implemented as $abcd$ and the third term is wrongly split as $a + c$, $I_{DORF} = abcd + ac$ and $I_{DORF \ast CORF - 1} = abcd + a + c$. Note that $TC_{DORF} = UTP_1(S) \cup UTP_2(S) = \{1100, 1101, 1110, 0011, 0111, 1011\}$, and $TC_{DORF \ast CORF - 1} = \{i \in UTP_1(S) \cup UTP_2(S) : a + c = 0\} = 0$ because $a + c = 1$ on all points in $UTP_1(S) \cup UTP_2(S)$. Hence, all points in $TC_{DORF}$ cannot be used to detect $I_{DORF \ast CORF - 1}$.

(B) Relationship between $DC_{CORF}$ and $DC_{DORF \ast CORF - 1}$: The relationship R4 holds because:

\[
DC_{CORF} = \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1\}
\]
\[
\subseteq \{\vec{t} \in (UTP_{i_1}(S) \cup UTP_{i_1+1}(S)) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_2} = 1\}
\]
\[
= TC_{DORF \ast CORF - 1}.
\]
Hence any point satisfying $DC_{CORF}$ satisfies $DC_{DORF \ast CORF - 1}$. 
Case 2  The double-fault expression is equivalent to Expression (25) in Table 1 where \(i_1 = i_2\). The detection condition, denoted by \(DC_{DORF \times CORF -2}\), is “any point in \(UTP_{i_1+1}(S)\) such that \(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 0\) or any point in \(FP(S)\) such that \(p_{i_1,1,j_1} = 1\)”. We use \(TC_{DORF \times CORF -2}\) to denote the set of all points that satisfy \(DC_{DORF \times CORF -2}\), that is \(TC_{DORF \times CORF -2} = \{\vec{t} \in UTP_{i_1+1}(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 1\}\).

(A) Relationship between \(DC_{DORF}\) and \(DC_{DORF \times CORF -2}\): \(TC_{DORF} = UTP_{i_1}(S) \cup UTP_{i_1+1}(S)\) and 
\(TC_{DORF \times CORF -2} = \{\vec{t} \in UTP_{i_1+1}(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 1\}\).

Hence, the relationship between \(DC_{DORF}\) and \(DC_{DORF \times CORF -2}\) depends on whether there are points in \(UTP_{i_1+1}\) such that \(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 0\). There are two cases:

(a) Some points in \(UTP_{i_1+1}(S)\) are such that \(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 0\). Then R2 holds because of the following reasons:

(1) Some points in \(TC_{DORF}\) can satisfy \(DC_{DORF \times CORF -2}\). For example, we can find a point \(\vec{t}\) such that \(\vec{t} \in UTP_{i_1+1}(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 0\) \(\subseteq UTP_{i_1+1}(S) \subseteq TC_{DORF}\). Note that, \(\vec{t}\) satisfies \(DC_{DORF \times CORF -2}\) because \(\vec{t} \in UTP_{i_1+1}(S)\) and \(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 0\).

(2) Some points in \(TC_{DORF}\) cannot satisfy \(DC_{DORF \times CORF -2}\). We can find a point \(\vec{t}\) such that \(\vec{t} \in UTP_{i_1}(S) \subseteq TC_{DORF}\). Note that, \(\vec{t}\) does not satisfy \(DC_{DORF \times CORF -2}\) because \(\vec{t} \notin UTP_{i_1+1}(S)\) and \(\vec{t} \notin FP(S)\).

(b) All points \(\vec{t}\) in \(UTP_{i_1+1}(S)\) are such that \(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} = 1\). Then R3 holds. Here is an example, let \(S = ab + ac + d\). If the first two terms \(ab\) and \(ac\) are implemented as \(abac\) and the first term is wrongly split as \(a + b\), \(I_{DORF} = abac + d\) and \(I_{DORF \times CORF -2} = a + bac + d\). Note that, \(TC_{DORF} = UTP_{1}(S) \cup UTP_{2}(S) = \{1100, 1010\}\). The two points \(1100\) and \(1010\) can distinguish \(S\) from \(I_{DORF}\) because \(S\) and \(I_{DORF}\) evaluate to \(1\) and \(0\) on these points, respectively. However, they cannot be used to distinguish \(S\) from \(I_{DORF \times CORF -2}\) because both \(S\) and \(I_{DORF \times CORF -2}\) evaluate to \(1\) on these points.
(B) Relationship between $DC\text{\textsubscript{CORF}}$ and $DC\text{\textsubscript{DORF\times CORF}}$–2: The relationship R5 holds because of the following reasons:

(a) Some points in $TC\text{\textsubscript{CORF}}$ can satisfy $DC\text{\textsubscript{DORF\times CORF}}$–2. Since $S$ is in IDNF, $NFP_{i,j}(S) \neq \emptyset$ for all $i$ and $j$. By definition of $NFP_{i,j}(S)$, all literals in $p_{ii}$ evaluate to 1 except $x_{ki}^i = 0$. Hence $p_{i,1,j_1} + p_{i,1,j_1+k_1} = 1 + 0 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i,j_1}(S) \subseteq \{\vec{t} \in FP(S) : p_{i,1,j_1} + p_{i,1,j_1+k_1} = 1\} \subseteq TC\text{\textsubscript{CORF}}$. Note that, $\vec{t}$ satisfies $DC\text{\textsubscript{DORF\times CORF}}$–2 because $\vec{t}$ is a false point of $S$ and $p_{i,1,j_1} = 1$ on $\vec{t}$.

(b) Some points in $TC\text{\textsubscript{CORF}}$ cannot satisfy $DC\text{\textsubscript{DORF\times ENF}}$. Since $S$ is in IDNF, $NFP_{i,j}(S) \neq \emptyset$ for all $i$ and $j$. By definition of $NFP_{i,j}(S)$, all literals in $p_{ii}$ evaluate to 1 except $x_{ki}^i = 0$. Hence $p_{i,1,j_1} + p_{i,1,j_1+k_1} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i,j_1}(S) \subseteq \{\vec{t} \in FP(S) : p_{i,1,j_1} + p_{i,1,j_1+k_1} = 1\} \subseteq TC\text{\textsubscript{CORF}}$. Note that, $\vec{t}$ cannot satisfy $DC\text{\textsubscript{DORF\times CORF}}$–2 because $p_{i,1,j_1} = 0$.

Hence, only some, but not all, points satisfying $DC\text{\textsubscript{CORF}}$ can satisfy $DC\text{\textsubscript{DORF\times CORF}}$–2.

5.1.15 CORF and CORF

In this section, we analyse the relationship between the double fault involving two CORFs. Since there are two CORFs, we use CORF1 and CORF2 to identify the two different CORFs in double fault CORF1 × CORF2 for ease of understanding.

Let $S$ be a Boolean expression in IDNF. As discussed in Section 2.2, if the $i_1$-th term, $p_{i_1}$, in $S$ is wrongly implemented as $p_{i_1,j_1} + p_{i_1,j_1+k_1}$ where $p_{i_1} = p_{i_1,j_1} \cdot p_{i_1,j_1+k_1}$, the corresponding detection condition, denoted by $DC\text{\textsubscript{CORF1}}$, is “any point in $FP(S)$ such that $p_{i_1,j_1} + p_{i_1,j_1+k_1} = 1$”. We use $TC\text{\textsubscript{CORF1}}$ to denote the set of all points that satisfy $DC\text{\textsubscript{CORF1}}$, that is $TC\text{\textsubscript{CORF1}} = \{\vec{t} \in FP(S) : p_{i_1,j_1} + p_{i_1,j_1+k_1} = 1\}$.

Similarly, if the $i_2$-th term, $p_{i_2}$, in $S$ is wrongly implemented as $p_{i_2,j_2} + p_{i_2,j_2+k_2}$ where $p_{i_2} = p_{i_2,j_2} \cdot p_{i_2,j_2+k_2}$, the corresponding detection condition, denoted by $DC\text{\textsubscript{CORF2}}$, is “any point in
$FP(S)$ such that $p_{i2,1,j2} + p_{i2,j2+1,k2} = 1$. We use $TC_{CORF2}$ to denote the set of all points that satisfy $DC_{CORF2}$, that is $TC_{CORF2} = \{ \vec{t} \in FP(S) : p_{i2,1,j2} + p_{i2,j2+1,k2} = 1 \}$.

For $CORF1 \times CORF2$, there are two subcases.

**Case 1** The double-fault expression is equivalent to Expression (26) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{CORF1 \times CORF2 - 1}$, is “any point in $FP(S)$ such that $p_{i1,1,j1} + p_{i1,j1+1,k1} + p_{i2,1,j2} + p_{i2,j2+1,k2} = 1$”. We use $TC_{CORF1 \times CORF2 - 1}$ to denote the set of all points that satisfy $DC_{CORF1 \times CORF2 - 1}$, that is $TC_{CORF1 \times CORF2 - 1} = \{ \vec{t} \in FP(S) : p_{i1,1,j1} + p_{i1,j1+1,k1} + p_{i2,1,j2} + p_{i2,j2+1,k2} = 1 \}$.

(A) Relationship between $DC_{CORF1}$ and $DC_{CORF1 \times CORF2 - 1}$: The relationship R1 holds because:

\[
TC_{CORF1} = \{ \vec{t} \in FP(S) : p_{i1,1,j1} + p_{i1,j1+1,k1} = 1 \} \\
\subseteq \{ \vec{t} \in FP(S) : p_{i1,1,j1} + p_{i1,j1+1,k1} + p_{i2,1,j2} + p_{i2,j2+1,k2} = 1 \} = DC_{CORF1 \times CORF2 - 1}.
\]

Hence, any point satisfying $DC_{CORF1}$ satisfies $DC_{CORF1 \times CORF2 - 1}$.

(B) Relationship between $DC_{CORF2}$ and $DC_{CORF1 \times CORF2 - 1}$: The relationship R4 holds because:

\[
TC_{CORF2} = \{ \vec{t} \in FP(S) : p_{i2,1,j2} + p_{i2,j2+1,k2} = 1 \} \\
\subseteq \{ \vec{t} \in FP(S) : p_{i1,1,j1} + p_{i1,j1+1,k1} + p_{i2,1,j2} + p_{i2,j2+1,k2} = 1 \} = DC_{CORF1 \times CORF2 - 1}.
\]

Hence, any point satisfying $DC_{CORF1}$ satisfies $DC_{CORF1 \times CORF2 - 1}$.

**Case 2** The double-fault expression is equivalent to Expression (27) in Table 1 where $i_1 = i_2$ and $j_1 < j_2$. The detection condition, denoted by $DC_{CORF \times CORF - 2}$, is “any point in $FP(S)$ such that $p_{i1,1,j1} + p_{i1,j1+1,j2} + p_{i1,j2+1,k1} = 1$”. We use $TC_{CORF1 \times CORF2 - 2}$ to denote the set of all points...
that satisfy $DC_{CORF1\times CORF2−2}$, that is $TC_{CORF1\times CORF2−2} = \{ \vec{t} \in FP(S) : p_{i_1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1 \}$.

(A) Relationship between $DC_{CORF}$ and $DC_{CORF\times CORF−2}$: The relationship R1 holds because:

$$TC_{CORF_1}$$

$$= \{ \vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1 \}$$

$$\subseteq \{ \vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1 \}$$

(Note that, either all literals of $p_{i_1,1,j_1}$ or all literals of $p_{i_1,j_1+1,k_{i_1}}$ evaluate to 1 on these points.)

$$= DC_{CORF1\times CORF2−2}$$

Hence, any point satisfying $DC_{CORF}$ satisfies $DC_{CORF\times CORF−2}$.

(B) Relationship between $DC_{CORF}$ and $DC_{CORF\times CORF−2}$: The relationship R4 holds because:

$$TC_{CORF_2}$$

$$= \{ \vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1 \}$$

$$= \{ \vec{t} \in FP(S) : p_{i_1,1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1 \} \quad (i_1 = i_2)$$

$$\subseteq \{ \vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1 \}$$

(Note that, either all literals of $p_{i_1,1,j_2}$ or all literals of $p_{i_1,j_2+1,k_{i_1}}$ evaluate to 1 on these points.)

$$= DC_{CORF1\times CORF2−2}$$

Hence, any point satisfying $DC_{CORF}$ satisfies $DC_{CORF\times CORF−2}$.

5.2 Fault Coupling on 4 Remaining Faulty Implementations

As mentioned previously, in double fault with ordering, there are 4 double-fault expressions that do not have their counterparts in the double faults without ordering. In this section, we analyse the relationships between their detection conditions and those of their corresponding single faults.
5.2.1 TOF and DORF

In this section, we analyse the relationship between the double fault TOF and DORF and its corresponding single faults with respect to the situation of Expression (53) in Table 1.

Let $S$ be a Boolean expression in IDNF. The double fault expression for TOF and then DORF corresponding to Expression (53) can be derived from the situation where the $i_1$-th term, $p_{i_1}$, is omitted first and then the two consecutive terms $p_{i_1-1}$ and $p_{i_1+1}$ are wrongly implemented as $p_{i_1-1} \cdot p_{i_1+1}$.

As discussed in Section 2.2, if the $i_1$-th term, $p_{i_1}$, in $S$ is wrongly omitted, the corresponding detection condition, denoted by $DC_{TOF}$, is “any point in $UTP_{i_1}(S)$”. We use $TC_{TOF}$ to denote the set of all points that satisfy $DC_{TOF}$, that is $TC_{TOF} = UTP_{i_1}(S)$.

Similarly, if the subexpression $p_{i_1-1} + p_{i_1+1}$ in $S$ is wrongly implemented as $p_{i_1-1} \cdot p_{i_1+1}$, the corresponding detection condition, denoted by $DC_{DORF}$, is “any point in $UTP_{i_1-1}(S)$ or any point in $UTP_{i_1+1}(S)$”. We use $TC_{DORF}$ to denote the set of all points that satisfy $DC_{DORF}$, that is $TC_{DORF} = UTP_{i_1-1}(S) \cup UTP_{i_1+1}(S)$.

Hence, the corresponding detection condition of Expression (53), denoted by $DC_{TOF \times DORF}$, is “any point in $\left( (TP_{i_1-1}(S) \cup TP_{i_1}(S)) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1 \text{ or } i_1+1}}^m TP_i(S) \right) \cup \left( (TP_{i_1}(S) \cup TP_{i_1+1}(S)) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1 \text{ or } i_1+1}}^m TP_i(S) \right)$”.

We use $TC_{TOF \times DORF}$ to denote the set of all points that satisfy $DC_{TOF \times DORF}$, that is $TC_{TOF \times DORF} = \left( (TP_{i_1-1}(S) \cup TP_{i_1}(S)) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1 \text{ or } i_1+1}}^m TP_i(S) \right) \cup \left( (TP_{i_1}(S) \cup TP_{i_1+1}(S)) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1 \text{ or } i_1+1}}^m TP_i(S) \right)$.

(A) Relationship between $DC_{TOF}$ and $DC_{TOF \times DORF}$: The relationship R1 holds because:
\[ \text{TC}_{\text{TOF}} \]
\[ = \text{UTP}_{i_1}(S) \]
\[ = \text{TP}_{i_1}(S) \setminus \left( \bigcup_{i \neq i_1}^{m} \text{TP}_{i}(S) \cup \text{TP}_{i+1}(S) \right) \]
\[ \subseteq \text{TP}_{i_1}(S) \setminus \left( \bigcup_{i=1}^{m} \text{TP}_{i}(S) \right) \quad \text{(By using } (A \setminus (B \cup C)) \subseteq (A \setminus B)) \]
\[ \subseteq \left( \text{TP}_{i_1}(S) \cup \text{TP}_{i+1}(S) \right) \setminus \left( \bigcup_{i \neq i_1}^{m} \text{TP}_{i}(S) \right) \quad \text{(By using } (A \setminus B) \subseteq (A \cup C) \setminus B) \]
\[ \subseteq \left( \text{TP}_{i_1-1}(S) \cup \text{TP}_{i_1}(S) \right) \setminus \bigcup_{i \neq i_1-1,i_1} \text{TP}_{i}(S) \cup \left( \text{TP}_{i_1}(S) \cup \text{TP}_{i+1}(S) \right) \setminus \bigcup_{i \neq i_1,i_1+1} \text{TP}_{i}(S) \]
\[ = \text{TC}_{\text{TOF}} \oplus \text{DORF} \quad \text{(By using } B \subseteq A \cup B) \]

Hence, any point satisfying \( \text{DC}_{\text{TOF}} \) satisfies \( \text{DC}_{\text{TOF}} \oplus \text{DORF} \).

(B) Relationship between \( \text{DC}_{\text{DORF}} \) and \( \text{DC}_{\text{TOF}} \oplus \text{DORF} \): The relationship R4 holds because:
\[ DC_{DORF} \]
\[ = UTP_{i_1-1}(S) \cup UTP_{i_1+1}(S) \]
\[ = \left( TP_{i_1-1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup \left( TP_{i_1+1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \]
\[ = \left( TP_{i_1-1}(S) \setminus \left( \bigcup_{i=1}^{m} \left( TP_i(S) \cup TP_i(S) \right) \right) \right) \cup \left( TP_{i_1+1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup \right. \]
\[ TP_{i_1}(S) \left. \right) \]
\[ \subseteq \left( TP_{i_1-1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup \left( TP_{i_1+1}(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \]
\[ \text{(By using } (A \setminus (B \cup C)) \subseteq (A \setminus B) \text{)} \]
\[ \subseteq \left( (TP_{i_1-1}(S) \cup TP_{i_1}(S)) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \cup \left( (TP_{i_1}(S) \cup TP_{i_1+1}(S)) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \right) \]
\[ \text{(By using } (A \setminus B) \subseteq (A \cup C) \setminus B) \text{)} \]
\[ = TC_{TOF \times DORF} \]

Hence, any point satisfying \( DC_{DORF} \) satisfies \( DC_{TOF \times DORF} \).

### 5.2.2 CORF and ENF

In this section, we analyse the relationship between the double fault CORF×ENF and its corresponding single faults.

Let \( S \) be a Boolean expression in IDNF. The double fault expression given by Expression (73) can be derived from the situation where the \( i_1 \)-th term, \( p_{i_1} \), is wrongly implemented as \( p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} \) where \( p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_1} \) and then the subexpression \( p_{i_1,j_1+1,k_1} + \cdots + p_{h_1} \) is wrongly negated.

As discussed in Section 2.2, if the \( i_1 \)-th term, \( p_{i_1} \), in \( S \) is wrongly implemented as \( p_{i_1,1,j_1} + p_{i_1,j_1+1,k_1} \) where \( p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_1} \), the corresponding implementation, denoted by \( I_{CORF} \), will be equiv-
alent to $p_1 + \cdots + p_{i-1} + p_{i,1,j_1} + p_{i,1,j_1+1,k_i} + p_{i+1} + \cdots + p_{h_1} + \cdots + p_m$, and its detection condition, denoted by $DC_{CORF}$, is “any point in $FP(S)$ such that $p_{i,1,j_1} + p_{i,1,j_1+1,k_i} = 1$”. We use $TC_{CORF}$ to denote the set of all points that satisfy $DC_{CORF}$, that is $TC_{CORF} = \{\bar{t} \in FP(S) : p_{i,1,j_1} + p_{i,1,j_1+1,k_i} = 1\}$.

Furthermore, given $I_{CORF}$, if the subexpression $p_{i,1,j_1+1,k_i} + \cdots + p_{h_1}$ is wrongly negated, the corresponding implementation, denoted by $I_{CORF \times ENF}$, will be equivalent to $p_1 + \cdots + p_{i-1} + p_{i,1,j_1} + p_{i,1,j_1+1,k_i} + p_{i+1} + \cdots + p_{h_1} + \cdots + p_m$, and its detection condition with respect to $I_{CORF}$, denoted by $DC_{ENF}$, is “any point in $\{\bar{t} \in \mathbb{B}^n : p_{i,1,j_1+1,k_i} + p_{i+1} + \cdots + p_{h_1} = 1$ and $p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\}$ or any point in $\{\bar{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\}$”. We use $TC_{ENF}$ to denote the set of all points that satisfy $DC_{ENF}$, that is $TC_{ENF} = (\{\bar{t} \in \mathbb{B}^n : p_{i,1,j_1+1,k_i} + p_{i+1} + \cdots + p_{h_1} = 1$ and $p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\}) \cup (\{\bar{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\})$. For ease of comparison, we express the condition in terms of $S$ as follows:

$$\begin{align*}
\left(\{\bar{t} \in \mathbb{B}^n : p_{i,1,j_1+1,k_i} + p_{i+1} + \cdots + p_{h_1} = 1$ and $p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\}\right) \cup \left(\{\bar{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\}\right)
\end{align*}$$

$$\equiv$$

$$\left(\{\bar{t} \in \mathbb{B}^n : p_{i,1,j_1+1,k_i}(p_{i+1} + \cdots + p_{h_1}) + p_{i+1} + \cdots + p_{h_1} = 1$ and $p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0$ and $p_{i,1,j_1} = 0\}\right) \cup \left(\{\bar{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0$ and $p_{i,1,j_1} = 0\}\right) \cup \left(\{\bar{t} \in FP(S) : p_{i,1,j_1} = p_{i,1,j_1+1,k_i} = 0\}\right)$$

$$\equiv$$

$$\left(\{\bar{t} \in \mathbb{B}^n : p_{i,1,j_1+1,k_i}(p_{i+1} + \cdots + p_{h_1}) = 1$ and $p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0$ and $p_{i,1,j_1} = 0\}\right) \cup \left(\{\bar{t} \in \mathbb{B}^n : p_{i,1,j_1} + \cdots + p_{h_1} = 1$ and $p_1 + \cdots + p_{i,1,j_1} + p_{h_1+1} + \cdots + p_m = 0$ and $p_{i,1,j_1} = 0\}\right) \cup \left(\{\bar{t} \in FP(S) : p_{i,1,j_1} = p_{i,1,j_1+1,k_i} = 0\}\right)$$

$$\equiv$$

$$\left(\{\bar{t} \in \mathbb{B}^n : p_{i,1,j_1+1,k_i} = 1$ and $p_1 + \cdots + p_{i,1,j_1} + \cdots + p_{h_1} + \cdots + p_m = 0$ and $p_{i,1,j_1} = 0\}\right) \cup \left(\{\bar{t} \in (\bigcup_{i=1}^{m} TP_i(S)) \setminus (\bigcup_{i \notin i_1+1 \ldots,i_1}^{m} TP_i(S)) : p_{i,1,j_1} = 0\}\right) \cup \left(\{\bar{t} \in FP(S) : p_{i,1,j_1} = p_{i,1,j_1+1,k_i} = 0\}\right)$$
\(
\equiv (\{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_1} = 1 \text{ and } p_{i_1,1,j_1} = 0\}) \cup (\{\vec{t} \in (\bigcup_{i=i_1+1}^{k_1} TP_i(S)) \setminus (\bigcup_{i=i_1+1}^{k_1} \bigcup_{j=1}^{m} TP_j(S)) : p_{i_1,j_1} = 0\}) \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0\})
\)

Now, the corresponding detection condition of Expression (70), denoted by \(DC_{\text{CORF} \times \text{ENF}}\), is “any point in (\bigcup_{i=i_1+1}^{k_1} TP_i(S)) \setminus (\bigcup_{i=i_1+1}^{k_1} \bigcup_{j=1}^{m} TP_j(S))\) such that \(p_{i_1,j_1} = 0\) or any point in \(FP(S)\) such that \(p_{i_1,j_1+1,k_1} = 0\). We use \(TC_{\text{CORF} \times \text{ENF}}\) to denote the set of all points that satisfy \(DC_{\text{CORF} \times \text{ENF}}\), that is

\(TC_{\text{CORF} \times \text{ENF}} = \{\vec{t} \in (\bigcup_{i=i_1+1}^{k_1} TP_i(S)) \setminus (\bigcup_{i=i_1+1}^{k_1} \bigcup_{j=1}^{m} TP_j(S)) : p_{i_1,j_1} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_1} = 0\}\).

(A) Relationship between \(DC_{\text{CORF}}\) and \(DC_{\text{CORF} \times \text{ENF}}\): The relationship R2 holds because of the following reasons:

(a) Some points in \(TC_{\text{CORF}}\) can satisfy \(DC_{\text{CORF} \times \text{ENF}}\). Since \(S\) is in IDNF, \(NFP_{i_1,j_1}(S) \neq \emptyset\) for all \(i\) and \(j\). By definition of \(NFP_{i_1,j_1}(S)\), all literals in \(p_{i_1}\) evaluate to 1 except \(x_{k_1}^{i_1} = 0\). Hence \(p_{i_1,j_1} + p_{i_1,j_1+1,k_1} = 1 + 0 = 1\) on \(\vec{t}\). Therefore, we can find a point \(\vec{t}\) such that \(\vec{t} \in NFP_{i_1,j_1}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,j_1} + p_{i_1,j_1+1,k_1} = 1\} \subseteq TC_{\text{CORF}}\). Note that, \(\vec{t}\) satisfies \(DC_{\text{CORF} \times \text{ENF}}\) because \(\vec{t}\) is a false point of \(S\), \(x_{k_1}^{i_1} = 0\) and \(p_{i_1,j_1+1,k_1} = 0\) on \(\vec{t}\).

(b) Some points in \(TC_{\text{CORF}}\) cannot satisfy \(DC_{\text{CORF} \times \text{ENF}}\). Since \(S\) is in IDNF, \(NFP_{i_1,j_1}(S) \neq \emptyset\) for all \(i\) and \(j\). By definition of \(NFP_{i_1,j_1}(S)\), all literals in \(p_{i_1}\) evaluate to 1 except \(x_{k_1}^{i_1} = 0\). Hence \(p_{i_1,j_1} + p_{i_1,j_1+1,k_1} = 0 + 1 = 1\) on \(\vec{t}\). Therefore, we can find a point \(\vec{t}\) such that \(\vec{t} \in NFP_{i_1,j_1}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,j_1} + p_{i_1,j_1+1,k_1} = 1\} \subseteq TC_{\text{CORF}}\). Note that, \(\vec{t}\) cannot satisfy \(DC_{\text{CORF} \times \text{ENF}}\) because \(p_{i_1,j_1+1,k_1} = 1\) on \(\vec{t}\).

Hence, only some, but not all, points satisfying \(DC_{\text{CORF}}\) can satisfy \(DC_{\text{CORF} \times \text{ENF}}\).

(B) Relationship between \(DC_{\text{ENF}}\) and \(DC_{\text{CORF} \times \text{ENF}}\):
For clarification and ease of discussion, let \( X = \{ \vec{t} \in \bigcup_{i=1}^{h_1} TP_i(S) \} \) and \( Y = \{ \vec{t} \in FP(S) : p_{i_1,j_1} = 0 \} \). We observe \( TC_{ENF} \cap TC_{CORF \times ENF} \)

\[
= (X \cup \{ \vec{t} \in FP(S) : p_{i_1,j_1} = 0 \}) \cap (X \cup \{ \vec{t} \in FP(S) : p_{i_1,j_1+1,k_i} = 0 \})
\]

\[
= X \cup \{ \vec{t} \in FP(S) : p_{i_1,j_1} = p_{i_1,j_1+1,k_i} = 0 \} = X \cup Y.
\]

The relationship between \( DC_{ENF} \) and \( DC_{CORF \times ENF} \) depends on whether both \( X \) and \( Y \) are empty. There are two cases:

(a) Both \( X \) and \( Y \) are empty. Then R6 holds. Here is an example. Let \( p_{i_1,j_1} = 0 \) and \( Y = \{ \vec{t} \in FP(S) : p_{i_1,j_1} = p_{i_1,j_1+1,k_i} = 0 \} \).

We observe \( TC_{ENF} \cap TC_{CORF \times ENF} \)

\[
= (X \cup \{ \vec{t} \in FP(S) : p_{i_1,j_1} = 0 \}) \cap (X \cup \{ \vec{t} \in FP(S) : p_{i_1,j_1+1,k_i} = 0 \})
\]

\[
= X \cup \{ \vec{t} \in FP(S) : p_{i_1,j_1} = p_{i_1,j_1+1,k_i} = 0 \} = X \cup Y.
\]

(b) Any one of \( X \) and \( Y \) is not empty. Then R5 holds.

(i) R5 holds when \( X \) is not empty because of the following reasons:

(1) Some points in \( TC_{ENF} \) can satisfy \( DC_{CORF \times ENF} \). Since \( X \) is not empty, we can find a point \( \vec{t} \) such that \( \vec{t} \in X \subseteq TC_{ENF} \), \( \vec{t} \) also satisfies \( DC_{CORF \times ENF} \).

(2) Some points in \( TC_{ENF} \) cannot satisfy \( DC_{CORF \times ENF} \). Since \( S \) is in IDNF, \( NFP_{i_1,j}(S) \neq \emptyset \) for all \( i \) and \( j \). By definition of \( NFP_{i_1,j}(S) \), all literals in \( p_{i_1} \) evaluate to 1 except \( x_{i_1}^j = 0 \). Hence \( p_{i_1,j_1} + p_{i_1,j_1+1,k_i} = 0 + 1 = 1 \) on \( \vec{t} \). Therefore, we can find a point \( \vec{t} \) such that \( \vec{t} \in NFP_{i_1,j}(S) \subseteq \{ \vec{t} \in FP(S) : p_{i_1,j_1} = 0 \} \subseteq TC_{ENF} \). Note that, \( \vec{t} \) cannot satisfy \( DC_{CORF \times ENF} \) because \( p_{i_1,j_1+1,k_i} = 1 \) on \( \vec{t} \).

(ii) R5 holds when \( Y \) is not empty because of the following reasons

(1) Some points in \( TC_{ENF} \) can satisfy \( DC_{CORF \times ENF} \). Since \( Y \) is not empty, we can find a point \( \vec{t} \) such that \( \vec{t} \in Y \subseteq TC_{ENF} \), \( \vec{t} \) also satisfies \( DC_{CORF \times ENF} \).
(2) Some points in $TC_{ENF}$ cannot satisfy $DC_{CORF \times ENF}$. Since $S$ is in IDNF, $NFP_{i,j}(S) \neq \emptyset$ for all $i$ and $j$. By definition of $NFP_{i,1}(S)$, all literals in $p_i$ evaluate to 1 except $x_i^j = 0$. Hence $p_{i,1,j} + p_{i,j+1,k_i} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i,1}(S) \subseteq \{ \vec{t} \in FP(S) : p_{i,1,j} = 0 \} \subseteq TC_{ENF}$. Note that, $\vec{t}$ cannot satisfy $DC_{CORF \times ENF}$ because $p_{i,j+1,k_i} = 1$ on $\vec{t}$.

Hence, only some, but not all, points satisfying $DC_{ENF}$ can satisfy $DC_{CORF \times ENF}$.

5.2.3 CORF and TNF

In this section, we analyse the relationship between the double fault $CORF \times TNF$ and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. The double fault expression given by Expression (73) can be derived from the situation where the $i_{th}$ term, $p_i$, is wrongly implemented as $p_{i,1,j} + p_{i,j+1,k_i}$ where $p_i = p_{i,1,j} \cdot p_{i,j+1,k_i}$ and then the subexpression $p_{i,j+1,k_i}$ is wrongly negated.

As discussed in Section 2.2, if the $i_{th}$ term, $p_i$, in $S$ is wrongly implemented as $p_{i,1,j} + p_{i,j+1,k_i}$ where $p_i = p_{i,1,j} \cdot p_{i,j+1,k_i}$, the corresponding implementation, denoted by $I_{CORF}$ will be equivalent to $p_1 + \cdots + p_{i-1,j} + p_{i,1,j} + p_{i,j+1,k_i} + p_{i+1} + \cdots + p_m$, and its detection condition, denoted by $DC_{CORF}$, is “any point in $FP(S)$ such that $p_{i,1,j} + p_{i,j+1,k_i} = 1$”. We use $TC_{CORF}$ to denote the set of all points that satisfy $DC_{CORF}$, that is $TC_{CORF} = \{ \vec{t} \in FP(S) : p_{i,1,j} + p_{i,j+1,k_i} = 1 \}$.

Furthermore, given $I_{CORF}$, if the term $p_{i,j+1,k_i}$ is wrongly negated, the corresponding implementation, denoted by $I_{CORF \times TNF}$, will be equivalent to $p_1 + \cdots + p_{i-1,j} + \overline{p_{i,j+1,k_i}} + p_{i+1} + \cdots + p_m$, and its detection condition with respect to $I_{CORF}$, denoted by $DC_{TNF}$, is “any point in $\{ \vec{t} \in \mathbb{B}^n : p_{i,j+1,k_i} = 1$ and $p_1 + \cdots + p_{i-1,j} + p_{i+1} + \cdots + p_m = 0 \}$ or any point in $\{ \vec{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i-1,j} + p_{i,j+1,k_i} + p_{i+1} + \cdots + p_m = 0 \}$”. We use $TC_{TNF}$ to denote the set of all points that satisfy $DC_{TNF}$, that is $TC_{TNF} = \{ \vec{t} \in \mathbb{B}^n : p_{i,j+1,k_i} = 1$ and $p_1 + \cdots + p_{i-1,j} + p_{i+1} + \cdots + p_m = 0 \} \cup \{ \vec{t} \in \mathbb{B}^n : p_{i,j+1,k_i} = 1$ and $p_1 + \cdots + p_{i,j+1} + p_{i+1} + \cdots + p_m = 0 \}$.
\[ \mathbb{B}^n : p_1 + \cdots + p_{i_1, j_1} + p_{i_1, j_1+1, k_1} + \cdots + p_m = 0 \} . \]

For ease of comparison, we express the condition in terms of \( S \):

\[ \{ \vec{i} \in \mathbb{B}^n : p_{i_1, j_1+1, k_1} = 1 \text{ and } p_1 + \cdots + p_{i_1, j_1} + p_{i_1+1} + \cdots + p_m = 0 \} \cup \{ \vec{i} \in \mathbb{B}^n : p_{i_1, j_1} + p_{i_1, j_1+1, k_1} + p_{i_1+1} + \cdots + p_m = 0 \} \equiv \{ \vec{i} \in \mathbb{B}^n : p_{i_1, j_1+1, k_1} = 1 \} \cup \{ \vec{i} \in \mathbb{B}^n : p_{i_1, j_1} + p_{i_1, j_1+1, k_1} + p_{i_1+1} + \cdots + p_m = 0 \} \]

\( \equiv \{ \vec{i} \in FP(S) : p_{i_1, j_1} = 0 \} \cup \{ \vec{i} \in FP(S) : p_{i_1, j_1+1, k_1} = 0 \} \)

Now, the detection condition of Expression (73), denoted by \( DC_{CORF \times TNF} \), is “any point in \( FP(S) \) such that \( p_{i_1, j_1+1, k_1} = 0 \)”. We use \( TC_{CORF \times TNF} \) to denote the set of all points that satisfy \( DC_{CORF \times TNF} \), that is \( TC_{CORF \times TNF} = \{ \vec{i} \in FP(S) : p_{i_1, j_1+1, k_1} = 0 \} \).

(A) Relationship between \( DC_{CORF} \) and \( DC_{CORF \times TNF} \): The relationship R2 holds because of the following reasons:

(a) Some points in \( TC_{CORF} \) can satisfy \( DC_{CORF \times TNF} \). Since \( S \) is in IDNF, \( NFP_{i_1, j_1}(S) \neq \emptyset \) for all \( i \) and \( j \). By definition of \( NFP_{i_1, j_1}(S) \), all literals in \( p_{i_1} \) evaluate to \( 1 \) except \( x^j_{i_1} \). Hence, \( p_{i_1, j_1+1, k_1} = 0 \) and \( p_{i_1, j_1} + p_{i_1, j_1+1, k_1} = 1 + 0 = 1 \) on \( \vec{i} \). Therefore, we can find a point \( \vec{i} \) such that \( \vec{i} \in NFP_{i_1, j_1}(S) \subseteq \{ \vec{i} \in FP(S) : p_{i_1, j_1} + p_{i_1, j_1+1, k_1} = 1 \} \subseteq TC_{CORF} \). Note that, \( \vec{i} \) satisfies \( DC_{CORF \times TNF} \) because \( \vec{i} \) is a false point of \( S \) and \( p_{i_1, j_1+1, k_1} = 0 \) on \( \vec{i} \).

(b) Some points in \( TC_{CORF} \) cannot satisfy \( DC_{CORF \times TNF} \). Since \( S \) is in IDNF, \( NFP_{i_1, j_1}(S) \neq \emptyset \) for all \( i \) and \( j \). By definition of \( NFP_{i_1, j_1}(S) \), all literals in \( p_{i_1} \) evaluate to \( 1 \) except \( x^j_{i_1} \). Hence, \( p_{i_1, j_1+1, k_1} = 1 \) and \( p_{i_1, j_1} + p_{i_1, j_1+1, k_1} = 0 + 1 = 1 \) on \( \vec{i} \). Therefore, we can find a point \( \vec{i} \) such that \( \vec{i} \in NFP_{i_1, j_1}(S) \subseteq \{ \vec{i} \in FP(S) : p_{i_1, j_1} + p_{i_1, j_1+1, k_1} = 1 \} \subseteq TC_{CORF} \). Note that, \( \vec{i} \)
cannot satisfy $DC_{CORF\times TNF}$ because $p_{i_1,j_1+1,k_i} = 1$ on $\vec{t}$.

Hence, only some, but not all, points satisfying $DC_{CORF}$ can satisfy $DC_{CORF\times TNF}$.

(B) Relationship between $DC_{TNF}$ and $DC_{CORF\times TNF}$:

For ease of references, let $X = \{\vec{t} \in FP(S) : p_{i_1,j_1} = p_{i_1,j_1+1,k_i} = 1\}$.

We observe $TC_{TNF} \cap TC_{CORF\times TNF}$

$= \{\vec{t} \in FP(S) : p_{i_1,j_1} = 0\} \cap \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_i} = 0\}$

$= \{\vec{t} \in FP(S) : p_{i_1,j_1} = p_{i_1,j_1+1,k_i} = 0\}$

$= X$.

The relationship between $DC_{TNF}$ and $DC_{CORF\times TNF}$ depends on whether $X$ is empty. There are two cases:

(a) $X$ is empty. Then, R6 holds. Here is an example. Let $S = ab + \bar{a}b$. If the first term $ab$ is split into $a + b$ and $b$ is then wrongly negated, $I_{CORF} = a + b + \bar{a}b$ and $I_{CORF\times TNF} = a + \bar{b} + \bar{a}b$.

Now, we have $TC_{TNF} = \{\vec{t} \in FP(S) : a = 0\} = \{01\}$; and $TC_{CORF\times TNF} = \{\vec{t} \in FP(S) : b = 0\} = \{10\}$. Hence, all points satisfy $TC_{ENF}$ do not satisfy $TC_{ENF\times CORF}$.

(b) $X$ is not empty. Then, R5 holds because of the following reasons:

1. Some points in $TC_{TNF}$ can satisfy $DC_{CORF\times TNF}$. Since $X$ is not empty, we can find a point $\vec{t}$ such that $\vec{t} \in X \subseteq TC_{TNF}$. Note that, $\vec{t}$ also satisfies $DC_{CORF\times TNF}$.

2. Some points in $TC_{TNF}$ cannot satisfy $DC_{CORF\times TNF}$. Since $S$ is in IDNF, $NFP_{i,j}(S) \neq \emptyset$ for all $i$ and $j$. By definition of $NFP_{i,j}(S)$, all literals in $p_{i,j}$ evaluate to 1 except $x_i^j = 0$.

Hence, $p_{i_1,j_1+1,k_i} = 1$ and $p_{i_1,j_1} + p_{i_1,j_1+1,k_i} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i,j}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,j_1} = 0\} \subseteq TC_{TNF}$. Note that, $\vec{t}$ cannot satisfy $DC_{CORF\times TNF}$ because $p_{i_1,j_1+1,k_i} = 1$ on $\vec{t}$.
5.2.4 CORF and TOF

In this section, we analyse the relationship between the double fault CORF×TOF and its corresponding single faults.

Let $S$ be a Boolean expression in IDNF. The double fault expression given by Expression (76) can be derived from the situation where the $i_1$-th term, $p_{i_1}$, is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1,1,k_1}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1,1,k_1}$ and then the subexpression $p_{i_1,j_1,1,k_1}$ is wrongly omitted.

As discussed in Section 2.2, if the $i_1$-th term, $p_{i_1}$, in $S$ is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1,1,k_1}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1,1,k_1}$, the corresponding implementation, denoted by $I_{\text{CORF}}$, will be equivalent to $p_1 + \cdots + p_{i_1-1} + p_{i_1,1,j_1} + p_{i_1,j_1,1,k_1} + p_{i_1+1} + \cdots + p_m$, and its detection condition, denoted by $DC_{\text{CORF}}$, is “any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1,1,k_1} = 1$”. We use $TC_{\text{CORF}}$ to denote the set of all points that satisfy $DC_{\text{CORF}}$, that is $TC_{\text{CORF}} = \{ \vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1,1,k_1} = 1 \}.$

Furthermore, given $I_{\text{CORF}}$, if the subexpression $p_{i_1,j_1,1,k_1}$ is wrongly omitted, the corresponding implementation, denoted by $I_{\text{CORF} \times \text{TOF}}$, will be equivalent to $p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m$, and its detection condition with respect to $I_{\text{CORF}}$, denoted by $DC_{\text{TOF}}$, is “any point in $\{ \vec{t} \in \mathbb{B}^n : p_{i_1,j_1,1,k_1} = 1$ and $p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0 \}$. We use $TC_{\text{TOF}}$ to denote the set of all points that satisfy $DC_{\text{TOF}}$, that is $TC_{\text{TOF}} = \{ \vec{t} \in \mathbb{B}^n : p_{i_1,j_1,1,k_1} = 1$ and $p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0 \}.$

For ease of comparison, we express the condition in terms of $S$:

$$\{ \vec{t} \in \mathbb{B}^n : p_{i_1,j_1,1,k_1} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0 \}$$

$$\equiv \{ \vec{t} \in \mathbb{B}^n : p_{i_1,j_1,1,k_1} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0 \}$$

$$\equiv \{ \vec{t} \in FP(S) : p_{i_1,j_1,1,k_1} = 1 \}.$$

Now, the detection condition of Expression (76), denoted by $DC_{\text{CORF} \times \text{TOF}}$, is “any point in $FP(S)$ such that $p_{i_1,1,j_1} = 1$”. We use $TC_{\text{CORF} \times \text{TOF}}$ to denote the set of all points that satisfy $DC_{\text{CORF} \times \text{TOF}}$. 
that is $TC_{CORF \times TOF} = \{ \vec{t} \in FP(S) : p_{i_1,1,j_1} = 1 \}$.

(A) Relationship between $DC_{CORF}$ and $DC_{CORF \times TOF}$: The relationship R2 holds because of the following reasons:

(a) Some points in $TC_{CORF}$ can satisfy $DC_{CORF \times TOF}$. Since $S$ is in IDNF, $NFP_{i,j}(S) \neq \emptyset$ for all $i$ and $j$. By definition of $NFP_{i,j}(S)$, all literals in $p_{i,j}$ evaluate to 1 except $x^j_{k_{i_1}} = 0$. Hence, $p_{i_1,1,j_1} = 1$ and $p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_{i_1}} = 1 + 0 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,j_1}(S) \subseteq \{ \vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_{i_1}} = 1 \} \subseteq TC_{CORF}$. Note that, $\vec{t}$ satisfies $DC_{CORF \times TOF}$ because $\vec{t}$ is a false point of $S$ and $p_{i_1,1,j_1} = 1$ on $\vec{t}$.

(b) Some points in $TC_{CORF}$ cannot satisfy $DC_{CORF \times TOF}$. Since $S$ is in IDNF, $NFP_{i,j}(S) \neq \emptyset$ for all $i$ and $j$. By definition of $NFP_{i,j}(S)$, all literals in $p_{i,j}$ evaluate to 1 except $x^j_{k_{i_1}} = 0$. Hence, $p_{i_1,1,j_1} = 0$ and $p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_{i_1}} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,j_1}(S) \subseteq \{ \vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,1,j_1+1,k_{i_1}} = 1 \} \subseteq TC_{CORF}$. Note that, $\vec{t}$ cannot satisfy $DC_{CORF \times TOF}$ because $p_{i_1,1,j_1} = 0$ on $\vec{t}$.

Hence, only some, but not all, points satisfying $DC_{CORF}$ can satisfy $DC_{CORF \times TOF}$.

(B) Relationship between $DC_{TOF}$ and $DC_{CORF \times TOF}$: The relationship R6 holds because:

$$TC_{TOF} \cap TC_{CORF \times TOF} = \{ \vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 1 \} \cap \{ \vec{t} \in FP(S) : p_{i_1,1,j_1} = 1 \} = \emptyset \text{ (because when } p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 1, p_i = p_{i_1,1,j_1}p_{i_1,j_1+1,k_{i_1}} = 1, \text{ hence, } S(\vec{t}) = 1 \text{ which is impossible for any false point of } S)$$

Hence, points satisfying $DC_{TOF}$ cannot satisfy $DC_{CORF \times TOF}$.

For ease of reading and understanding, Table 3 summarizes the relationship between the detection conditions of single fault classes and their corresponding double fault classes. Since our objective is to determine which double fault classes can always be detected by test cases that can detect individual single fault classes, we are particularly interested in R1 or R4. From Table 3, 15 out
Table 3: Relationship of detection conditions for single and double faults

(a) Double-fault expressions (1)–(27) due to double faults without ordering

<table>
<thead>
<tr>
<th>Fault class</th>
<th>ENF</th>
<th>TNF</th>
<th>TOF</th>
<th>DORF</th>
<th>CORF</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENF</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>1: R2, R5;</td>
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<td></td>
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<tr>
<td>2: R2, R6</td>
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<tr>
<td>3: R2, R5;</td>
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<tr>
<td>4: R2, R6</td>
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<tr>
<td>5: R1, R6;</td>
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<tr>
<td>6: R2, R6</td>
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<td>7: R1, R6;</td>
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<tr>
<td>8: R2, R5;</td>
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<tr>
<td>9: R2, R6;</td>
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<tr>
<td>10: R2, R6</td>
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<tr>
<td>(R1,R2)</td>
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<tr>
<td>a, R4;</td>
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<tr>
<td>11: (R1,R2)</td>
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<tr>
<td>a, R4;</td>
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<tr>
<td>12: R2, R6</td>
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<tr>
<td>TNF</td>
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<tr>
<td>13: R2, R5</td>
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<td>14: R1, R6</td>
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<td>15: R1, R6;</td>
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<td>16: R2, R5</td>
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<td>17: (R1,R2)</td>
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<tr>
<td>a, R4;</td>
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<td>18: R2, R6</td>
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<td>TOF</td>
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<td>19: R1, R4</td>
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<td>20: R1, R4</td>
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<td>21: (R1,R2)</td>
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<tr>
<td>b, R4;</td>
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<td>DORF</td>
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<td>22: R1, R4;</td>
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<td>23: R1, R4</td>
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<td>24: (R1,R2)</td>
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<tr>
<td>b, R4;</td>
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<tr>
<td>25: (R2,R3)</td>
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<tr>
<td>b, R5</td>
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<tr>
<td>CORF</td>
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<tr>
<td>26: R1, R4;</td>
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<tr>
<td>27: R1, R4</td>
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</tr>
</tbody>
</table>

(b) Four extra double-fault expressions (53), (70), (73) and (76) due to double faults with ordering

<table>
<thead>
<tr>
<th>First fault class A</th>
<th>ENF</th>
<th>TNF</th>
<th>TOF</th>
<th>DORF</th>
</tr>
</thead>
<tbody>
<tr>
<td>TOF</td>
<td></td>
<td></td>
<td></td>
<td>53: R1, R4</td>
</tr>
<tr>
<td>CORF</td>
<td>70: R2, (R5, R6)</td>
<td></td>
<td>73: R2, (R5, R6)</td>
<td>76: R2, R6</td>
</tr>
</tbody>
</table>

a “(R1,R2)” means that either R1 or R2 holds. Please see the corresponding discussion for details.
b The interpretation is similar to that of “(R1, R2)”. 
of 31 double-fault expressions have R1 or R4. These 15 expressions can always be detected by test cases that collectively detect the two individual single fault classes. For the remaining expressions, there is no guarantee that those test cases that detect each one of the two individual single faults will detect the double-fault expressions. It is interesting to note that R3 and R6 do not hold simultaneously. Hence, regarding the double fault classes studied in this report, there is always a chance that test cases that collectively detect individual single fault classes may also detect those double fault classes.

6 Comparing Existing Testing Strategies

Instead of concentrating on specified test set to detect studied double fault classes, we are more interested to find some test case selection strategies which can guarantee to detect them. Therefore, following questions are considered in this section:

Many test case selection strategies have been developed to detect all the individual single fault classes described in Section 2. Can these strategies also detect all the double fault classes considered in this report? If yes, which one can be used. If no, which double fault classes can be detected and which one cannot?

Existing test case selection strategies for detecting faults in Boolean expressions include the BOR strategy [13, 14], the BASIC meaningful impact strategy (or simply the BASIC strategy) [17], the MUMCUT strategy [19] and the modified condition/decision coverage (MC/DC) criterion [3]. Since the BOR strategy requires every variable in the expression to occur only once, it is not widely applicable to all Boolean expressions in IDNF. It has been shown in [18] that MC/DC cannot guarantee to detect single-fault expressions due to TOF, DORF and CORF. As shown in [1, 19], both the BASIC and MUMCUT strategies can detect all single fault classes described in Section 2. Hence, we consider these two strategies further.
Given a Boolean expression $S$ in IDNF, the BASIC strategy selects (1) a unique true point from every $UTP_i(S)$, and (2) a near false point from every $NFP_{i,j}(S)$. The following three theorems prove that the test set selected by the BASIC strategy, and hence, any strategy that subsumes it, satisfies all detection conditions of all double-fault expressions considered in this report. A testing criterion $C_1$ is said to subsume another criterion $C_2$ if any test set that satisfies $C_1$ must also satisfy $C_2$. We need the following lemma to proceed.

**Lemma 6.1 ([1, Theorem 1])** Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Then, we have

1. $UTP_i(S) \neq \emptyset$ for all $i = 1, \ldots, m$

2. $NFP_{i,j}(S) \neq \emptyset$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, k_i$, where $k_i$ is the number of literals in the term $p_i$.

**Theorem 6.1** Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Suppose that $T$ is the set of near false points formed by selecting a near false point from $NFP_{i,j}(S)$ for every $i$ and $j$. Then $T$ satisfies the following conditions:

1. There exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$.

2. For all possible $i_2$ and $j_2$ pair where $1 \leq i_2 \leq m$, $1 \leq j_2 < k_{i_2}$ and $k_{i_2}$ is the number of literals of the $i_2$-th term $p_{i_2}$ in $S$, there exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ and $p_{i_2,1,j_2} + p_{i_2,2,j_2+1,k_{i_2}} = 1$.

3. For all possible $i_1$ and $j_1$ pair where $1 \leq i_1 \leq m$, $1 \leq j_1 < k_{i_1}$ and $k_{i_1}$ denotes the number of literals of the $i_1$-th term $p_{i_1}$ in $S$, there exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ and $p_{i_1,1,j_1+1,k_{i_1}} = 0$.

4. For all possible $i_1$ and $j_1$ pair where $1 \leq i_1 \leq m$, $1 \leq j_1 < k_{i_1}$ and $k_{i_1}$ denotes the number of literals of the $i_1$-th term $p_{i_1}$ in $S$, there exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ and $p_{i_1,1,j_1} = 1$. 
5. For all possible \(i_1, i_2, j_1\) and \(j_2\) where \(1 \leq i_1 < i_2 \leq m, 1 \leq j_1 < k_{i_1}, 1 \leq j_2 < k_{i_2}\) and \(k_{i_1}\) and \(k_{i_2}\) denote the numbers of literals of the \(i_1\)-th and \(i_2\)-th terms, \(p_{i_1}\) and \(p_{i_2}\), in \(S\) respectively, there exists \(\bar{t} \in T\) such that \(\bar{t} \in FP(S)\) and \(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\).

6. For all possible \(i_1, j_1\) and \(j_2\) where \(1 \leq i_1 \leq m, 1 \leq j_1 < k_{i_1}\), and \(k_{i_1}\) denotes the number of literals of the \(i_1\)-th term \(p_{i_1}\) in \(S\), there exists \(\bar{t} \in T\) such that \(\bar{t} \in FP(S)\) and \(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\).

**Proof:** By Lemma 6.1, \(T\) is non-empty because \(NFP_{i,j}(S)\) is non-empty for every possible \(i\) and \(j\) where \(1 \leq i \leq m, 1 \leq j \leq k_i\) and \(k_i\) is the number of literals of \(p_i\) in \(S\).

1. By definition of \(T\), any \(\bar{t} \in T \subset NFP(S) \subset FP(S)\). Hence the result follows.

2. For any \(i_2\) and \(j_2\) pair where \(1 \leq i_2 \leq m, 1 \leq j_2 < k_{i_2}\) and \(k_{i_2}\) is the number of literals of \(p_{i_2}\), there exists \(\bar{t} \in T\) such that \(\bar{t} \in NFP_{i_2,j_2}(S) \subset FP(S)\) by definition of \(T\) and Lemma 6.1. Therefore, all literals of \(p_{i_2}(x_1^{i_2}, \ldots, x_{k_{i_2}}^{i_2})\) evaluate to \(1\) except \(x_{j_2}^{i_2}\) which evaluates to \(0\) on \(\bar{t}\).

Thus, \(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = x_1^{i_2} \ldots x_{j_2 - 1}^{i_2} x_{j_2}^{i_2} x_{j_2+1}^{i_2} \ldots x_{k_{i_2}}^{i_2} = 0 + 1 = 1\). Hence, the result follows.

3. For any \(i_1\) and \(j_1\) pair where \(1 \leq i_1 \leq m, 1 \leq j_1 < k_{i_1}\) and \(k_{i_1}\) is the number of literals of \(p_{i_1}\) in \(S\), there exists \(\bar{t} \in T\) such that \(\bar{t} \in NFP_{i_1,j_1}(S) \subset FP(S)\) by definition of \(T\) and Lemma 6.1.

Therefore, all literals of \(p_{i_1}(x_1^{i_1}, \ldots, x_{k_{i_1}}^{i_1})\) evaluate to \(1\) except \(x_{j_1+1}^{i_1}\) which evaluates to \(0\) on \(\bar{t}\).

Therefore, \(p_{i_1,j_1+1,k_{i_1}} = x_1^{i_1} \ldots x_{j_1+1}^{i_1} = 0\). Hence, the result follows.

4. For any \(i_1\) and \(j_1\) pair where \(1 \leq i_1 \leq m, 1 \leq j_1 < k_{i_1}\) and \(k_{i_1}\) is the number of literals of \(p_{i_1}\) in \(S\), there exists \(\bar{t} \in T\) such that \(\bar{t} \in NFP_{i_1,j_1}(S) \subset FP(S)\) by definition of \(T\) and Lemma 6.1.

Therefore, all literals of \(p_{i_1}(x_1^{i_1}, \ldots, x_{k_{i_1}}^{i_1})\) evaluate to \(1\) except \(x_{j_1+1}^{i_1}\) which evaluates to \(0\) on \(\bar{t}\).

Therefore, \(p_{i_1,1,j_1} = x_1^{i_1} \ldots x_{j_1+1}^{i_1} = 1\). Hence, the result follows.

5. For all possible \(i_1, i_2, j_1\) and \(j_2\) where \(1 \leq i_1 < i_2 \leq m, 1 \leq j_1 < k_{i_1}, 1 \leq j_2 < k_{i_2}\) and \(k_{i_1}\) and \(k_{i_2}\) are the numbers of literals of \(p_{i_1}\) and \(p_{i_2}\) respectively, there exists \(\bar{t} \in T\) such that \(\bar{t} \in NFP_{i_1,j_1+1}(S) \subset FP(S)\) by definition of \(T\) and Lemma 6.1. It should be noted that the
choice of \( i \) is independent of \( i_2 \) and \( j_2 \). Therefore, all literals of \( p_i(x_{i_1}^1, \ldots, x_{k_i}^1) \) evaluate to 1 except \( x_{j_1+1}^1 \) which evaluates to 0 on \( i \). Thus, \( p_{i_1,1,j_1} + p_{i_1,j_1+1,k_i} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_i} = x_1^1 \cdots x_{j_1+1}^1 \cdot x_{j_1+2}^1 \cdots x_{k_i}^1 = 1 + 0 = 1. \) Hence, the result follows.

6. For all possible \( i_1, j_1 \) and \( j_2 \) where \( 1 \leq i_1 \leq m, 1 \leq j_1 < j_2 < k_i \) and \( k_i \) is the number of literals of \( p_{i_1} \), there exists \( i \in T \) such that \( i \in NFP_{i,j_1+1}(S) \subseteq FP(S) \) by definition of \( T \) and Lemma 6.1. Therefore, all literals of \( p_{i_1}(x_{i_1}^1, \ldots, x_{k_i}^1) \) evaluate to 1 except \( x_{j_1+1}^1 \) which evaluates to 0 on \( i \). Therefore, \( p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_i} = x_1^1 \cdots x_{j_1+1}^1 \cdot x_{j_1+2}^1 \cdots x_{j_2}^1 + x_{j_2}^1 \cdots x_{k_i}^1 = 1 + 0 + 1 = 1. \) Hence, the result follows.

\[ \square \]

**Theorem 6.2** Let \( S = p_1 + \cdots + p_m \) be a Boolean expression in irredundant disjunctive normal form. Suppose that \( T \) is the set of unique true points formed by selecting a unique true point from \( UTP_i(S) \) for every \( i \). Then \( T \) satisfies the following conditions:

1. For any \( i_2 \) where \( 1 \leq i_2 \leq m \), there exists \( i \in T \) such that \( i \in UTP_{i_2}(S) \).

2. For any \( i_1, h_1, i_2 \) and \( h_2 \) where \( 1 \leq i_1 \leq i_2 < h_2 \leq h_1 \leq m \) and \( \{i_2, i_2+1, \ldots, h_2-1, h_2\} \subset \{i_1, i_1+1, \ldots, h_1\} \), there exists \( i \in T \) such that \( i \in \bigcup_{i=i_1}^{h_1} TP_i(S) \setminus \left( \bigcup_{i=i_1}^{m} TP_i(S) \right) \).

3. For any \( i_1 \) where \( 1 \leq i_1 < m \), there exists \( i \in T \) such that \( i \in TP_{i_1}(S) \setminus \left( \bigcup_{i=i_1+1}^{m} TP_i(S) \right) \).

4. For any \( i_1 \) and \( i_2 \) pair where \( 1 \leq i_1 < i_2 < m \), there exists \( i \in T \) such that \( i \in \bigcup_{i=i_1}^{i_2} TP_i(S) \setminus \left( \bigcup_{i=i_1+1}^{m} TP_i(S) \right) \).

5. For any \( i_1 \) and \( h_1 \) where \( 1 \leq i_1 < h_1 \leq m \), there exists \( i \in T \) such that \( i \in \bigcup_{i=i_1}^{h_1} TP_i(S) \setminus \bigcup_{i=i_1+1}^{m} TP_i(S) \).
6. For any \( i_1, i_2 \) and \( h_1 \) where \( 1 \leq i_1 \leq i_2 \leq h_1 \leq m \) and \( i_1 < h_1 \), there exists \( \tilde{t} \in T \) such that 
\[
\tilde{t} \in \left( \bigcup_{i=i_1}^{h_1} \text{TP}_i(S) \right) \setminus \left( \bigcup_{i=i_1}^{m} \text{TP}_i(S) \right).
\]

7. For any \( i_1 \) and \( h_1 \) where \( 1 \leq i_1 < h_1 < m \), there exists \( \tilde{t} \in T \) such that 
\[
\tilde{t} \in \left( \bigcup_{i=i_1}^{h_1} \text{TP}_i(S) \right) \setminus \left( \bigcup_{i=i_1+1}^{m} \text{TP}_i(S) \right).
\]

8. For any \( i_1 \) and \( i_2 \) where \( 1 \leq i_1 < i_2 < m \), there exists \( \tilde{t} \in T \) such that 
\[
\tilde{t} \in \left( \bigcup_{i=i_1, i_1+1}^{m} \text{TP}_i(S) \right) \cup \left( \bigcup_{i=i_1, i_2, i_2+1}^{m} \text{TP}_i(S) \right).
\]

9. For any \( i_1 \) and \( i_2 \) where \( 1 \leq i_1 < i_2 < m \), there exists \( \tilde{t} \in T \) such that 
\[
\tilde{t} \in \left( \bigcup_{i=i_1, i_1+1, i_2, i_2+1}^{m} \text{TP}_i(S) \right) \cup \left( \bigcup_{i=i_1, i_1+1, i_2, i_2+1}^{m} \text{TP}_i(S) \right).
\]

10. For any \( i_1 \) where \( 1 \leq i_1 < m - 1 \), there exists \( \tilde{t} \in T \) such that 
\[
\tilde{t} \in \left( \bigcup_{i=i_1, i_1+1, i_1+2}^{m} \text{TP}_i(S) \right) \cup \left( \bigcup_{i=i_1, i_1+1, i_1+2}^{m} \text{TP}_i(S) \right).
\]

11. For any \( i_1 \) where \( 1 < i_1 < m \), there exists \( \tilde{t} \in T \) such that 
\[
\tilde{t} \in \left( \bigcup_{i=i_1-1, i_1}^{m} \text{TP}_i(S) \right) \cup \left( \bigcup_{i=i_1-1, i_1}^{m} \text{TP}_i(S) \right).
\]

**Proof**: By Lemma 6.1, \( T \) is non-empty because \( UTP_i(S) \) is non-empty for every possible \( i \) where \( 1 \leq i \leq m \).

1. For any \( i_2 \) where \( 1 \leq i_2 < m \), there is a \( \tilde{t} \in T \) such that \( \tilde{t} \in UTP_{i_2}(S) \) by definition of \( T \) and Lemma 6.1.
2. Let \( i_1, h_1, i_2 \) and \( h_2 \) be such that \( 1 \leq i_1 \leq i_2 < h_2 \leq h_1 \leq m \) and \( \{i_2, i_2 + 1, \ldots, h_2 - 1, h_2\} \subset \{i_1, i_1 + 1, \ldots, h_1\} \). Since the condition in the theorem is the detection condition of two subexpressions \( p_{i_1} + \cdots + p_{h_1} \) and \( p_{i_2} + \cdots + p_{h_2} \) being negated where \( p_{i_2} + \cdots + p_{h_2} \) is contained in \( p_{i_1} + \cdots + p_{h_1} \) and they are not identical, there is a term \( p_{i_3} \) in \( p_{i_1} + \cdots + p_{h_1} \) but not in \( p_{i_2} + \cdots + p_{h_2} \). By definition of \( T \) and Lemma 6.1, there is a \( \vec{t} \in T \) such that \( \vec{t} \in UTP_{i_1}(S) \). We then have \( \vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \) because \( i_3 \in \{i_1, \ldots, h_1\} \setminus \{i_2, \ldots, h_2\} \). Since \( \vec{t} \in UTP_{i_1}(S) \), \( \vec{t} \not\in TP_i(S) \) for any \( i \neq i_3 \). Hence, \( \vec{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \subset \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \). The result follows.

3. For any \( i_1 \) where \( 1 \leq i_1 < m \), there is a \( \vec{t} \in T \) such that \( \vec{t} \in UTP_{i_1}(S) \) by definition of \( T \) and Lemma 6.1. Therefore, \( \vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \). Since \( \vec{t} \in UTP_{i_1}(S) \), \( \vec{t} \not\in TP_i(S) \) for any \( i \neq i_1 \). Hence, \( \vec{t} \in \left( TP_{i_1}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \subset \left( TP_{i_1}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \). The result follows.

4. For any \( i_1 \) and \( i_2 \) pair where \( 1 \leq i_1 < i_2 < m \), there is a \( \vec{t} \in T \) such that \( \vec{t} \in UTP_{i_1}(S) \) by definition of \( T \) and Lemma 6.1. It should be noted that the choice of \( \vec{t} \) is independent of \( i_2 \). Therefore, \( \vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset TP_{i_1}(S) \cup TP_{i_2}(S) \). Since \( \vec{t} \in UTP_{i_1}(S) \), \( \vec{t} \not\in TP_i(S) \) for any \( i \neq i_1 \). Hence, \( \vec{t} \in \left( TP_{i_1}(S) \cup TP_{i_2}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \subset \left( TP_{i_1}(S) \cup TP_{i_2}(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \). The result follows.

5. For any \( i_1 \) and \( h_1 \) where \( 1 \leq i_1 < h_1 \leq m \), there is a \( \vec{t} \in T \) such that \( \vec{t} \in UTP_{i_1}(S) \) by definition of \( T \) and Lemma 6.1. It should be noted that the choice of \( \vec{t} \) is independent of \( h_1 \). Therefore, \( \vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \). Since \( \vec{t} \in UTP_{i_1}(S) \), \( \vec{t} \not\in TP_i(S) \) for any \( i \neq i_1 \). Hence, \( \vec{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \subset \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \). The result follows.

6. Let \( i_1, i_2 \) and \( h_1 \) be such that \( 1 \leq i_1 \leq i_2 < h_1 \leq m \) and \( i_1 < h_1 \). Since the condition in the theorem is the detection condition of both the subexpression \( p_{i_1} + \cdots + p_{h_1} \) and the term \( p_{i_2} \).
9. For any \( i \) and \( h_1 \) where \( 1 \leq i < h_1 < m \), there is a \( \vec{t} \in T \) such that \( \vec{t} \in UTP_{i_2}(S) \) by definition of \( T \) and Lemma 6.1. It should be noted that the choice of \( \vec{t} \) is independent of \( h_1 \). Therefore, 
\[
\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset \bigcup_{i=1}^{h_1} TP_i(S).
\]
Since \( \vec{t} \in UTP_{i_1}(S), \vec{t} \not\in TP_i(S) \) for any \( i \neq i_1 \). Hence, 
\[
\vec{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \subset \bigcup_{i=1}^{h_1} TP_i(S) \setminus \bigcup_{i=1}^{m} TP_i(S). \quad \text{The result follows.}
\]

7. For any \( i_1 \) and \( h_1 \) where \( 1 \leq i_1 < h_1 < m \), there is a \( \vec{t} \in T \) such that \( \vec{t} \in UTP_{i_1}(S) \) by definition of \( T \) and Lemma 6.1. It should be noted that the choice of \( \vec{t} \) is independent of \( h_1 \). Therefore, 
\[
\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset \bigcup_{i=1}^{h_1} TP_i(S).
\]
Since \( \vec{t} \in UTP_{i_1}(S), \vec{t} \not\in TP_i(S) \) for any \( i \neq i_1 \). Hence, 
\[
\vec{t} \in \left( \bigcup_{i=1}^{h_1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \subset \bigcup_{i=1}^{h_1} TP_i(S) \setminus \bigcup_{i=1}^{m} TP_i(S). \quad \text{The result follows.}
\]

8. For any \( i_1 \) and \( i_2 \) where \( 1 \leq i_1 < i_2 < m \), there is a \( \vec{t} \in T \) such that \( \vec{t} \in UTP_{i_1}(S) \) by definition of \( T \) and Lemma 6.1. It should be noted that the choice of \( \vec{t} \) is independent of \( i_2 \). Thus, 
\[
\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \cup TP_{i_2}(S).
\]
Since \( \vec{t} \in UTP_{i_1}(S), \vec{t} \not\in TP_i(S) \) for \( i \neq i_1 \). Hence, 
\[
\vec{t} \in \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \subset \bigcup_{i=1}^{m} TP_i(S) \setminus \bigcup_{i=1}^{m} TP_i(S) \cup \bigcup_{i=1}^{m} TP_i(S). \quad \text{The result follows.}
\]

9. For any \( i_1 \) and \( i_2 \) where \( 1 \leq i_1 < i_2 < m \), there is a \( \vec{t} \in T \) such that \( \vec{t} \in UTP_{i_1}(S) \) by definition of \( T \) and Lemma 6.1. It should be noted that the choice of \( \vec{t} \) is independent of \( i_2 \). Thus, 
\[
\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset \bigcup_{i=1}^{m} TP_i(S).
\]
Since \( \vec{t} \in UTP_{i_1}(S), \vec{t} \not\in TP_i(S) \) for any \( i \neq i_1 \). Hence, 
\[
\vec{t} \in \left( \bigcup_{i=1}^{m} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \subset \bigcup_{i=1}^{m} TP_i(S) \setminus \bigcup_{i=1}^{m} TP_i(S) \cup \bigcup_{i=1}^{m} TP_i(S). \quad \text{The result follows.}
\]

Since \( \vec{t} \in UTP_{i_1}(S), \vec{t} \) does not belong to any of these sets \( TP_{i_1+1}(S), TP_{i_2}(S) \) and \( TP_{i_2+1}(S) \).
Therefore, \( \vec{r} \not\in \left( (TP_{i_1}(S) \cap TP_{i_1+1}(S)) \cup (TP_{i_2}(S) \cap TP_{i_2+1}(S)) \right) \). Hence, \( \vec{r} \in \bigcup_{i=i_1, j_1+1, i_2, j_2+1} TP_i(S) \setminus \left( (TP_{i_1}(S) \cap TP_{i_1+1}(S)) \cup (TP_{i_2}(S) \cap TP_{i_2+1}(S)) \right) \). Therefore, we have \( \vec{r} \in \bigcup_{i=i_1, j_1+1, i_2, j_2+1} TP_i(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup \left( \bigcup_{i=1}^{m} TP_i(S) \right) \). The result follows.

10. For any \( i_1 \) where \( 1 \leq i_1 < m - 1 \), there is a \( \vec{r} \in T \) such that \( \vec{r} \in UTP_{i_1}(S) \) by definition of \( T \) and Lemma 6.1. Therefore, \( \vec{r} \in UTP_{i_1}(S) \subseteq TP_{i_1}(S) \subseteq \bigcup_{i=i_1, j_1+1, i_2, j_2+1} TP_i(S) \). Since \( \vec{r} \in UTP_{i_1}(S) \), \( \vec{r} \) does not belong to any of these two sets, \( TP_{i_1+1}(S) \) and \( TP_{i_2+2}(S) \). Thus, \( \vec{r} \not\in (TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_2+2}(S)) \).

Hence, \( \vec{r} \in \left( \bigcup_{i=i_1, j_1+1, i_2, j_2+1} TP_i(S) \right) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup \left( \bigcup_{i=1}^{m} TP_i(S) \right) \). Therefore, \( \vec{r} \in \bigcup_{i=i_1, j_1+1, i_2, j_2+1} TP_i(S) \setminus \left( \bigcup_{i=1}^{m} TP_i(S) \right) \cup \left( \bigcup_{i=1}^{m} TP_i(S) \right) \). The result follows.

11. For any \( i_1 \) where \( 1 < i_1 < m \), there is a \( \vec{r} \in T \) such that \( \vec{r} \in UTP_{i_1}(S) \) by definition of \( T \) and Lemma 6.1. Thus, \( \vec{r} \in UTP_{i_1}(S) \subseteq TP_{i_1}(S) \subseteq (TP_{i_1}(S) \cup TP_{i_1-1}(S)) \). Since \( \vec{r} \in UTP_{i_1}(S) \), \( \vec{r} \not\in TP_{i_1}(S) \) for \( i \not= i_1 \). Hence, \( \vec{r} \in (TP_{i_1}(S) \cup TP_{i_1-1}(S)) \setminus \bigcup_{i=1}^{m} TP_i(S) \subseteq (TP_{i_1}(S) \cup TP_{i_1-1}(S)) \setminus \bigcup_{i=1}^{m} TP_i(S) \).

Thus, we have \( \vec{r} \in (TP_{i_1}(S) \cup TP_{i_1-1}(S)) \setminus \bigcup_{i=1}^{m} TP_i(S) \). The result follows.

\( \square \)
Table 4: Conclusions in Theorems 6.1 and 6.2 satisfying detection conditions in Table 2

<table>
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<tr>
<th>Double-fault expression</th>
<th>Detection condition</th>
<th>Conclusion in Theorem 6.1</th>
<th>Conclusion in Theorem 6.2</th>
</tr>
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<tbody>
<tr>
<td>(1), (3), (5)–(7), (9)–(11), (13)–(15), (17)</td>
<td>(C2)</td>
<td>(1)</td>
<td>–</td>
</tr>
<tr>
<td>(2)</td>
<td>(C1)</td>
<td>–</td>
<td>(2)</td>
</tr>
<tr>
<td>(4)</td>
<td>(C1)</td>
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<td>(12)</td>
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<td>(19)</td>
<td>(C1)</td>
<td>–</td>
<td>(4)</td>
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<td>(20)</td>
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**Theorem 6.3** Let \( S = p_1 + \cdots + p_m \) be a Boolean expression in irredundant disjunctive normal form. Then, the BASIC meaningful impact strategy can detect all 31 double-fault expressions considered in this report.

**Proof:** As a reminder, the BASIC meaningful impact strategy selects (1) one unique true point from every \( UTP_i(S) \), and (2) one near false point from every \( NFP_{i,j}(S) \). Table 4 indicates that all detection conditions of all 31 double-fault expressions in Table 2 can be collectively satisfied by the corresponding conclusions in Theorems 6.1 and 6.2. Hence, the result follows.

**Theorem 6.4** Let \( S = p_1 + \cdots + p_m \) be a Boolean expression in irredundant disjunctive normal form. Then, any test case selection strategy that subsumes the BASIC meaningful impact strategy can detect all double-fault expressions considered in this report.

**Proof:** By Theorem 6.3, the BASIC meaningful impact strategy can detect all 31 double-fault expressions considered in this report. As a result, any test case selection strategy that subsumes the BASIC meaningful impact strategy can detect them.

Among the existing test case selection strategies proposed in research literature, those that subsume the BASIC strategy include the MANY-A strategy, the MAX-A strategy and the MUMCUT strategy [17, 19]. By Theorem 6.4, all these strategies can also detect all double fault classes considered in this report. The empirical study in [17] shows that, for a group of Boolean expressions under study, the BASIC, MANY-A and MAX-A strategies require, on average, test sets of sizes 9.8%, 19.3% and 40.6%, respectively, of the entire input domain. As reported in [19], the sizes of test sets generated by the MUMCUT strategy for the same group of Boolean expressions are, on average, approximately 12.0% of the entire input domain. Hence, both the BASIC and MUMCUT strategies generate much smaller test sets than the MANY-A and MAX-A strategies do, and that they can detect all double faults related to terms in Boolean expressions. Although the MUMCUT strategy requires slightly larger test sets, it can detect more single fault classes (such as literal insertion fault and literal reference fault) than that of the BASIC strategy [1, 19].
7 Conclusion

In this report, we study the detection conditions on double faults related to terms within a Boolean expression. For double fault related to terms, 49 out of 53 faulty expressions of double faults with ordering are equivalent to the 27 distinct double-fault expressions due to double faults without ordering, and the 4 remaining double-fault expressions are not equivalent to any of 27 faulty expressions. Altogether, there are 31 different double-fault expressions among all double fault classes considered in this report.

We also study the fault coupling between single fault classes and their corresponding double fault classes via analysing the relationship between detection conditions of single and double fault classes. We find that 15 out of the 31 double-fault expressions can always be detected by test cases that detect single fault classes. Moreover, for the remaining 16 double-fault expressions, some but not all test cases that detect each individual single fault class will also detect the double-fault expressions.

Based on the detection conditions, we prove that any test case selection strategy that subsumes the BASIC meaningful impact strategy can detect all double fault classes considered in this report. This is very interesting because none of these strategies were originally developed for detecting double faults. Among existing test case selection strategies based on Boolean expressions, both the BASIC strategy and the MUMCUT strategy generate much smaller test sets than other strategies that subsume the BASIC strategy.

Some other classes of single faults related to literals in Boolean expressions have not been studied in this report. We are extending our work to explore detection conditions of double faults related to literals and to further analyse whether existing test case selection strategies are able to detect all these double fault classes. We intend to complete the analysis of detection conditions so as to understand more precisely the behaviour of multiple faults for Boolean expressions.
References


