Thermomagnetic convection in a vertical layer of ferromagnetic fluid

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Linear stability of convection flow of ferromagnetic fluid between two vertical differentially heated plates placed in a uniform external magnetic field perpendicular to the plates is studied. Complete stability diagrams for two- and three-dimensional disturbances are presented. It is shown that two distinct mechanisms, thermogravitational and magnetic, are responsible for the appearance of three instability modes. The physical nature of all three modes is investigated in detail and the most prominent features are identified to provide guidance for future experimental investigation. Depending on the governing parameters, the instability patterns are shown to consist of vertical stationary magnetoconvection rolls and/or vertically or obliquely counterpropagating thermogravitational or thermomagnetic waves. © 2008 American Institute of Physics. [DOI: 10.1063/1.2952596]

I. INTRODUCTION

Common nonconducting artificial magnetic fluids consist of magnetite colloids which contain ferromagnetic nanoparticles suspended in a carrier fluid, usually synthetic oil, water, or kerosene. To prevent formation of magnetite aggregates and their subsequent sedimentation, a surfactant such as oleic acid is frequently used. The magnetic properties of such colloids have been studied for a considerable time since the 1930s.1 The interest in magnetic fluids was significantly boosted in the 1960s when their industrial production became possible.2 Nowadays a significant body of literature exists devoted to the properties of ferrocolloids, see, for example, Refs. 2–6 and references therein. Artificially manufactured ferrofluids respond to an external magnetic field similarly to natural paramagnetic and diamagnetic fluids (e.g., water, protein solutions, paramagnetic melts) and gases (e.g., oxygen). However, the degree of the magnetization which can be achieved in artificial ferrofluids is many orders of magnitude higher than that in natural magnetic fluids. Therefore noticeable magnetic effects on fluid flows can be observed in magnetic fields created by ordinary permanent or electromagnets which makes these fluids suitable for a wide range of technological applications.

The practical uses of artificially manufactured magnetocolloids continue to multiply as their quality is improved.7 One of the relatively new applications of nonconducting ferrofluids is as a heat exchanger in heat exchangers operating in reduced gravity conditions on orbital stations where cooling by natural gravitational convection cannot be achieved.5,8–10 Nonuniform heating results in a nonuniform magnetization of ferrofluid placed in an external magnetic field. Subsequently, a ponderomotive force arises which drives stronger magnetized fluid particles to the regions with a stronger magnetic field. This phenomenon is known as magnetoconvection and is not associated with gravitational buoyancy forces. It is not unique to artificial ferrocolloids and is found in natural paramagnetic11–13 and diamagnetic14,15 fluids. However, the intensity of a thermomagnetic convection in ferrofluids is much higher, which enables them to be used in realistic heat exchange systems.

Several authors studied thermomagnetic convection both theoretically and experimentally in various geometries and magnetic conditions, see Refs. 16–23 to name a few. Since this type of convection can occur in the absence of gravity, the easiest way to study this flow mechanism theoretically is by using a mathematical model which neglects buoyancy forces. This was done, for example, in Ref. 18. However, in realistic spacecraft conditions, the effective gravity is almost never zero due to various microaccelerations associated with equipment vibrations (high frequency gravity modulation) or maneuvering the ship (quasistatic gravity modulation). Moreover it is much easier and cheaper to conduct experimental investigation of magnetoconvective motions in Earth conditions, i.e., at normal gravity provided that the influences of the two driving mechanisms are reliably distinguished. Therefore it is important to study combined convection caused by the competing gravitational and magnetic mechanisms—this is the goal of the current work.

For the current study, the simple geometry of a vertical wide and tall fluid layer, heated from one side and placed in a perpendicular magnetic field, is chosen. Such a configuration is easy to recreate experimentally and it enables one to focus on investigating the physical mechanisms leading to a nontrivial fluid motion without being distracted by the complicated boundary effects. We follow the major steps of the analysis presented in Ref. 16 but show that the flow patterns arising in a vertical configuration are much more complicated than those found in the horizontal layer investigated in Ref. 16.

Experimental investigation of ferrofluid convection in a differentially heated vertical layer in a horizontal transverse magnetic field has been reported previously.24–26 However, these observations were conducted for a thick layer of fluid
II. PROBLEM DEFINITION AND GOVERNING EQUATIONS

Consider a vertical layer of ferromagnetic fluid which fills a gap between two infinitely long and wide parallel plates. The plates are separated by distance 2d and are maintained at constant different temperatures $T_a \pm \Theta$. An external uniform magnetic field $\mathbf{H}_{\infty}=(H^x,0,0)$ is applied perpendicular to the layer. This field causes an internal magnetic field $\mathbf{H}$ within the layer so that $|\mathbf{H}|=H$. Since the fluid is ferromagnetic, the external field leads to its magnetization $\mathbf{M}$ which is assumed to be codirected with the internal magnetic field: $\mathbf{M}=\chi_s\mathbf{H}$, where the nondimensional parameter $\chi_s$ is the magnetic susceptibility of the ferromagnetic fluid. In general it is a function of magnetic field and temperature. The magnitude of magnetization is $|\mathbf{M}|=M$.

Choose the right-hand system of coordinates $(x,y,z)$ with the origin in the midplane of the layer in such a way that the vertical plates are located at $x=\pm d$, the gravity vector $\mathbf{g}$ has components $(0,-g,0)$, and the $z$ axis is horizontal and parallel to the plates. Assuming that the temperature difference $2\Theta$ between the walls and the variation in magnetic field in the flow domain are sufficiently small, we adopt the Boussinesq approximation of the governing continuity, Navier–Stokes, and thermal energy equations so that they read

$$\nabla \cdot \mathbf{v} = 0, \quad (1)$$

$$\rho_s \frac{\partial \mathbf{v}}{\partial t} + \rho_s \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \eta_s \nabla^2 \mathbf{v} + \rho g + \mu_0 M \nabla H, \quad (2)$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa_s \nabla^2 T. \quad (3)$$

In the above equations, $\rho_s=\rho(T_a)$ is the fluid density at the average temperature $T_a$. The dynamic viscosity $\eta_s$ of working fluid is assumed to be constant. As discussed in Ref. 2, this assumption is sufficiently accurate when both the temperature and magnetic field variations are small within the flow domain. The variation in fluid density $\rho$ with temperature $T$ is accounted for only in the buoyancy term and is approximated linearly as

$$\rho = \rho_s[1 - \beta_s(T - T_a)], \quad (4)$$

where $\beta_s$ is the coefficient of thermal expansion. As discussed, for example, in Ref. 2, the last term in Eq. (2) describes a ponderomotive force which acts on a magnetized fluid in a nonuniform magnetic field (i.e., magnetized fluid tends to move in the direction of increasing magnetic field). The coefficient $\mu_0=4\pi \times 10^{-7}$ H/m is the magnetic constant and $\kappa_s$ is the (constant) thermal diffusivity of the fluid.
Equations (1)–(3) describe the so-called inductionless flow of magnetic fluid when magnetization and magnetic field are not influenced by the motion of the fluid and are given parametrically. However, in this work the interplay between fluid motions caused by thermal convection and by the variation in magnetic field and magnetization due to the temperature gradient in the layer is of primary interest. Therefore following Ref. 16, a set of Maxwell equations describing the magnetic field in the absence of induction currents is introduced (it is assumed that the working fluid is a perfect dielectric),

$$\nabla \times \mathbf{H} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

(5)

where the magnetic induction is

$$\mathbf{B} = \mu_0 (\mathbf{M} + \mathbf{H}).$$

(6)

The two-way coupling between Eqs. (1)–(3) and Eqs. (5) and (6) is established when the magnetization is allowed to depend not only on the intensity of the magnetic field but also on the temperature which is nonuniform across the layer. In the quasistationary limit, the magnetic moment of small magnetic particles suspended in a ferrocolloid adjusts to the applied magnetic field almost immediately (see Ref. 7, pp. 88 and 89) so that the fluid magnetization $\mathbf{M}$ and the magnetic field $\mathbf{H}$ remain collinear. Thus the constitutive equation relating the magnetization and magnetic fields is written as

$$\mathbf{M} = M(H, T) \frac{\mathbf{H}}{H}.$$

(7)

To close the problem, a magnetic equation of state is required which is assumed to be in the simplest linearized form valid for small temperature and field variations,

$$M = M_s + \chi \Delta H - K \Delta T, \quad \Delta H = H - H_s, \quad \Delta T = T - T_s.$$

(8)

Here $M_s$ and $H_s$ are the magnetization and the magnitude of the magnetic field at the location with temperature $T_s$ (i.e., in the midplane of the layer as will become clear from Sec. III), $\chi = \partial M / \partial H|_{H_s}$, and $K = - \partial M / \partial T|_{T_s}$. Using Eq. (8), rewrite Eq. (7) as

$$\mathbf{M} = \frac{M_s + \chi \Delta H - K \Delta T}{H} \mathbf{H}.$$

(9)

However, since Eq. (8) itself is a linearization of a fully nonlinear magnetic equation of state in the vicinity of $(H_s, T_s)$, the consistency of approximation requires that Eq. (9) be also linearized about the same point, i.e., $H = H_s + \Delta H$ and $H = H_s + \Delta H$, so that $|H_s| = H_s$ is the constant vector representing the major direction of the magnetic field and $\Delta$ denotes small increments. This leads to

$$\mathbf{M} = [M_s + (\chi - \chi_s) \Delta H - K \Delta T] e_s + \chi_s \Delta H,$$

(10)

where $e_s = \mathbf{H}_s / H_s$ and $\chi_s = M_s / H_s$. Quantities $\Delta H$ and $\Delta \mathbf{H}$ are related as $\Delta H = \Delta \mathbf{H} \cdot e_s$. Using Eq. (10) to eliminate magnetization from the second equation in Eq. (5), we obtain an equation for the magnetic field,

$$(1 + \chi) \nabla \cdot \mathbf{H} + (\chi - \chi_s) \nabla \cdot \mathbf{H} \cdot e_s - K \nabla \cdot e_s = 0.$$

(11)

It follows from this equation that thermomagnetic coupling only occurs when the temperature gradient and magnetic field have components in the same direction as is the case in the considered problem.

Note that if $H_s \parallel \Delta \mathbf{H}$, then Eqs. (10) and (11) become

$$\mathbf{M} = (M_s + \chi \Delta H - K \Delta T) e_s,$$

$$[(1 + \chi) \nabla \cdot \mathbf{H} - K \nabla \cdot e_s] = 0.$$

(12)

In the limit of small magnetic fields, when the working fluid is far from saturation, according to Langevin’s magnetization law, the dependence of magnetization on magnetic field is linear, $M = M(H) = \chi H$, and therefore, $\chi = (\partial M / \partial H)|_{H_s} = \chi_s$.

Equations (10) and (11) then take the forms of

$$\mathbf{M} = (M_s - K \Delta T) e_s + \chi \Delta \mathbf{H},$$

$$\nabla \cdot \mathbf{H} - K \nabla \cdot e_s = 0.$$

(13)

Processes of magnetization or demagnetization of a ferromagnetic fluid placed in a variable magnetic field lead to changes in the fluid’s internal energy and temperature. This is known as a magnetocalorific effect. The thermal energy equation (3) then has to be modified to describe these processes. Such an equation has been derived by various authors (e.g., Refs. 3 and 16 and references therein). However, as discussed in Ref. 6, the magnetocalorific effect is most noticeable near the Curie point. Away from this temperature, its influence on fluid temperature is negligible. Therefore, in this work, the thermal energy equation in the form of Eq. (3) will be used. Also, the effects of sedimentation are neglected upon the assumption that magnetite particles are uniformly distributed in a ferrocolloid so that the magnetic properties of the fluid are only affected by the thermal inhomogeneities. Finally, the fluid’s viscosity variation due to magnetoviscous effects within the flow domain is also assumed to be insignificant. This is a reasonable approximation if the concentration of the magnetite particles in a ferrofluid is not very high,\textsuperscript{3,36} the applied magnetic field is steady and close to uniform,\textsuperscript{37} and the flow shear rate is large enough to break agglomerates of magnetic particles formed by the applied magnetic field.\textsuperscript{38} The justification of these assumptions and their validity ranges is a complicated research area in its own right. The reader is referred to the discussions by Shliomis (see Ref. 7, pp. 85–111) and Odenbach and Thur (see Ref. 7, pp. 185–201) which contain a concise overview of this field and provide further references to a number of significant works in the area.

It is convenient to redefine pressure $p$ entering the momentum equation (2) so that it includes both a hydrostatic
component and a gradient component of the magnetic force (see also Ref. 7, pp. 86 and 87). In order to do this, equations of state (4) and (8) are used to write
\begin{equation}
\rho g + \mu_0 M \nabla H = \rho_0 [1 - \beta_s (T - T_a)] \mathbf{g} \\
+ \mu_0 [M_0 + \chi (H - H_a) - K (T - T_a)] \nabla H \\
= \nabla \left\{ \rho_0 (\mathbf{r} \cdot \mathbf{g}) + \mu_0 \left[ M_0 H + \frac{1}{2} \chi (H - H_a)^2 \right] \right\} \\
- (\rho_0 \beta_s \mathbf{g} + \mu_0 K \nabla H) (T - T_a),
\end{equation}
where coordinate vector \( \mathbf{r} = (x, y, z) \). Then introducing
\begin{equation}
P = p - \rho_0 (\mathbf{r} \cdot \mathbf{g}) - \mu_0 \left[ M_0 H + \frac{1}{2} \chi (H - H_a)^2 \right],
\end{equation}
Eq. (2) is rewritten as
\begin{equation}
\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \rho_0 \mathbf{v} \cdot \nabla \mathbf{v} = - \nabla P + \eta_0 \nabla^2 \mathbf{v} - \rho_0 \beta_s (T - T_a) \mathbf{g} \\
- \mu_0 K (T - T_a) \nabla H.
\end{equation}

The no-slip/no-penetration conditions require that all velocity components vanish at solid walls and the assumption of a thermodynamic equilibrium requires that the fluid temperature at the walls be equal to the wall temperature. Therefore standard hydrodynamic and thermal boundary conditions become
\begin{equation}
\mathbf{v} = 0, \quad T = T_w \pm \theta \quad \text{at} \quad x = \pm d.
\end{equation}
The boundary conditions for magnetic field and magnetization in the case of no surface currents are obtained from the following conditions:
\begin{equation}
(\mathbf{H}^e - \mathbf{H}) \times \mathbf{n} = 0, \quad (\mathbf{B}^e - \mathbf{B}) \cdot \mathbf{n} = 0 \quad \text{at} \quad x = \pm d,
\end{equation}
where superscript \( e \) denotes fields outside the layer and \( \mathbf{n} = (1, 0, 0) \) is the normal vector to the walls. Using Eqs. (6) and (10), the second of the conditions in Eq. (17) is rewritten as
\begin{equation}
[H^e - [(1 + \chi) H_a + \chi (H_a - H)] \Delta H \mp K \Theta] e_z \\
- (1 + \chi) \Delta H] \cdot \mathbf{n} = 0
\end{equation}
at \( x = \pm d \).

The governing equations and boundary conditions are nondimensionalized using
\begin{align*}
\mathbf{r} &= d \mathbf{r}', & \mathbf{v} = (\eta_0 / \rho_0 d) \mathbf{v}', \\
t &= (\rho_0 d^2 / \eta_0) t', & P = (\eta_0^2 / \rho_0 d^2) P', \\
T - T_a &= \Theta \theta', & \mathbf{H} = [K \Theta / (1 + \chi)] \mathbf{H}', \\
H &= [K \Theta / (1 + \chi)] H', & \mathbf{M} = [K \Theta / (1 + \chi)] \mathbf{M}', \\
M &= [K \Theta / (1 + \chi)] \mathbf{M}', & \mathbf{g} = g e_z.
\end{align*}

Then, omitting primes for simplicity of notation, we obtain
\begin{equation}
\nabla \cdot \mathbf{v} = 0,
\end{equation}
\begin{table}
\caption{Typical values of experimental parameters and fluid properties for which magnetoconvection patterns shown in Fig. 1 were observed.}
\begin{tabular}{|l|l|l|}
\hline
Notation & Parameter & Typical value \\
\hline
\hline
\rho_0 & Density & 1.25 \times 10^3 \text{ kg/m}^3 \\
\beta_s & Coefficient of thermal expansion & 8.6 \times 10^{-3} \text{ K}^{-1} \\
\kappa_v & Thermal diffusivity & 5 \times 10^{-6} \text{ m}^2 \text{s}^{-1} \\
\eta_0 & Dynamic viscosity & 8 \times 10^{-5} \text{ kg/m s} \\
\kappa_m & Integral magnetic susceptibility & 5 \\
\chi & Differential magnetic susceptibility & 5 \\
K & Pyromagnetic coefficient & \sim 10^2 \text{ A/m K} \\
B & External magnetic field & 4 \times 10^6 \text{ A/m} \\
T_s & Average temperature in the layer & 300 K \\
\Theta & Temperature difference between the walls & 1-15 K \\
2d & Distance between the walls & 3.5 mm \\
g & Gravity & 9.81 \text{ m/s}^2 \\
\hline
\end{tabular}
\end{table}

The dimensionless parameters appearing in the problem,
\begin{equation}
\text{Gr} = \frac{\rho_0^2 \beta_s \Theta d^3}{\eta_0^2}, \quad \text{Gr}_m = \frac{\rho_0 \mu_0 K^2 \Theta^2 d^2}{\eta_0^2 (1 + \chi)}, \\
\text{Pr} = \frac{\eta_0}{\rho_0 \kappa_v}, \quad N = \frac{H_s (1 + \chi)}{K \Theta},
\end{equation}
are the thermal and magnetic Grashof numbers characterizing the importance of buoyancy and magnetic forces, Prandtl number characterizing the ratio of viscous and thermal diffusion, and parameter \( N \) characterizing the theromagnetic properties of a working fluid, respectively. Using the data from Table I, we estimate the characteristic values of the above nondimensional parameters for preliminary experiments, some of which are illustrated in Fig. 1 as
Governing parameters.

Unless specified otherwise in the text or figures, the computational results will be presented for the above values of the governing parameters.

III. BASIC FLOW

Equations (19)–(26) admit steady solution of the form
\[ \mathbf{v}_0 = (0, v_0(x), 0), \quad \theta_0 = \theta_0(x), \quad P_0 = P_0(x), \quad H_0 = (H_0(x), 0, 0), \text{ and } \mathbf{e}_r = (1, 0, 0), \]
which satisfy
\[ DP_0 = -\text{Gr}_m \theta_0 DH_0, \quad D^2 v_0 = -\text{Gr} \theta_0, \quad (28) \]
\[ D^2 \theta_0 = 0, \quad D \left[ H_0 - \theta_0 \right] = 0, \quad (29) \]
\[ v_0 = 0, \quad \theta_0 = \pm 1, \quad (30) \]
\[ \text{He} = (1 + \chi)(H - 1) - (\chi - \chi_s)N \quad \text{at} \quad x = \pm 1, \]
where \( D = d/dx \). The basic state is therefore given by
\[ \theta_0 = -x, \quad v_0 = (\text{Gr}/6)(x^3 - x), \quad P_0 = -\text{Gr}_m(x^2/2) + C, \quad (31) \]
\[ H_0 = [\text{He} + (\chi - \chi_s)N]/(1 + \chi) - x = N - x, \quad (32) \]

\[ (\mathbf{v}, P, T, \mathbf{H}, M, M) = (\mathbf{v}_0, P_0, T_0, H_0, H_0, M_0, M_0) + [(\mathbf{v}_1(x), P_1(x), T_1(x), H_1(x), H_1(x), M_1(x), M_1(x))] \exp(\sigma t + i\alpha y + i\beta z + \text{c.c.}). \]

where \( \sigma = \omega^2 + i\sigma^T \) is the complex amplification rate, \( \alpha \) and \( \beta \) are real wavenumbers in the \( y \) and \( z \) directions, respectively, and c.c. denotes the complex conjugate of the expression in brackets. To satisfy Eq. (22) identically, it is convenient to introduce perturbation \( \phi_1(x) \exp(\sigma t + i\alpha y + i\beta z) \) of a magnetic potential so that
\[ H_1(x) = (D\phi_1(x), i\alpha \phi_1(x), i\beta \phi_1(x))^T, \]
\[ H_1(x) = D\phi_1(x), \quad (34) \]
\[ M_1(x) = (\chi D\phi_1(x) - (1 + \chi)\theta_1(x), i\alpha \chi \phi_1(x), i\beta \chi \phi_1(x))^T, \]
\[ M_1(x) = \chi D\phi_1(x) - (1 + \chi)\theta_1(x). \]

The linearization of Eqs. (19)–(26) about the basic state leads to
\[ Du_1 + i(\alpha v_1 + \beta w_1) = 0, \quad (35) \]
\[ \sigma u_1 + (\alpha^2 + \beta^2 + i\alpha v_0 - D^2)u_1 + DP_1 + \text{Gr}_m D\phi_1 \theta_1 + \text{Gr}_m \Theta_0 D^2 \phi_1 = 0, \quad (36) \]
\[ \sigma v_1 + Dv_0 u_1 + (\alpha^2 + \beta^2 + i\alpha v_0 - D^2)v_1 + i\beta P_1 - \text{Gr} \theta_1 + i\alpha \text{Gr}_m \Theta_0 D \phi_1 = 0, \quad (37) \]
\[ M_0 = [\chi \text{He} - (\chi - \chi_s)N]/(1 + \chi) + x = \chi e + N - x, \quad (33) \]

where \( C \) is an arbitrary constant. The basic flow temperature and velocity profiles are not affected by the magnetic field. In deriving expressions (32) and (33), it has been taken into account that by definition, \( M_0(0)/H_0(0) = \chi e \), and therefore, \( \text{He} = (1 + \chi e)N \). In other words, the external uniform magnetic field of dimensional magnitude \( \text{He} \) applied perpendicular to the vertical layer of the magnetic fluid creates field of magnitude \( H_0 = \text{He} / (1 + \chi e) \) in the midplane of the layer.

Solutions (32) and (33) for the magnetic field and magnetization are equivalent to those given in Ref. 16. Note also the peculiar behavior of pressure \( P_0 \). It is symmetric with respect to the midplane of the layer and attains its maximum at \( x = \pm 1 \). Therefore it tends to isolate regions near the cold and hot walls from each other, preventing a horizontal momentum exchange between them. As seen from the definitions of \( \text{Gr}_m \), this effect is solely due to the thermal variation in magnetization in the flow region: the pressure is constant across the layer in nonmagnetic fluids.

IV. LINEARIZED PERTURBATION EQUATIONS

Next we investigate linear stability of the basic state discussed in Sec. III with respect to infinitesimal disturbances which are periodic in the \( y \) and \( z \) directions. We use a standard normal mode approach and write perturbed quantities as

\[ \sigma \phi_1 + (\alpha^2 + \beta^2 + i\alpha v_0 - D^2)\phi_1 + i\beta P_1 + i\beta \text{Gr}_m \Theta_0 D \phi_1 \]
\[ = 0, \quad (38) \]
\[ \sigma \theta_1 + D\Theta_0 \mu_1 + \left( \frac{\alpha^2 + \beta^2 - D^2}{\text{Pr}} + i\alpha v_0 \right) \phi_1 = 0, \quad (39) \]
\[ \left( D^2 - \frac{1 + \chi e}{1 + \chi} (\alpha^2 + \beta^2) \right) \phi_1 = D \theta_1 = 0. \quad (40) \]

Note that parameter \( N \) does not appear in the perturbation equations, which is consistent with the conclusion of Ref. 16 that the magnitude of the uniform component of the external magnetic field does not influence the stability properties of the flow directly.39

The disturbance velocity and temperature fields are subject to standard homogeneous boundary conditions
\[ u_1 = v_1 = w_1 = \theta_1 = 0 \quad \text{at} \quad x = \pm 1. \quad (41) \]

As was discussed in Ref. 16, in the case of nonmagnetic boundaries, a perturbation of a magnetic field within a fluid layer causes perturbation of the external field. If there are no induced currents outside the layer and the ambient space is filled with nonmagnetic medium (air), then the external magnetic field has a potential \( \phi_1(x) \exp(\sigma t + i\alpha y + i\beta z) \) which, as
follows from Eqs. (5) and (6), satisfies Laplace’s equation
\[(D^2 - \alpha^2 - \beta^2) \phi' = 0\]
in regions \(x < -1\) and \(x > 1\). A physically relevant bounded solution then is given by
\[
\phi'(x) = \begin{cases} 
C_1 e^{\alpha x + \beta x}, & x < -1, \\
C_2 e^{-\alpha x + \beta x}, & x > 1.
\end{cases}
\]  
(42)
Linearization of the first of conditions in Eq. (17) and boundary condition (26) then leads to
\[
\phi_0 = \phi_1, \quad D\phi_1 = (1 + \chi)D\phi \quad \text{at} \quad x = \pm 1.
\]  
(43)
Eliminating \(C_{1,2}\) from Eqs. (42) and (43), the following mixed boundary conditions for \(\phi_1\) are obtained:
\[
(1 + \chi)D\phi_1 \pm \alpha\beta \phi_1 = 0 \quad \text{at} \quad x = \pm 1.
\]  
(44)
Note that a similar set of boundary conditions was considered in Ref. 40, but Ref. 16 contains a misprint (opposite signs).

Introduce the following generalized Squire’s transformation:
\[
(x,y,z) = (\tilde{x},\tilde{y},\tilde{z}), \quad \alpha = \tilde{\alpha}, \quad \alpha^2 + \beta^2 = \tilde{\alpha}^2, \quad \beta = \tilde{\beta},
\]
\[
u = \tilde{\nu}, \quad \beta \nu_1 = \tilde{\beta} \nu_1 = \tilde{\nu}, \quad \nu_1 = \tilde{\nu},
\]
\[
\theta_1 = \tilde{\theta}, \quad \beta \theta_1 = \tilde{\beta} \theta_1 = \tilde{\theta},
\]
\[
\alpha \rho = \tilde{\alpha} \rho, \quad \rho = \tilde{\rho}, \quad \chi = \tilde{\chi}, \quad \chi_1 = \tilde{\chi}_1,
\]
and note that \(\alpha \nu_1 = \tilde{\alpha} \nu_1\), where \(\tilde{\nu}_1 = \frac{\tilde{\nu}}{\tilde{\alpha}}\), \(\Theta_0 = \tilde{\Theta}_0\), and \(H_0 = \tilde{H}_0\). Then Eqs. (35)–(40) become
\[
D\tilde{\nu} + i\tilde{\alpha} \tilde{\nu} = 0,
\]  
(46)
\[
\tilde{\alpha} \tilde{\nu} + (\tilde{\alpha}^2 + i\tilde{\alpha} \tilde{\nu}_0 - D^2) \tilde{\nu} + D\tilde{P} + \tilde{\rho} \tilde{\nu} + \tilde{\rho} \tilde{\theta} \tilde{\nu} = 0,
\]  
(47)
\[
\tilde{\alpha} \tilde{\nu} + (\tilde{\alpha}^2 + i\tilde{\alpha} \tilde{\nu}_0 - D^2) \tilde{\nu} + i\tilde{\beta} \tilde{\nu} + \tilde{\beta} \tilde{\rho} \tilde{\theta} \tilde{\nu} = 0,
\]  
(48)
\[
\tilde{\alpha} \tilde{\theta} + D\tilde{\theta} \tilde{\nu} + \left(\tilde{\alpha}^2 - D^2 \frac{\tilde{\rho}}{\tilde{\beta}} + i\tilde{\alpha} \tilde{\nu}_0\right) \tilde{\theta} = 0,
\]  
(49)
\[
\left(D^2 - \tilde{\alpha}^2 - \tilde{\beta}^2\right) \tilde{\theta} - D\tilde{\theta} = 0,
\]  
(50)
where Eq. (48) is obtained in a standard way by multiplying Eq. (37) by \(\alpha\), Eq. (38) by \(\beta\), adding them together, and dividing the result by \(\tilde{\alpha}\). Note that only Eq. (49) contains \(\tilde{\nu}\) and thus it can be solved for \(\tilde{\nu}\) for any specified \(\tilde{\beta}\) after \(\tilde{\alpha}\). \(\tilde{\theta}\) and \(\tilde{\rho}\) are found from Eqs. (46)–(48), (50), and (51) which comprise an equivalent two-dimensional problem.

Consider two limiting cases. If \(\beta = 0\), i.e., if the solution is periodic only in the vertical direction, Eq. (49) is fully decoupled from the rest of the system and becomes
\[
\tilde{\sigma} \tilde{\nu} = \left(D^2 - \tilde{\alpha}^2 - i\tilde{\alpha} \tilde{\nu}_0\right) \tilde{\nu}.
\]  
(52)
It has a trivial solution, \(\tilde{\nu} = 0\), unless \(\tilde{\sigma}\) is an eigenvalue of Eq. (52). Upon multiplying Eq. (52) by \(\tilde{\nu}^*\), the complex conjugate of \(\tilde{\nu}\), integrating it across the fluid layer by parts, and taking the real part of the resulting expression, we obtain
\[
\tilde{\sigma}^2 = -\tilde{\alpha}^2 - \frac{\int_{-1}^1 |D\tilde{\nu}|^2 d\tilde{x}}{\int_{-1}^1 |\tilde{\nu}|^2 d\tilde{x}},
\]
which is negative regardless of the actual function \(\tilde{\nu}\). Therefore a classical result that any aperiodic motion in the horizontal \(\tilde{z}\) direction must decay exponentially quickly due to viscous dissipation follows so that for \(\beta = 0\), the asymptotic solution \(\tilde{\nu} = 0\) holds.

Note that in the other limit of \(\alpha = 0\) when the solution is periodic in the horizontal \(\tilde{z}\) but not vertical \(\tilde{y}\) direction, the Squire transformation requires that \(\tilde{\rho} = 0\) for any finite \(\tilde{\rho}\). This means that thermal convection characterized by the Grashof number plays no role at all in establishing a periodic flow pattern in the horizontal direction. It is fully defined by the thermomagnetic effects and by the value of \(\tilde{\rho}\).

V. NUMERICAL RESULTS

Equations (46)–(51) are discretized using the pseudospectral Chebyshev collocation method as introduced in Refs. 41 and 42 and implemented in Refs. 43 and 44. This spatial approximation converges exponentially quickly to the exact solution. Fifty-two collocation points used in the current computations guarantee that all digits in the reported numerical results are significant and accurate. Upon discretization and exclusion of Eq. (49), system (46)–(51) results in a generalized algebraic eigenvalue problem for the complex amplification rate \(\tilde{\sigma}\),
\[
(A + \tilde{\sigma}B)q = 0,
\]  
(53)
where \(A = A(\tilde{\alpha}, \tilde{\rho}, \tilde{\rho}_m, \tilde{P}, \tilde{\chi}, \tilde{\chi}_1)\) and \(B\) are matrices obtained after discretization of the perturbation equations and eigenvector \(q\) contains discretized components of \((\tilde{u}, \tilde{v}, \tilde{\rho}, \tilde{\theta}, \tilde{\phi})^T\). The eigenvalue problem is solved using a standard LAPACK subroutine ZGGEV as implemented in the Intel Math Kernel Library. Once both \(\tilde{\sigma}\) and \(q\) are found, the equation
\[
(D^2 - \tilde{\alpha}^2 - \tilde{\beta}^2 - i\tilde{\alpha} \tilde{\nu}_0)\tilde{\nu} = i\tilde{\beta} (\tilde{\rho} + \tilde{\rho}_m \tilde{\theta} D \tilde{\phi})
\]  
(54)
is solved for \(\tilde{\nu}\). The inverse Squire transformation (45) then recovers full three-dimensional solutions for perturbations.

A. Numerical accuracy check

In order to test the numerical code, the gravitational convection threshold (for \(\tilde{\rho}_m = 0\)) is computed for \(\tilde{P} = 0.71\). The critical values of \(\tilde{\rho}_m = 502.35\) and \(\tilde{\alpha}_c = 1.405\) are obtained which, after multiplying by the corresponding scaling factors of 16 and 2, respectively, agree with previously known ac-
accurate results (e.g., Ref. 43). The critical values for the magnetic convection threshold (for \( \overline{Gr}=0 \)) for the case of \( \tilde{X}_s=\tilde{X}_c=4 \) were also computed to obtain \( \overline{Gr}_{mc}=1.3934 \) and \( \tilde{\alpha}_c=1.9278 \). These compare well with the values of \( \overline{Gr}_{mc}=1.3846 \) and \( \tilde{\alpha}_c=1.95 \) computed from the corresponding data reported in Ref. 16 (a slight discrepancy is attributed to a lower spatial resolution used in Ref. 16). As an additional test, the onset of magnetic convection (\( \overline{Gr}=0 \)) between ferromagnetic plates is considered. As discussed by Gotoh and Yamada, in this case, the appropriate boundary conditions for the perturbation part of the magnetic potential are \( \phi_0(\pm 1)=0 \). Unfortunately, the authors do not report the numerical values of their critical magnetic Rayleigh numbers, presenting them only graphically. However, it can be shown that this problem results in equations similar to those for convection in a vertical layer of a dielectric fluid between two vertical plates considered in Ref. 46 and to that of vibrational convection in a vertical layer \(^{47,48} \) (see also discussion in Ref. 32). Therefore the threshold value of \( \overline{Gr}_{mc} \) and \( \tilde{\alpha}_c \) is estimated from the corresponding vibrational and dielectric Rayleigh and wavenumbers reported in Refs. 48 and 46, respectively. These estimations are \((\overline{Gr}_{m1}, \tilde{\alpha}_{c1})=(1.024, 1.61)\) and \((\overline{Gr}_{m2}, \tilde{\alpha}_{c2})=(1.02341, 1.613)\). They are in excellent agreement with our values of \((\overline{Gr}_{mc}, \tilde{\alpha}_c)=(1.02367, 1.613)\).

### B. Stability results for an equivalent two-dimensional problem

In order to clearly identify all possible physical mechanisms leading to the onset of convection in the considered geometry, we first report the stability results obtained for an equivalent two-dimensional problem given by Eqs. (46)–(48), (50), and (51). Formally, it corresponds to \( \beta=0 \), i.e., to disturbance patterns which are periodic in the vertical \( y \) direction but not in the horizontal \( z \) direction (horizontal convection rolls). For each set of physical governing parameters, the problem is solved for a range of wavenumbers \( \tilde{\alpha} \) to locate the maximum of the disturbance amplification rate \( \sigma^* \), see, for example, the left plot in Fig. 3. Then the values of \( \overline{Gr}_m \) or \( \overline{Gr} \) are iteratively changed until the set of values is found such that the maximum value of \( |\sigma^*| \) becomes smaller than the given threshold \((10^{-7})\). This set of parameters then gives a point on the marginal stability boundary. Repeating the computational process, we obtain a full stability diagram presented in Fig. 2(a).

The stability region for an equivalent two-dimensional problem is bounded by three lines each representing different types of instability characterized by its own wavenumber, see Fig. 2(b). The solid line originates at \( \overline{Gr}_m=0 \) and thus corresponds to the onset of a classical thermogravitational convection. For large-Prandtl-number fluid, this type of convection is represented by two counterpropagating thermal waves. Numerically, this is seen from the right plot in Fig. 3: The eigenvalues of the linearized problems appear as the complex conjugate pairs (the \( \sigma^* \) values have equal magnitudes but opposite signs). In turn the corresponding wavespeeds \( \tilde{c} \) shown by solid lines in Fig. 2(c) have opposite signs as well. It is noteworthy that the wavespeeds of disturbance thermal waves leading to thermogravitational convection are larger than the maximum speed of the basic flow, see points A and B in Table II. On the other hand, when the role of magnetic effects increases, the disturbance wavespeed decreases, see point C in Table II. Such a wavespeed behavior is indicative of the gradual change in the instability mechanism from thermogravitational to thermomagnetic as the ratio \( \overline{Gr}_m/\overline{Gr} \) increases. This will be discussed in more detail further in this section.

Typical disturbance velocity and thermal fields for thermomagnetoconvection are shown as a series of snapshots in Figs. 4 and 5. The fields are composed of two counterpropagating waves whose amplitudes (which are undetermined in the linear stability framework) are assumed to be equal so that the resulting pattern remains symmetric. As noted in Ref. 32, such a symmetry is indeed observed in fully nonlinear simulations provided that the boundary conditions are also symmetric. The instability velocity pattern
instability boundary approximately below point C in Fig. 4. During the transition, central vortices are clearly change the direction of rotation, as seen in a series of snapshots in Fig. 4. During the transition, central vortices are destroyed and replaced by pairs of short-lived vortices which appear near the walls and propagate in the direction of the local basic flow before they are replaced by central vortices rotating in the opposite direction again. As expected, the direction of the vortex rotation is correlated with the location of warmer fluid regions: Less dense warmer fluid (see light areas in Fig. 5) tends to rise, confirming the thermogravitational buoyancy-driven nature of this instability. In contrast to the stationary central vortex system seen in Fig. 4, the thermal field snapshots clearly demonstrate the presence of two counterpropagating thermogravitational waves: The alternating warm and cold fluid regions shift upwards along the hot left wall and downwards along the cold right wall.

Using a disturbance energy analysis, it was shown in Ref. 35 that when the value of $\tilde{Gr}$ decreases along the solid instability boundary approximately below point C in Fig. 2(a), the thermogravitational instability waves are replaced with thermomagnetic waves. While the transition between the two types of waves is continuous along the solid line in Fig. 2(a) and their appearances are similar, the physical mechanisms causing them are quite different. The thermogravitational waves are due to a buoyancy force acting in the vertical direction, while thermomagnetic waves are caused by a magnetic force acting across the gap. To avoid any ambiguity, the nature of this force is discussed in detail in the following paragraph in the context of stationary pure thermomagnetic convection. However, it is important to keep in mind that the physical mechanism discussed next is dominant for both stationary and wave instability regimes occurring at the relatively small values of $\tilde{Gr}$, see Ref. 35.

The stability region in Fig. 2(a) is bounded from below by the dashed line which originates at $\tilde{Gr}=0$ and therefore corresponds to the onset of magnetoconvection. As seen from Figs. 6 and 2(b), magnetoconvection is characterized by stationary patterns since the corresponding $\tilde{\sigma} = \tilde{c} = 0$. This observation is consistent with findings of Refs. 16 and 49. As seen from Fig. 2(b), stationary magnetoconvection patterns have larger wavenumbers than those of thermogravitational convection. This distinction is most evident for smaller values of the magnetic Grashof number when the thermomagnetic convection rolls have a characteristic size about 1.5 times smaller than their thermogravitational counterparts. Typical disturbance fields for magnetoconvective instability are shown in Fig. 7. Similar to thermogravitational instability, this instability leads to the appearance of central vortices seen in the leftmost plot in Fig. 7. While these vortices are also stationary, they do not change the direction of their rotation. Since the regions of warm and cool fluids are centrally located above each other, they cannot cause buoyancy-

![Graph showing disturbance temporal amplification rates and frequencies as functions of the combined wavenumber $\tilde{a}$ for $\tilde{Gr}_{en,Gr} = (0.40, 0.974)$ (onset of thermogravitational convection).]

**TABLE II.** Selected critical values for Pr=130 and $\tilde{\chi}=\chi_s=5$. Here $\tilde{v}_{0,\text{max}}$ is the maximum speed of the basic flow and $\tilde{c} = -\tilde{\sigma}/\tilde{a}$ is the disturbance wavespeed. The corresponding points A–D are shown in Fig. 2.

<table>
<thead>
<tr>
<th>$\tilde{Gr}_{en}$</th>
<th>$\tilde{Gr}_{mc}$</th>
<th>$\tilde{a}$</th>
<th>$\tilde{c}$</th>
<th>$\tilde{v}_{0,\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>40.974</td>
<td>0</td>
<td>1.2384</td>
<td>$\pm 2.692$</td>
</tr>
<tr>
<td>B</td>
<td>39.976</td>
<td>1.398</td>
<td>1.2563</td>
<td>$\pm 2.622$</td>
</tr>
<tr>
<td>C</td>
<td>16.69</td>
<td>15.775</td>
<td>1.6964</td>
<td>$\pm 1.033$</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>1.398</td>
<td>1.9365</td>
<td>0</td>
</tr>
</tbody>
</table>
driven motion of fluid and thus cannot be the reason for the vortex’s appearance. Instead the pressure gradient directed from dark to light regions in the third plot is responsible for the vortical motion. For example, consider the region near the hot left wall near $y=1.5$. As seen from the third plot in Fig. 7, the value of the $y$ component of the pressure gradient $-\frac{\partial P}{\partial y}$ attains its positive maximum at this location. Responding to such a driving force, the fluid moves upwards. 

---

**FIG. 4.** Typical disturbance velocity fields for thermogravitational instability for $(Gr_m, Gr) = (0.40, 974)$. Snapshots are for the indicated times where $T = 1.88$ is the period of oscillations.

**FIG. 5.** Typical disturbance thermal fields for the same set of parameters as in Fig. 4. Lighter areas correspond to higher temperature.
here, as illustrated in the leftmost plot. At the same time the value of \(-\partial P_1/\partial y\) reaches its negative minimum near the cold wall around \(y=1.5\), driving the fluid downwards. As a result, a vortex is formed which is seen near \(y=1.5\) in the leftmost plot.

While providing the most obvious explanation for the disturbance flow pattern, the above discussion of the driving pressure gradient gives only a secondary reason for the existence of magnetoconvection. Indeed the definition of pressure \(P\) given by Eq. (14) is introduced for mathematical convenience and contains a combination of thermodynamic, hydrostatic, and magnetic components. It is not directly seen which of them dominates in any particular regime. In order to disclose the true nature of the primary physical mechanism driving convection, we refer to the last term in Eq. (2). It represents the force which acts on a magnetized fluid driving it to the regions with a stronger magnetic field. The primary magnetic field equation (32) weakens with the distance away from the hot wall. Therefore stronger magnetized fluid particles tend to move towards the hot wall. According to Eq. (34), the disturbance magnetization field is affected by two factors: the induced disturbance magnetic field \(H_1\) and the disturbance thermal field (more precisely, the negative of it). A comparison of the second and fourth plots in Fig. 7 shows that in the thermomagnetic convection regime, fluid magnetization is strongly correlated with the disturbance thermal field: Highly magnetized regions correspond to cooler locations. Therefore a larger thermal gradient leads to more pro-

![FIG. 6. (Color online) Same as in Fig. 3 but for \((\overline{Gr}, \overline{Gr})=(1.398,0)\) (onset of stationary magnetoconvection).](image_url)

![FIG. 7. Disturbance fields for magnetoconvective instability for \((\overline{Gr}, \overline{Gr})=(1.398,0)\).](image_url)
nounced variations in the local magnetization. On the other hand, the induced magnetic field (the rightmost plot) tends to weaken the local magnetization gradient and thus thermomagnetic effects. The disturbance energy analysis presented in Ref. 35 confirms that the induction of the magnetic field by the displacement of magnetized particles always plays a stabilizing role. Yet the magnitude of this effect is always smaller than the destabilization due to the thermomagnetic effects. Therefore the details of thermomagnetic convection mechanism are as follows. Thermal perturbations lead to the formation of cooler stronger magnetized regions in the flow domain (light areas in the fourth plot in Fig. 7). The fluid in these regions is then driven towards the hot wall where the basic magnetic field is stronger (see the major flow direction between \( y = 0.5 \) and \( y = 1 \) in the leftmost plot). This impinging jet of magnetized fluid hits the wall and creates a high pressure region there, see the third plot. The so-created vertical pressure gradient is, subsequently, responsible for driving the fluid away from the impingement point. The fluid conservation then requires that warmer less magnetized fluid is displaced towards the cold wall, thus creating the vortices seen in the leftmost plot in Fig. 7.

There also exists a small parametric region in the lower right corner of Fig. 2(a) [see also the close-up in Fig. 12(b)] where the stability region is bounded from below by the dash-dotted line. Unlike the solid and dashed lines corresponding to the thermogravitational/thermomagnetic waves and stationary magnetoconvection, respectively, the nature of this dash-dotted boundary is not apparent from Fig. 2(a) alone. Thus we refer to the growth rate spectrum plots in Fig. 8 to identify it. It follows from Fig. 2(b) that the dash-dotted line corresponds to the rightmost top maximum in the left plot in Fig. 8. It is also seen from Fig. 8 that this maximum is continuously connected to the leftmost maximum which, according to the wavenumber and wavespeed data in Figs. 2(b) and 2(c), corresponds to the onset of thermomagnetic wave instability (see the discussion of solid lines in Fig. 2 above). Therefore we conclude by continuity that the dash-dotted segment of the stability boundary in Figs. 2(a) and 12(b) still corresponds to thermomagnetic waves but characterized by a larger wavenumber. Thus it is expected that at larger values of the magnetic Grashof number, the unsteady thermomagnetic convection may reveal itself as a combination of two pairs of counterpropagating thermomagnetic waves [see solid and dash-dotted lines in Fig. 2(c)] with different wavenumbers and wavespeeds. The disturbance fields corresponding to the left and right maxima in Fig. 8 are qualitatively similar to those shown in Figs. 4 and 5 and are not presented here.

Similarly, we deduce that the middle maximum in the left plot in Fig. 8 corresponds to a stationary magnetoconvection mode (the corresponding values of \( \sigma = c = 0 \)) with disturbance fields similar to those shown in Fig. 7.

It is worthwhile noting that unlike the dashed and solid marginal stability lines in Fig. 2(a), the dash-dotted segment of the stability boundary has a finite extent. The reasons for this and become clear from Figs. 9 and 10. At the right end of the dash-dotted segment, the \( \sigma^d \) maximum seen in Fig. 8 for large wavenumbers shifts to the left until it disappears, blending with the right “slope” of the maximum existing for smaller wavenumbers. As a result the range of unstable disturbance wavenumbers widens, forming a plateau seen in the left plot in Fig. 9 and the distinction between short and long wavelength modes becomes obscured. All disturbances with wavenumbers \( 1.5 \leq \tilde{a} \leq 3 \) have approximately the same growth rate and cannot be distinguished on this ground alone. This is in contrast to the regime depicted in Fig. 8 where disturbances corresponding to the three discrete wavenumbers (three maxima) are expected to dominate the instability pattern.

At the left end of the dash-dotted stability boundary segment, the middle maximum seen in the left plot of Fig. 8 dominates. This maximum corresponds to a stationary magnetoconvection instability mode. It leaves little room for disturbance waves with large wavenumbers and suppresses waves with small wavenumbers completely, see Fig. 10. Two
real eigenvalues $\sigma$ (one of which corresponds to a stationary thermomagnetic instability mode) collide at $\tilde{\alpha} \approx 3.6$, see the left plot in Fig. 10, and form a pair of complex conjugate eigenvalues which correspond to two counterpropagating thermomagnetic waves. The real part $\tilde{\sigma}^R$ of these complex conjugate eigenvalues remains negative for $\overline{Gr}_m \approx 11.8$ which corresponds to the left end of the dash-dotted line in Fig. 2(a). This is the smallest value of $\overline{Gr}_m$ for $Pr=130$ for which thermomagnetic waves can exist.

Overall, the computational stability results show that magnetic effects play a strongly destabilizing role in thermodiffugravitational convection. The critical Grashof number decreases rapidly with the increasing magnetic Grashof number, see the solid line in Fig. 2(a). A stronger magnetic field also slows down thermal waves as seen from Fig. 2(c) and Table II. On the other hand, thermodiffugravitational effects tend to inhibit magnetoconvection: a stronger magnetic field is required for the onset of magnetoconvection when $\overline{Gr}$ is increased. This effect was also confirmed by the disturbance energy analysis in Ref. 35.

C. Three-dimensional results

In Sec. V B the stability results obtained by solving an equivalent two-dimensional problem (46)–(48), (50), and (51) were discussed. Physically, these results correspond to convection patterns which are periodic in the vertical $y$ direction and independent of the horizontal $z$ coordinate. In other words instability patterns consisting of horizontal convection rolls with $\beta=w_1=0$ were considered in Sec. V B and

![Figure 9](https://example.com/fig9.png)

**FIG. 9.** (Color online) Same as in Fig. 3 but for $(\overline{Gr}_m, \overline{Pr})=(16.7, 6.5)$ (thermomagnetic waves with a wide wavenumber range).

![Figure 10](https://example.com/fig10.png)

**FIG. 10.** (Color online) Same as in Fig. 3 but for $(\overline{Gr}_m, \overline{Pr})=(11.8, 1.8)$. In the left plot: Modes 1 and 2 correspond to stationary magnetoconvection rolls and large-wavenumber thermomagnetic waves, respectively.
the physical mechanisms causing them were identified. While the physical reasons for the detected instabilities do not depend on whether two- or three-dimensional patterns have been considered, the parametric stability region is very sensitive to the spatial orientation of perturbation patterns. In this section we will systematically describe various regions of three-dimensional instability which unfold from the two-dimensional stability diagram of Fig. 2(a) upon the inverse transformation (45). The most essential feature of this transformation is that the magnetic Grashof number remains invariant while the thermogravitational Grashof number for three-dimensional patterns is necessarily larger than its equivalent two-dimensional counterpart. Namely,

$$\text{Gr} = \frac{\tilde{\alpha}}{G_r}, \quad \alpha = \sqrt{\alpha^2 - \beta^2}.$$  \hspace{1cm} (55)

Geometrically, this means that under the inverse transformation, all lines not crossing the horizontal axis in the equivalent two-dimensional stability diagram of Fig. 2(a) will be shifted upwards and only points with $\text{Gr}=0$ will remain fixed. Therefore if a line separates the stability region below and the instability region above, then upon transformation, the stability region will be enlarged. Equivalently this means that two-dimensional disturbance structures consisting of horizontal convection rolls are most dangerous. Conversely, if a line separates an instability region below and a stability region above, then under the inverse transformation, the instability region will be enlarged, meaning that oblique or vertical convection patterns are more dangerous than horizontal convection rolls. Thus the instability observed experimentally will be represented by three-dimensional patterns.

Figure 11 provides another illustration for the meaning of transformation (55). Note that $\gamma = \pm \arccos(\sqrt{\alpha^2 - \beta^2} / \tilde{\alpha}) = \pm \arccos(Gr / Gr)$ is the inclination angle of convection rolls with respect to the horizontal direction so that $\beta=0$ (or $\alpha=\tilde{\alpha}$) corresponds to horizontal patterns (solid lines). Within a linear analysis framework, both positive and negative inclination angles are equally possible. For $\beta=0$, the amplification rate $\sigma^2 < 0$ and the basic flow is stable with respect to vertically periodic two-dimensional patterns in the range $3.2 \leq \text{Gr} \leq 30$, see solid lines crossing the $\sigma^2=0$ level in Fig. 11(a). However, if inclined three-dimensional patterns are considered, the $\sigma^2 > 0$ region bounded from the left by the $\text{Gr}=0$ axis and from the right by the solid line 1 expands to the right (see the dashed line 1). In the limit of $\alpha=0$, $\gamma = \pm 90^\circ$ (i.e., the pattern consists of the vertical magnetoconvection rolls) and the $\sigma^2 > 0$ region bounded by the dashed line 1 region will cover the stability region found for two-dimensional disturbances completely. The solid line 3 corresponding to thermomagnetic waves will also move to the right into the two-dimensional stability region. This means that three-dimensional oblique thermomagnetic waves and vertical stationary magnetoconvection rolls are more dangerous than two-dimensional vertically propagating thermomagnetic waves and horizontal magnetoconvection rolls. On the other hand, under transformation (55), the instability region found for vertically propagating thermogravitational waves at $\text{Gr} \approx 30$ [solid line 2 in Fig. 11(a)] will move towards the larger values of $\text{Gr}$. This implies that the two-dimensional vertically propagating thermogravitational waves are more dangerous than similar three-dimensional oblique waves.

Such a geometrical consideration enables us to identify 16 regions shown in Fig. 12 which are characterized by distinct instability patterns (or combinations of patterns). These parametric regions are separated by the corresponding segments of the two-dimensional marginal stability boundaries and vertical dotted lines. They are described in detail next.

Region [1]. This is the only region of true linear stability. It corresponds to $G_{\infty} \leq 1.398$ and $G_{\infty} \leq 40.974$. Neither thermogravitational nor magnetic convection can develop for this range of parameters so that the basic flow remains parallel with the linear temperature profile.

Region [2]. Here the thermogravitational convection sets. The most dangerous disturbances correspond to two counterpropagating waves similar to those found in natural convection of large-Prandtl-number fluids. They lead to a formation of waves whose snapshot is shown as a set of horizontal convection rolls in Fig. 13. The observation plane is chosen to be at $\alpha=0.95$ near the cold wall in order to simplify comparison with future experiments: In practice, thermal fields are observed through a transparent cold wall.
while the hot wall is attached to a nontransparent heater. The thermal pattern seen in the right plot of Fig. 13 propagates downwards, while its counterpart near the hot wall propagates upwards. In contrast, the velocity field shown in the left snapshot in Fig. 13 consists of rolls which do not propagate but rather periodically change the direction of their rotation. This confirms that the combination of two counterpropagating thermogravitational waves is responsible for the formation of a standing velocity wave. The application of a magnetic field does not change the orientation of convection rolls. However, it has an overall destabilizing effect, reducing the value of the critical Grashof number below $Gr = 40,974$.

**Region [3]**. In this region only a magnetoconvective instability is present. As discussed in Sec. V B, increasing the value of $Gr$ tends to inhibit magnetoconvection. Therefore the fastest amplification of magnetoconvective disturbances is achieved along the horizontal axis of Fig. 2(a). According to transformations (45), this amplification rate will be observed for three-dimensional disturbances at any value of the Grashof number defined by Eq. (55). Since $\bar{Gr} \to 0$ for the most dangerous disturbances, the finite value of $Gr$ can only be obtained if $\alpha \to 0$ and $\bar{\alpha} \to \beta$. This means that in region, the vertical stationary magnetoconvection rolls are expected to dominate the instability pattern yet weaker inclined rolls may also exist. This is in qualitative agreement with experimental results shown in Fig. 1. It is also worthwhile noting that the similarity of the governing equations for magnetic and vibrational types of convection briefly noted in Sec. V A has an experimental confirmation at least in region [3]: Experiments reported in Ref. 50 showed that similarly to magnetoconvection considered in the present work, the vertical orientation of convection rolls is also preferred in a differentially heated vertical layer of fluid subject to horizontal vibrations in the plane of the layer.

**Region [4]**. The instability pattern here is a combination of horizontal rolls resulting from a pair of thermogravitational waves and stationary vertical magnetoconvection rolls.

**Region [5]**. This region is a stability region for horizontal convection rolls. However, as follows from the discussion of Fig. 11 above, the parallel basic flow here is unstable with respect to the stationary vertical and oblique magnetoconvection rolls. This region is similar to region [3], but oblique thermomagnetic instability waves may be found here which correspond to the large-wavenumber instability (dash-dotted line in Fig. 12). Yet as seen from Fig. 11(a) the amplification rate of this mode is significantly smaller than that of stationary vertical rolls. Therefore the large-wavenumber thermo-

![FIG. 12. (Color online) (a) Three-dimensional unfoldings of the stability diagram for an equivalent two-dimensional problem; (b) close-up of a region with three types of instability modes. Line types are the same as in Fig. 2 discussed in text. To improve readability, small parametric regions (Refs. 13–16) are only marked in the close-up.](image1)

![FIG. 13. Three-dimensional instantaneous disturbance velocity (left) and temperature (right) fields near the cold wall in the plane $x=0.95$ for $(Gr_Gm_Gr)=0.41)$. Thermogravitational instability. Lighter areas correspond to warmer fluid.](image2)
magnetic pattern might not be easy to observe experimentally. Yet it should be possible to detect its unsteady signature since it corresponds to a pair of thermomagnetic waves propagating over a stationary vertical magnetoconvection background.

Region [6]. In this region three instability patterns coexist: stationary vertical magnetoconvection rolls, vertically propagating thermogravitational waves, and larger wavenumber oblique thermomagnetic waves. Again stationary vertical magnetoconvective rolls are expected to be the most prominent feature of the resulting mixed instability pattern at least near the lower boundary of this region since the corresponding disturbances have the largest amplification rate.

Region [7]. This region corresponds to basic flow which is stable with respect to the horizontal convection rolls. In order to better understand its three-dimensional stability features, refer to Fig. 14. As an example let us consider the point \((Gr_0, Gr_m)=(15,15)\) which belongs to this parametric domain. Note that the solid \(\sigma^a(Gr)\) line in Fig. 14(a) which shows the maximum possible linear amplification rates for various two-dimensional instability modes at \(Gr_m=15\) has three maxima for \(Gr<15\): at \(Gr_1=0\), \(Gr_2=2\), and \(Gr_3=7\). It is straightforward to deduce from Figs. 2(c) and 14(b) that these maxima correspond to stationary magnetoconvection and large- and small-wavenumber thermomagnetic waves, respectively. Using transformation (55) we then deduce that despite the fact that the flow is stable with respect to any two-dimensional vertically periodic disturbance pattern, it is unstable with respect to stationary thermomagnetic waves with the inclination angle of \(\gamma_1=\pm \arccos(Gr_1/Gr_0)\approx \pm 90^\circ\), i.e., vertical rolls, see Fig. 15, and large- and small-wavenumber counterpropagating thermomagnetic waves forming rolls inclined at \(\gamma_2=\pm \arccos(Gr_2/Gr_0)\approx \pm 82^\circ\) and \(\gamma_3=\pm \arccos(Gr_3/Gr_0)\approx \pm 62^\circ\), respectively, see Figs. 16 and 17. Note also that the larger the value of \(Gr\) in region [7], the larger the inclination angle of thermomagnetic waves. For example, the upper boundary of region [7] at \(Gr_m=15\) corresponds to \(Gr\approx 23\) [at which vertically propagating thermogravitational waves appear, see the solid line 2 in Fig. 14(a)]. The inclination angles for this parametric regime are then expected to be \(\gamma_1=\pm 90^\circ\) (vertical stationary magnetoconvection rolls), \(\gamma_2=\pm 85^\circ\) (large-wavenumber thermomagnetic waves), and \(\gamma_3=\pm 72^\circ\) (small-wavenumber thermomagnetic waves). It is also seen from Fig. 14(a) that the amplification rates of stationary thermomagnetic convection rolls (lines 1) is significantly larger than that for thermomagnetic waves (lines 3 and 4). Therefore vertical rolls are expected to dominate the convection pattern. The presence of oblique thermomagnetic waves should be possible to detect by observing the relatively weak “blinking” superposed on developed vertical rolls. Such blinking was indeed seen in preliminary experiments.

Region [8]. If the Grashof number is increased so that one enters region [8], then, in addition to three vertical and

![FIG. 14. (Color online) Same as in Fig. 11 but for \(Gr_m=15\). Labels in plot (a) denote (1) stationary magnetoconvection rolls, (2) thermogravitational waves, (3) large-wavenumber thermomagnetic waves, and (4) small-wavenumber thermomagnetic waves.](image)

![FIG. 15. Three-dimensional disturbance velocity (left) and temperature (right) fields near the cold wall in the plane \(z=0.95\) for \((Gr_m,Gr)=(15,15)\). Stationary magnetoconvective instability. Lighter areas correspond to warmer fluid.](image)
near vertical oblique convection patterns described above, a pair of thermogravitational vertically propagating waves also appear. As seen from Fig. 14(a) for \( \text{Gr} \gtrsim 30 \), the amplification rate of such a horizontal pattern becomes larger than that of oblique patterns. Therefore it is expected that the overall flow pattern will primarily be a combination of stationary vertical rolls and horizontal rolls formed by vertically propagating thermogravitational waves. However, this pattern will be less regular than in region [6] due to the presence of weak oblique waves.

**Region [9].** In this region vertically propagating thermomagnetic waves lead to the appearance of horizontal convection rolls. However, their amplification rate is significantly smaller than that of oblique waves and of vertical magnetoconvection rolls, see Fig. 14(a), which are expected to dominate the flow.

**Region [10].** Here the instability is caused by the magnetoconvection mechanism alone. The orientation of the resulting stationary convection rolls can be arbitrary, however, for any nonzero value of \( \text{Gr} \), the basic flow tends to cause the vertical alignment of the disturbance patterns.

**Region [11].** This region is similar to region [5] with the difference that large-wavenumber thermomagnetic waves are expected to propagate at a smaller inclination angle. Vertical stationary magnetoconvection rolls are still expected to dominate while small-wavenumber thermomagnetic waves are not present.


**Region [13].** The instability pattern here consists of the small-wavenumber thermomagnetic waves propagating vertically, slightly oblique large-wavenumber thermomagnetic waves, and predominantly vertical stationary magnetoconvection rolls which can form oblique or even horizontal patches.

**Region [14].** Same as region [13] but stationary magnetoconvection rolls cannot be horizontal here.

**Region [15].** In this region the superposition of vertical stationary magnetoconvection rolls, vertically propagating small-wavenumber thermomagnetic waves, and weak obliquely or vertically propagating large-wavenumber thermomagnetic waves is expected to define the flow pattern.

**Region [16].** This is the smallest parametric region in the diagram. It is characterized by the presence of the predominantly vertically propagating large-wavenumber thermomagnetic waves and mostly vertical stationary magnetoconvection rolls. Small-wavenumber thermomagnetic waves are not present.

To conclude, the above analysis demonstrates that magnetoconvection plays a major destabilizing role for \( \text{Gr}_m > 1.4 \). Stationary vertical convection rolls should be experimentally observable in parametric regions [3]–[16]. Vertically counterpropagating thermogravitational waves should be visible in regions [2], [4], [6], and [8], while oblique thermomagnetic waves should be seen in regions [5], [7], [9], and [11]–[15].

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**FIG. 16.** Same as in Fig. 15, but for large-wavenumber thermomagnetic waves. The shown patterns correspond to \( \gamma = -82^\circ \).

**FIG. 17.** Same as in Fig. 15 but for small-wavenumber thermomagnetic waves. The shown patterns correspond to \( \gamma = -62^\circ \).
VI. CONCLUSIONS

As a result of this comprehensive linear stability study of mixed thermogravitational and magnetic convections in a differentially heated vertical layer of ferrofluid placed in a uniform perpendicular magnetic field, 16 distinct parametric regions have been identified, each characterized by their own instability patterns. The patterns are shown to consist of vertical stationary magnetoconductive rolls, vertically counterpropagating thermogravitational waves which are typically found in convection of large-Prandtl-number fluids, and/or counterpropagating oblique thermomagnetic waves. It is also shown that for 14 out of 16 identified regions, the growth rate of the stationary magnetoconvective instability is larger than that for the thermogravitational or thermomagnetic waves. Therefore, it is expected that vertical or oblique rather than horizontal rolls will dominate the complex instability patterns for $Gr_{th} > 1.4$. Preliminary experiments conducted in the Laboratory of Magnetic Fluids (LMF) at the Department of General Physics at Perm State University (PSU), Russia, confirm this observation. The exact type of resulting mixed convection patterns cannot be determined within a linear approach alone. Therefore nonlinear analysis of interacting patterns is currently being undertaken by the author. The preparation for a detailed quantitative experimental study is also underway at LMF, PSU which among other goals will help in estimating the significance of physical factors (such as gravitational sedimentation and magnetoviscous effects) not accounted for in this initial computational investigation.

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39. The indirect influence is felt through the values of $\chi$ and $\chi_*$ which, in general, depend on the strength of the applied field.


