Towards accurate modelling of galaxy clustering on small scales: testing the standard $\Lambda$CDM + halo model

Manodeep Sinha,1,2,3★ Andreas A. Berlind,1★ Cameron K. McBride,4
Roman Scoccimarro,5 Jennifer A. Piscionere1,2 and Benjamin D. Wibking1,6

1Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235, USA
2Centre for Astrophysics & Supercomputing, Swinburne University of Technology, 1 Alfred St, Hawthorn, VIC 3122, Australia
3ARC Centre of Excellence for All Sky Astrophysics in 3 Dimensions (ASTRO 3D)
4Harvard-Smithsonian Center for Astrophysics, 60 Garden St, Cambridge, MA 02138, USA
5Center for Cosmology and Particle Physics, New York University, New York, NY 10003, USA
6Department of Astronomy, The Ohio State University, Columbus, OH 43210, USA

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ABSTRACT

Interpreting the small-scale clustering of galaxies with halo models can elucidate the connection between galaxies and dark matter haloes. Unfortunately, the modelling is typically not sufficiently accurate for ruling out models statistically. It is thus difficult to use the information encoded in small scales to test cosmological models or probe subtle features of the galaxy–halo connection. In this paper, we attempt to push halo modelling into the ‘accurate’ regime with a fully numerical mock-based methodology and careful treatment of statistical and systematic errors. With our forward-modelling approach, we can incorporate clustering statistics beyond the traditional two-point statistics. We use this modelling methodology to test the standard $\Lambda$ cold dark matter ($\Lambda$CDM) + halo model against the clustering of Sloan Digital Sky Survey (SDSS) seventh data release (DR7) galaxies. Specifically, we use the projected correlation function, group multiplicity function, and galaxy number density as constraints. We find that while the model fits each statistic separately, it struggles to fit them simultaneously. Adding group statistics leads to a more stringent test of the model and significantly tighter constraints on model parameters. We explore the impact of varying the adopted halo definition and cosmological model and find that changing the cosmology makes a significant difference. The most successful model we tried (Planck cosmology with $M_{\text{vir}}$ haloes) matches the clustering of low-luminosity galaxies, but exhibits a 2.3$\sigma$ tension with the clustering of luminous galaxies, thus providing evidence that the ‘standard’ halo model needs to be extended. This work opens the door to adding interesting freedom to the halo model and including additional clustering statistics as constraints.

Key words: methods: numerical – galaxies: evolution – galaxies: groups: general – galaxies: haloes – dark matter – large-scale structure of Universe.

1 INTRODUCTION

The observed spatial distribution of galaxies contains a richness of information about the initial conditions and subsequent evolution of the matter density field in the universe. Moreover, the dependence of the spatial distribution of galaxies on their observed properties contains information about the physics of galaxy formation and evolution. The study of galaxy clustering has thus proved to be a fruitful avenue for constraining cosmology and galaxy formation theory and it has provided the primary motivation for the construction of large astronomical galaxy surveys.

The specific information content of galaxy clustering depends on the physical scales considered. On large scales, galaxy clustering provides fairly clean constraints on cosmology because galaxies are simple tracers of the matter density field (e.g. Scherrer & Weinberg 1998; Narayanan, Berlind & Weinberg 2000). Prime examples of such constraints are measurements of the large-scale galaxy power spectrum (e.g. Tegmark et al. 2004) and the Baryon Acoustic Oscillation (BAO) signature (Eisenstein et al. 2005). On small scales ($\lesssim 10h^{-1}\text{ Mpc}$), galaxy clustering depends both on...
cosmology and on the detailed relationship between the galaxy and dark matter density fields, i.e. the bias, which is non-trivial and is set by the physics of galaxy formation. Although in principle galaxy clustering measurements on small scales have the potential to constrain both cosmological and galaxy formation theories, in practice this is extremely challenging because the process of galaxy formation is very complex and we currently lack a complete predictive theory for it. Moreover, the most accurate theories that do exist require computationally expensive hydrodynamic simulations to make predictions, and are thus not suited for exploring and constraining large parameter spaces.

Halo-based models that begin with the assumption that galaxies form and live inside dark matter haloes provided the breakthrough that made it possible to quantitatively model galaxy clustering on small scales. These models rely on the fact that the statistical properties of dark matter haloes are easy to predict with collisionless N-body simulations where the only important physical process is gravity. Halo models then adopt a parametrization to connect galaxies to haloes, thus bypassing the need to understand galaxy formation physics. In the halo model framework, there is a convenient conceptual and operational division between the roles of cosmology and galaxy formation: cosmology dictates the dark matter halo distribution while galaxy formation determines how exactly galaxies occupy haloes. This division is not perfect since gas physics can affect the properties of haloes (e.g. Cui et al. 2012); however, this is a second-order effect.

The connection between galaxy halo occupation and galaxy clustering was first made by semi-analytic models of galaxy formation that modelled the formation and evolution of galaxies inside haloes. Since these haloes resided within a larger density field in cosmological N-body simulations, it was straightforward to predict the clustering of the semi-analytic galaxies (Kauffmann, Nuss & Steinmetz 1997; Baugh et al. 1999; Kauffmann et al. 1999). Jing, Mo & Börner (1998) and Benson et al. (2000) took this a step further by realizing that galaxy clustering did not necessarily depend on all the details of galaxy formation, but rather only cared about halo occupation statistics as a function of halo mass. A series of papers then built upon the earlier work of Neyman & Scott (1952) and Scherrer & Bertschinger (1991) to develop a full analytic machinery for combining parametrized halo properties with occupation statistics to calculate the correlation function and power spectrum of galaxies on all scales (e.g. Peacock & Smith 2000; Seljak 2000; Scoccimarro et al. 2001; Cooray & Sheth 2002). These analytic models became generally known as the ‘halo model’. Berlind & Weinberg (2002) focused on the ‘halo occupation distribution’ (HOD), which is the complete parametrization connecting galaxies of a given class to haloes, and they investigated how the HOD affects several galaxy clustering statistics and laid out a road-map for empirically constraining the HOD with measurements of galaxy clustering using data from large surveys.

The halo model has since been used by a large number of studies to model galaxy clustering data in several galaxy redshift surveys: the 2dF Galaxy Redshift Survey (2dFGRS; Colless et al. 2001), the Sloan Digital Sky Survey (SDSS; York et al. 2000), the 6dF Galaxy Redshift Survey (6dFGRS; Jones et al. 2004), and the SDSS III Baryon Oscillation Spectroscopic Survey (BOSS; Dawson et al. 2013). Some studies investigated the two-point correlation function of low-redshift galaxies (Maggiocchi & Porciani 2003; Zehavi et al. 2004, 2005, 2011; Collister & Lahav 2005; Tinker et al. 2005; Watson et al. 2012; Beutler et al. 2013; Piscionere et al. 2015), others investigated the same for red galaxies (Blake, Collister & Lahav 2008; Brown et al. 2008; Zheng et al. 2009; Watson et al. 2010; White et al. 2011; Nikoloudakis, Shanks & Sawangwit 2013; Parejko et al. 2013; Guo et al. 2014, 2015a; Reid et al. 2015), or other special classes of galaxies such as AGN or radio galaxies (Wake et al. 2008; Mandelbaum et al. 2009; Richardson et al. 2013). Many studies modelled the two-point correlation function of Lyman Break and other types of high-redshift galaxies (Bullock, Wechsler & Somerville 2002; Moustakas & Somerville 2002; Hamana et al. 2004; Zheng 2004; Lee et al. 2006; Tinker, Wechsler & Zheng 2010; Jose et al. 2013; Kim et al. 2014), while others combined modelling of galaxies at different redshifts in order to learn about the co-evolution of galaxies and haloes (Yan, Madgwick & White 2003; Cooray 2006; Zheng, Coil & Zehavi 2007; Abbas et al. 2010; Tinker & Wetzel 2010; Wake et al. 2011; Tinker et al. 2013). Though the vast majority of studies have focused on the two-point correlation function, a few have applied the halo model to other clustering statistics, like the three-point correlation function (Marín 2011), or the galaxy–mass cross-correlation function as measured by galaxy–galaxy lensing (Guzik & Seljak 2002; Cacciato et al. 2013). Closely related to the HOD approach is the conditional luminosity function (CLF) approach that includes galaxy luminosity in the halo occupation parametrization (e.g. van den Bosch, Yang & Mo 2003; Yang, Mo & van den Bosch 2003; Vale & Ostriker 2004; Leauthaud et al. 2011, 2012).

All of these studies successfully used the halo model to translate clustering statistics into constraints on the relation between galaxy properties and the dark matter haloes they inhabit. For the most part, the constraints are focused on the relation between the luminosity or stellar mass of galaxies and the mass of their haloes (e.g. Zehavi et al. 2011). However, some studies focused on other aspects of the HOD, such as the radial distribution of galaxies within haloes (e.g. Watson et al. 2010, 2012; Piscionere et al. 2015), the velocity distribution of galaxies within haloes (Guo et al. 2015a,b,c), or galaxy assembly bias (Zentner et al. 2016). These results have been quite illuminating and have helped to explain many observed features of the galaxy population, such as the morphology–density relation and the different clustering strengths of different galaxy types. Halo modelling across different redshifts has even constrained the competing roles of merging and star formation in the stellar mass build-up of galaxies (e.g. Tinker et al. 2013).

In most halo model analyses, the statistical methodology employed is sophisticated. Clustering uncertainties and their correlations are quantified in estimated covariance matrices, parameter searches are executed using Monte Carlo Markov Chain (MCMC) methods, model parameter values are reported with their full probability distributions, and the goodness of fit for the halo model is typically reported using the $\chi^2$ statistic. However, the systematic errors in these analyses have been largely unquantified. Systematic errors exist in the estimation of covariance matrices that typically use the Jackknife method (Norberg et al. 2009). Systematic errors also exist in the halo model itself, since its implementation is almost always analytic and it contains several approximations. Without a robust characterization of systematic errors, it is impossible to interpret the published results in a statistical sense. For example, many of the published works contain model fits with reported values of $\chi^2$ that are high enough to warrant ruling the models out (e.g. some of the high-luminosity samples in Zehavi et al. 2011), but without a proper accounting of systematic errors in the modelling, it is not possible to determine whether the models have actually been ruled out. This problem is exacerbated by the ever shrinking statistical errors in clustering measurements due to the growing sample sizes provided by current galaxy surveys. In order to trust...
the goodness of fit reported by a given study, systematic errors in the modelling must be kept smaller than the statistical errors in the data measurements.

Since systematic errors in modelling are for the most part not quantified by published studies, many of the results in the literature should be interpreted with caution. In general, qualitative trends found are likely correct, but precise parameter values, error bars, or goodness-of-fit estimates are not necessarily reliable. This does not pose a problem for most published works because their goal was to uncover general trends rather than to test specific models. However, if we wish to use the halo model to probe more subtle features of the HOD, like the presence of assembly bias (e.g. see Zentner, Hearin & van den Bosch 2014), or if we wish to use galaxy clustering on small scales to constrain cosmological models, we will need to make our modelling methodology more accurate. A few studies have demonstrated the power of small scales to constrain cosmology (e.g. Abazajian et al. 2005; van den Bosch et al. 2007; Cacciato et al. 2013), but they will only be able to compete with more established probes of cosmology if the modelling is sufficiently accurate.

In this paper, we attempt to push the modelling of galaxy clustering on small scales into the ‘accurate regime’, where results can be trusted enough to confirm or rule out physical models. To be precise, by ‘accurate’ we mean that given a set of measured statistics from a galaxy survey, and a galaxy–halo connection model being tested, we can produce a reliable goodness of fit and reliable posterior probabilities for the model parameters. To achieve this, we need to (i) minimize the theoretical errors in the predicted distribution of haloes for the assumed cosmological model, (ii) forward-model the measured clustering statistics to incorporate all observational systematic errors, and (iii) robustly estimate the statistical errors and covariances in the measured clustering statistics from the model. In other words, our goal is to accomplish for small scales what is already routinely done in the study of large-scale clustering. This is an ambitious goal and we can only tackle it by adopting a fully numerical modelling framework that is based on large numbers of realistic mock galaxy catalogues. This data-intensive approach requires substantial computational effort. However, we are motivated to do this by the immense amount of information present in galaxy surveys on small scales, which is currently not being harnessed. In this first paper, we primarily assume that the cosmological model is known and we adopt the most widely used formulation of the HOD to model the clustering of SDSS galaxies. Our objective is thus to test the standard ΛCDM + halo model against the small-scale clustering of SDSS galaxies. Specifically, we use measurements of the projected correlation function and the group multiplicity function for two luminosity threshold samples from the SDSS DR7 (Abazajian et al. 2009) data set. However, this work is also intended to open the door for future studies where we will use additional clustering statistics and larger galaxy samples to test extensions of the standard HOD as well as variations in cosmology.

The layout of this paper is as follows. In Section 2, we describe the SDSS data samples that we use. In Section 3, we describe the N-body simulations and mock galaxy catalogues that make up the workhorse of our modelling methodology. In Section 4, we present our galaxy clustering measurements, and in Section 5, we present their associated errors and correlation matrices. In Section 6, we describe our model fitting methodology and present all of our modelling results. In Section 7, we discuss future improvements to our methodology. Finally, in Section 8, we summarize our results.

### Table 1. SDSS volume-limited sample parameters.

<table>
<thead>
<tr>
<th>$M^\text{lim}$</th>
<th>$z_{\text{min}}$</th>
<th>$z_{\text{max}}$</th>
<th>$z_{\text{median}}$</th>
<th>$V_{\text{eff}}$ ($10^6 h^{-1} \text{Mpc}^3$)</th>
<th>$\Omega_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−19</td>
<td>0.02</td>
<td>0.067</td>
<td>0.054</td>
<td>5.555</td>
<td>0.0149</td>
</tr>
<tr>
<td>−21</td>
<td>0.02</td>
<td>0.165</td>
<td>0.132</td>
<td>78.374</td>
<td>0.0012</td>
</tr>
</tbody>
</table>

### 2 SDSS GALAXY SAMPLES

The Sloan Digital Sky Survey (York et al. 2000) completed its original imaging and spectroscopic goals in 2008 with its seventh data release (DR7; Abazajian et al. 2009). The DR7 spectroscopic main galaxy sample (Strauss et al. 2002) is complete down to an apparent r-band Petrosian magnitude limit of 17.77 and contains over 900 000 galaxies. In this work, we use the large-scale structure samples from the NYU Value Added Galaxy Catalog (NYU-VAGC; Blanton et al. 2005). Specifically, we use a parent sample of just over 530000 galaxies, which only covers the northern part of the SDSS footprint and is cut back to $r < 17.6$ so that it is complete down to that magnitude limit across the sky. The reason for restricting the data to the northern footprint is that we can construct more independent mock catalogues using the north-only survey volume. Galaxy absolute magnitudes have been k-corrected to rest-frame magnitudes at redshift $z = 0.1$ (Blanton et al. 2003b) and corrected for passive luminosity evolution using the simple model described by Blanton (2006).

The SDSS spectroscopic sample has an incompleteness due to the mechanical restriction that spectroscopic fibres cannot be placed closer to each other than their own thickness. This fibre collision constraint makes it impossible to obtain redshifts for both galaxies in pairs that are closer than 55 arcsec on the sky. This restriction results in ~7 per cent of targeted galaxies not having a measured redshift. We assign fibre collided galaxies the redshift of the galaxy they collided with (i.e. the ‘nearest neighbour correction’; Zehavi et al. 2002). This correction recovers the true correlation function well on scales larger than the physical scale corresponding to 55 arcsec, which for the outer redshift limit of our samples corresponds to 0.1h$^{-1}$Mpc. There is some additional incompleteness due to bright foreground stars blocking background galaxies, but this is at the 1 per cent level.

In this study, we use two volume-limited subsamples of the full SDSS redshift sample that are each complete in a specified redshift range down to a limiting r-band absolute magnitude threshold. We construct each sample by choosing redshift limits $z_{\text{min}}$ and $z_{\text{max}}$ and only keeping galaxies whose evolved, redshifted spectra would still make the redshift survey’s apparent magnitude and surface brightness cuts at these limiting redshifts. Our low-luminosity sample is complete down to an r-band absolute magnitude of $−19$ in the redshift range of 0.02–0.067, while our high-luminosity sample is complete down to an r-band absolute magnitude of $−21$ in the redshift range of 0.02–0.165. The redshift limits, median redshift, effective volume, and number density of these two samples are listed in Table 1. The volumes and number densities of the samples are corrected for survey incompleteness.

Throughout this paper, comoving distances and absolute magnitudes for SDSS galaxies are calculated adopting a flat Λ cold dark matter (ΛCDM) cosmological model with $\Omega_m = 0.25$ and
$h = 1$. Our distances thus have units of $h^{-1}$Mpc and our absolute magnitudes are actually $M_i + 5 \log h$. We keep these data samples fixed regardless of what cosmological model we adopt when fitting to clustering statistics. Ideally, a change of cosmology in the model should also be reflected in the data measurements. However, we have checked that clustering statistics only change by ~1 per cent when switching cosmological models, which is negligible compared to the errors in our measurements.

3 Mock galaxy catalogues

The key to accurate forward halo modelling of galaxy clustering on small scales is to use mock catalogues both for predicting observed statistics and for estimating errors and covariances. Broadly, we can divide the challenge in creating an accurate model into three distinct pieces: (1) creating a reliable prediction of the halo population for a cosmological model, (2) comparing identical galaxy clustering statistics between the predicted model, and the observed data (3) correctly estimating covariances between all clustering statistics from the model. We take the following approach to ensure that systematic errors in the modelling are sub-dominant compared to observational errors, i.e. an accurate model. To obtain a reliable halo population, we use cosmological $N$-body simulations of sufficient volume and resolution. To compare identical clustering statistics, we create realistic mock catalogues (from the $N$-body simulations) that include systematic effects like survey geometry and redshift-space distortions. In addition, we use the exact same codes, when necessary, to measure the clustering statistics on both mock galaxies and observed data. To correctly estimate covariances, we use a large, independent ensemble of these realistic mocks instead of estimating errors from the observed data set. Skipping any one of these three steps compromises the ‘accurate’ part of the model. For instance, assuming an universal halo mass function, or an analytic non-linear bias fitting formula, or a Navarro–Frenk–White density profile for the halo has an impact on the predicted $w_p(r_p)$. Not including survey geometry in the modelling changes both the amplitude and the slope of the group multiplicity function (Berlind et al. 2006). Using jackknife resampling to build covariance matrices introduces scale dependent, systematic biases (Norberg et al. 2009). With our adopted methodology, we avoid all these systematic effects and push the small-scale modelling towards the accurate regime.

In this section, we describe the simulations and mock catalogue pipeline in detail.

3.1 N-body simulations and halo catalogues

3.1.1 For building covariance matrices

The bulk of simulations that we use are from the Large Suite of Dark Matter Simulations project (LasDamas; McBride et al. 2009). The LasDamas project focused on running many independent N-body realizations with the same cosmology but different initial phases. The cosmological parameters were roughly motivated by the WMAP3 constraints (Spergel et al. 2007) and are $\Omega_m = 0.25$, $\Omega_\Lambda = 0.75$, $\Omega_b = 0.04$, $h = 0.7$, $\sigma_8 = 0.8$, and $n_s = 1.0$. The LasDamas simulations were designed to model SDSS galaxies and consist of four different volume and resolution configurations that correspond to different luminosity samples. In this work, we use the Consuelo and Carmen configurations, which were designed to model galaxy samples with $r$-band absolute magnitude thresholds of ~19 and ~21, respectively.

All the simulations were seeded with second-order Lagrangian perturbation theory initial conditions using the code 2LPTIC (Scoccimarro 1998; Crocce, Pueblas & Scoccimarro 2006) and were evolved using the N-body code GADGET-2 (Springel 2005). Each Consuelo simulation evolved $140^3$ dark matter particles in a cubic volume of $420 h^{-1}$Mpc on a side, from a starting redshift of $z_{\text{init}} = 99$ to $z = 0$, with a gravitational force softening length of $8 h^{-1}$kpc. Each Carmen simulation evolved $112^3$ dark matter particles in a cubic volume of $1000 h^{-1}$Mpc on a side, from a starting redshift of $z_{\text{init}} = 49$ to $z = 0$, with a gravitational force softening length of $25 h^{-1}$kpc. The resulting particle masses of the Consuelo and Carmen simulations are $1.87 \times 10^9 h^{-1} M_\odot$ and $4.938 \times 10^8 h^{-1} M_\odot$, respectively. For the purpose of estimating covariance matrices, we use 50 realizations of each of these two boxes, which yield 200 mock catalogues per luminosity sample.

We identify haloes in the dark matter distributions using the simulation outputs corresponding to the median redshifts of the ~19 and ~21 samples, which are $z = 0.054$ and $z = 0.132$, respectively. For the fifty Consuelo and fifty Carmen simulations, haloes were identified with the NTROPY-FOFSV code (Gardner, Connolly & McBride 2007), which employs a friends-of-friends (FoF; Davis et al. 1985) algorithm. The FoF linking length was chosen to be 0.2 times the mean interparticle separation. Finally, we apply the Warren et al. (2006) correction to the FoF halo masses. For the purpose of placing central galaxies in haloes, we define the FoF halo centre to be at the location of the deepest part of the halo’s gravitational potential well. The mock catalogues that we use to construct covariance matrices are based on these FoF halo catalogues.

The mass resolutions of these simulations result in 150 particles in haloes of mass $\log M = 11.45$ (for Consuelo) and 122 particles in haloes of mass $\log M = 12.78$ (for Carmen), which, according to Zehavi et al. (2011), are the typical minimum masses we expect to host galaxies in our two luminosity samples. If we also consider the scatter in these minimum masses (the $\sigma_{\log M}$ parameter in Zehavi et al. 2011), then haloes with masses as low as $\log M = 11.26$ (for Consuelo) and $\log M = 12.10$ (for Carmen) will sometimes host a galaxy, which contain 97 and 25 particles, respectively. This is acceptable resolution for our purposes because these haloes only rarely host a single central galaxy and so we only need to roughly resolve their bulk properties (position, velocity, mass). Moreover, we expect that errors due to resolution will not impact the scatter among clustering measurements from different simulation realizations as much as it will impact the clustering measurements themselves.

3.1.2 For MCMC parameter exploration

For the purpose of generating predictions of clustering statistics within our MCMC framework, we have run a few additional simulations. The demands on simulation resolution are more stringent in this case because, during the halo model parameter search, sometimes galaxies are placed in significantly lower mass haloes than they are in the fiducial model that we use to construct covariance matrices. The simulations used in the MCMC parameter search must therefore resolve haloes down to lower masses. If we adopt the 2$\sigma$ low value of $\log M_{\text{min}}$ and 2$\sigma$ high value of $\sigma_{\log M}$ found by Zehavi et al. (2011), we can obtain a conservative estimate of the lowest mass haloes we must resolve. These masses are $\log M = 10.9$ and 11.6 for the ~19 and ~21 samples, which result in 42 and 8 particles in Consuelo and Carmen simulations, respectively. The Carmen simulations do not therefore have adequate resolution to serve as the basis for modelling the clustering of the $M_r < -21$ galaxies.
SDSS sample. We have thus run a higher resolution version of Carmen that we name CarmenHD, which contains 2240 particles and resolves haloes that have more than five times smaller mass than Carmen. In order to probe a variation in cosmology, we have also run new versions of Consuelo and CarmenHD that adopt the recent set of cosmological parameters given by the Planck experiment (Planck Collaboration XVI 2014). Specifically, these simulations, named Consuelo-plk and CarmenHD-plk, adopt the following parameter values: \( \Omega_m = 0.302, \Omega_k = 0.698, \Omega_b = 0.048, h = 0.681, \sigma_8 = 0.828, \) and \( n_s = 0.96. \) We have run two realizations each of CarmenHD, Consuelo-plk, and CarmenHD-plk since we use two boxes within our MCMC modelling framework. The box sizes, particle masses, force resolutions, and other summary information of all the simulations described above are listed in Table 2.

For the two Consuelo, CarmenHD, Consuelo-plk, and CarmenHD-plk simulations that we use in our MCMC modelling, haloes are identified with a spherical overdensity (SO; Lacey \& Cole 1994) algorithm using the ROCKSTAR code (Behroozi, Wechsler \& Wu 2013). We use two sets of SO halo definitions: the M200b definition where haloes are spheres of density 200 times the mean density of the universe and the Mvir definition where haloes have a density that depends on cosmology and redshift, as given by Bryan \& Norman (1998). We use these SO halo catalogues to predict galaxy clustering within our MCMC parameter searches. In the case of an SO halo, the halo centre is by definition at the centre of the halo sphere.¹

When evaluating the likelihood function during a model parameter exploration, it is desirable that the statistical errors in the model (due to cosmic variance) are much smaller than the errors in the SDSS data so that they do not add appreciably to the error budget. If this is not the case, it is equivalent to degrading the statistical power of the galaxy survey. Therefore, the clustering statistics calculated from the model should come from a larger volume of mock galaxies than the SDSS sample being modelled. On the other hand, generating too large a mock volume on the fly within an MCMC parameter search is computationally intractable. We achieve this balance by using either a single realization simulation cube in our modelling, or two realizations if it is necessary to faithfully reproduce the SDSS sample geometry. A single Consuelo or Carmen box contains about 13 times more total volume than the SDSS − 19 or −21 samples, respectively. Alternatively, if we need to include the full SDSS sample geometry in the mocks, two Consuelo or Carmen boxes can generate eight times more mock volume than their respective SDSS samples. This is sufficiently large to keep statistical errors in the modelling subdominant. However, we reduce the model errors further by carefully selecting the two simulation boxes we use. We do this by selecting the two boxes that exhibit a clustering signal closest to the mean of all 50 boxes. We do this in the following way. First, we populate all 50 Consuelo and Carmen boxes with the Zehavi et al. (2011) HOD model for −19 and −20 threshold samples, respectively. Then we measure the projected correlation function \( w_p(r_p) \) on each of these mocks, adopting one of the axes of the cube as the line-of-sight direction. We measure the mean clustering \( \langle w_p(r_p) \rangle \) from all 50 boxes as well as their standard deviation \( \sigma_{w_p}(r_p) \). We then compute a \( \chi^2 \) statistic for each box

\[
\chi^2_{\text{box}} = \sum_{r_p} \left( \frac{w_{p,\text{box}}(r_p) - w_p(r_p)}{\sigma_{w_p}(r_p)} \right)^2, \quad \forall \text{box} \in [1, 50].
\]

Finally, we identify the two realizations of Consuelo and Carmen that have the lowest values of \( \chi^2 \), which are essentially the two boxes whose random phases result in clustering that is closest to the mean of all 50 boxes. We use these same exact phases in all our modelling boxes: the same two sets of phases for the two Consuelo and two Consuelo-plk simulations and the same two sets of phases for the two CarmenHD and two CarmenHD-plk simulations. This procedure results in a significant reduction in cosmic variance errors. The average box-to-box scatter in \( w_p(r_p) \) across all scales is ∼10 per cent for Consuelo and ∼6 per cent for Carmen, while the average difference between our best two boxes and the mean of all 50 is only ∼5 per cent for Consuelo and ∼3 per cent for Carmen, representing a factor of 2 reduction in cosmic variance errors. This is the accuracy we would normally obtain with a much larger simulation volume.

### 3.2 From haloes to galaxies

We populate dark matter haloes in our simulations with mock galaxies using the ‘Halo Occupation Distribution’ (HOD) framework. In this framework, the number, positions, and velocities of galaxies within a halo are described statistically given a set of parametrized prescriptions. The key advantage of the HOD approach is that if the parametrization is sufficiently flexible it allows us to marginalize over the full uncertainty of galaxy formation theory. In this work, we adopt the ‘vanilla’ HOD model of Zheng et al. (2007), which has become somewhat of an industry standard in HOD modelling. Motivated by theoretical results (Kravtsov et al. 2004; Zheng et al. 2005), we split the galaxies into centrals and satellites within their haloes. The mean number of central galaxies as a function of halo mass \( M \) is given by

\[
\langle N_{\text{cen}} \rangle = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\log M - \log M_{\text{min}}}{\sigma_{\log M}} \right) \right],
\]

where \( M_{\text{min}} \) sets the minimum halo mass that can host a central galaxy, \( \sigma_{\log M} \) sets the scatter around this minimum halo mass, and \( \text{erf}(x) \) is the error function, \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-y^2) \, dy. \) The motivation behind this particular analytic form of \( \langle N_{\text{cen}} \rangle \) comes from assuming a lognormal distribution of central galaxy luminosity at fixed halo mass. If the luminosity–mass relation is a power-law \( L \propto M^p \) with a lognormal scatter \( \sigma_{\log L} \), then the scatter in mass at fixed luminosity is \( \sigma_{\log M} = \sqrt{2\sigma_{\log L}^2/p} \). The parameter \( \sigma_{\log M} \) in equation (2) represents this scatter at the luminosity limit of the sample and thus controls how quickly the probability of containing a galaxy above the luminosity limit rises from zero to one as halo mass increases. \( M_{\text{min}} \) is the mass at which this probability is one half, i.e. where half the haloes in the universe contain a central galaxy above the sample luminosity threshold, while the other half do not. The relation between central galaxy luminosity and halo mass is constrained by abundances to follow a roughly double power-law form with a steep slope at low mass, a transition near \( M = 10^{12} h^{-1} M_{\odot} \), and a shallow slope at high mass (e.g. Vale \& Ostriker 2004). For the −19 and −21 luminosity thresholds we consider in this paper, we expect the slope \( p \) to be approximately ∼1 and ∼0.3, respectively.

¹We note that the correlation matrices are derived from FoF haloes, while the modelling is done with SO haloes. This is because we did not have the SO halo catalogues for all the simulations while the paper was being prepared. We have since checked that our results do not change qualitatively with a correlation matrix derived from SO haloes.

²\( \log \) refers to \( \log_{10} \) everywhere in the text.
Previous work on satellite kinematics has constrained $\sigma_{\log L}$ to be approximately $\sim 0.15$ dex (More et al. 2009). We thus expect values of $\sigma_{\log M}$ that are $\sim 0.2$ and $\sim 0.7$ for our faint and bright samples, respectively.

In each halo we place a number of satellite galaxies drawn from a Poisson$^3$ distribution with a mean given by

$$\langle N_{\text{sat}} \rangle = \langle N_{\text{cen}} \rangle \times \left( \frac{M - M_0}{M_1} \right)^{\alpha},$$

where $M_0$ is the halo mass below which there are no satellite galaxies, $\alpha$ is the slope of the power-law occupation function at high masses, and $M_1$ sets the mass scale where haloes contain one satellite galaxy on average. The actual halo mass for which $\langle N_{\text{sat}} \rangle = 1$ is slightly higher than $M_1$ and also depends on the other parameter values. For some HOD parameter combinations, it is possible to have haloes that do not receive a central galaxy and yet still have a chance to get a satellite. We do not allow such cases in our modelling. Haloes are only eligible to receive satellite galaxies if they already have a central.

Once we have specified the number of centrals and satellites in a given halo, we need to give them positions and velocities within the halo. In this ‘vanilla’ HOD model we place the central galaxy at the centre of its halo. We then assign to it the mean velocity of the whole halo, which assumes that it is at rest relative to the halo. For the satellites, we choose their positions and velocities to be equal to those of randomly selected dark matter particles within the halo. This assumes that satellite galaxies trace the spatial and velocity distribution of dark matter within their halo.

Though the HOD is a statistical model that allows us to parametrize our ignorance of the detailed physics of galaxy formation, the various features of the HOD do contain information about galaxy formation. For example, the mass scale and scatter of the central galaxy occupation at low masses (parametrized by $M_{\text{min}}$ and $\sigma_{\log M}$) depend on the luminosity–mass relation in the vicinity of the luminosity threshold of the sample. The luminosity–mass relation depends on the efficiency with which haloes are able to cool gas and form stars as a function of mass. This efficiency is affected by several factors, such as supernova feedback, pre-heating caused by reionization, cold versus hot accretion of gas into the halo, merger history, environment, etc. On the other hand, the satellite occupation at higher masses depends primarily on the balance between the accretion and destruction of subhalos, which depends on the distributions of their infall times, masses and orbits, and on the physics of dynamical friction and tidal stripping (e.g. Watson, Berlind & Zentner 2011).

To summarize, the vanilla HOD model we use contains five free parameters that control the mean number of galaxies as a function of halo mass: $M_{\text{min}}$, $\sigma_{\log M}$, $M_0$, $M_1$, and $\alpha$. The vanilla model also makes the following set of simplifying assumptions:

1. All galaxies live inside dark matter haloes, using whatever halo definition we have adopted.
2. The number of galaxies in a halo only depends on the mass of the halo and not on other halo properties, such as age or concentration. In other words, there is no galaxy assembly bias.
3. The functional forms of $\langle N_{\text{cen}} \rangle$ and $\langle N_{\text{sat}} \rangle$ are given by equations (2) and (3).
4. The probability distribution $P(N_{\text{sat}}|N_{\text{sat}})$ of the number of satellite galaxies $N_{\text{sat}}$ given the mean number $\langle N_{\text{sat}} \rangle$ is a Poisson distribution.
5. The central galaxy in each halo lives at the halo centre and moves with the mean halo velocity. In other words, there is no central spatial or velocity bias.
6. Satellite galaxies in each halo trace the spatial and velocity distribution of dark matter within the halo. In other words, there is no satellite spatial or velocity bias.

These assumptions are all correct to first order; however, most of them are likely incorrect in detail. For example, galaxy assembly bias is usually considered for colour-selected samples, but it is probably also present to some degree for luminosity threshold samples (e.g. Zentner et al. 2014, 2016). We assume that the scatter in $N_{\text{sat}}$ at fixed mass is Poissonian. However, recent work with high-resolution N-body simulations suggests that, at least for subhalos, this assumption does not hold for all host-to-satellite mass ratios (Boylan-Kolchin et al. 2010; Mao, Williamson & Wechsler 2015). Finally, we assume that there is no spatial or velocity bias for centrals or satellites. However, in theory we expect some bias for satellites since they experience dynamical effects (friction and stripping) that do not affect individual dark matter particles. Moreover, there is some observational evidence that both centrals and satellites have some degree of velocity bias (van den Bosch et al. 2005; Guo et al. 2015c). These are the sorts of interesting second-order features of the HOD that can only be probed with an accurate modelling framework like the one we present in this paper. In this first work, we adopt the simple vanilla HOD model to test whether it can be ruled out by SDSS galaxy clustering data. Our strategy for future work is to add freedom to the model as demanded by the data. For example, if the vanilla HOD model is ruled out, then we will be justified to add freedom to it by relaxing the assumptions described above. If the model is not ruled out, then we will keep adding new

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3The function for generating Poisson random numbers in ‘Numerical Recipes in C’ code suffers from a floating underflow bug that sets in when $N_{\text{sat}} \geq 708$ – as can happen for faint samples. We used the ‘c’n’ routine that does not suffer from that particular bug.
clustering statistics until it is. Combining multiple statistics is a clear way forwards to break the degeneracies in the HOD model.

### 3.3 Adding survey realism

Once we have a simulation box populated with galaxies according to a given HOD model, we build mock catalogues that contain the main observational systematic effects that plague the galaxy clustering statistics we wish to model, namely redshift space distortions and sample geometry. First, we move the galaxy distribution from Cartesian to spherical coordinates by placing the mock observer at the centre of the simulation cube and converting Cartesian positions of galaxies into RA, Dec., and comoving distances. We then compute the line-of-sight peculiar velocities of galaxies and compute galaxy redshifts as \( z = z_{\text{cosm}}(1 + z_{\text{doppler}}) \), where \( z_{\text{cosm}} \) is the cosmological redshift and \( z_{\text{doppler}} \) is the redshift due to the radial peculiar velocity. Finally, we throw out mock galaxies that lie outside the redshift limits or the sky footprint of the SDSS sample. Since the volume of the SDSS sample is much smaller than that of the simulation used to model it, we can extract multiple mock catalogues from a single simulation cube. By applying a series of rotations on the box before converting it to spherical coordinates, we are able to extract four entirely independent mock catalogues from each box. Our 50 simulations for each volume-limited sample can thus produce 200 independent mock catalogues that we can use to construct covariance matrices.

Aside from survey geometry and redshift distortions, the other main source of systematic error present in the SDSS clustering data is incompleteness due to fibre collisions. Unfortunately, adding fibre collisions to our mock catalogues is not a trivial exercise. First, the mocks only represent volume-limited samples, whereas in the SDSS a galaxy can collide with any other galaxy in the full flux-limited sample. Secondly, the severity of fibre collisions depends on sky location in a complicated way due to the tiling of the survey footprint with spectroscopic plates (Blanton et al. 2003a). In regions where spectroscopic plates overlap, fibre collisions are substantially reduced, and these overlap regions correlate with the surface density of spectroscopic targets. For these reasons, we do not attempt to model fibre collisions, but we simply rely on the nearest-neighbour correction (Zehavi et al. 2002). We then restrict our clustering analysis to regimes where the correction works well, which include scales larger than 0.1h^{-1}\,\text{Mpc} in the correlation function (Zehavi et al. 2002) and groups with at least five members in the multiplicity function (Berlind et al. 2006).

### 4 CLUSTERING MEASUREMENTS

Our approach stems from a very simple and powerful idea: if we have a correct model describing the galaxy distribution, then we should be able to reproduce any clustering statistic that encodes information about the galaxy density field. If the model fails to provide a good fit to clustering statistics, then one or more of its assumptions need revising, e.g. the cosmological model, the halo definition, the features of the HOD, etc. If the model is able to provide a good fit to some set of clustering statistics, then we can keep adding more statistics until it fails. The galaxy clustering statistics we use do not need to have a physical basis or even be well established. We could define a new arbitrary statistic and use that instead. Since we can use the same codes and methods to compute statistics on both the data and the model, any statistic that can be measured from the data will do. What we want is statistics that contain a high degree of information about the galaxy density field, while still being computationally tractable. In this work, we use two fairly traditional statistics, the projected two-point correlation function \( w_p(r_p) \) and the group multiplicity function \( n(N) \), along with the overall number density of galaxies. In future work, we plan to extend to additional statistics, as well as perform more detailed studies to determine the optimal statistics that should be used for constraining a given model.

#### 4.1 The projected correlation function \( w_p(r_p) \)

The two-point correlation function is the most widely used galaxy clustering statistic and the one that is typically modelled with HOD models. The three-dimensional correlation function \( \xi(r) \) is the excess number of galaxy pairs above what is expected for a random distribution of points, as a function of pair separation \( r \). To deal with redshift distortions, galaxy pairs are decomposed into their line of sight and projected components \( \pi \) and \( r_p \), yielding the function \( \xi(r_p, \pi) \). This is then integrated over \( \pi \) to get the projected correlation function:

\[
w_p(r_p) = 2 \int_0^{\pi_{\text{max}}} \xi(r_p, \pi) \, d\pi.
\]

Since we will need to compute clustering statistics on the fly for mock catalogues within our MCMC framework, it is imperative that the computation be fast and efficient. We have developed and used a blazing fast new code Corrfunc\(^4\) (Sinha & Garrison 2017) that can compute \( w_p(r_p) \) for either simulation cubes or mock surveys in parallel. With this code we can measure \( w_p(r_p) \) out to 20h^{-1}\,\text{Mpc} for 10^6 galaxies in ~6 and 3 s of wallclock time on a single CPU core, for the number densities of the \( M_r < -19 \) and -21 samples, respectively.

We use the Zehavi et al. (2002) definitions of \( r_p \) and \( \pi \) and we count pairs in 14 equally spaced logarithmic bins of \( r_p \) whose centres range from 0.2 to 20h^{-1}\,\text{Mpc}. We estimate \( \xi(r_p, \pi) \) with the Landy–Szalay estimator (DD − 2DR + RR)/RR (Landy & Szalay 1993), where DD, DR, and RR are the properly normalized numbers of data–data, data–random, and random–random pairs, respectively. We use a random catalogue that contains ~10 times more points than its corresponding galaxy sample. Finally, we calculate \( w_p(r_p) \) by integrating out to \( \pi_{\text{max}} = 40 \) and 80h^{-1}\,\text{Mpc} for the \( M_r < -19 \) and -21 samples, respectively. The red points in Fig. 1 show our measurements for the two SDSS samples. Errors are calculated via jackknife resampling using 50 distinct regions on the sky (see McBride et al. 2011 for jackknifing details). These jackknife errors are only used in the initial steps for constructing a mock covariance matrix, as we describe in Section 5. Once we have a mock covariance matrix, we use that for all the main analysis in this paper. Fig. 1 also shows \( w_p(r_p) \) for a set of mock catalogues (grey lines and blue points), which we describe in Section 5.

We measure \( w_p(r_p) \) as described above for the SDSS samples and for the 200 mock catalogues per sample that are used to construct covariance matrices. However, we follow a slightly different approach when computing \( w_p(r_p) \) within a HOD parameter search. In this case, we do not use the Landy–Szalay estimator applied to mock catalogues that have the same spherical geometry as the SDSS. Rather, we use the much simpler ‘natural’ estimator DD/RR − 1 on full mock cubes that employ the plane parallel approximation, i.e. where one of the axes of the cube serves as the line-of-sight
direction. Specifically, we use the single ‘minimum cosmic variance’ box described in Section 3.1.2. This methodology is much faster computationally because it does not require a computation of the DR term in Landy–Szalay. Additionally, it does not contain any noise from a random catalogue because in a cubic geometry with periodic boundary conditions the RR term can be calculated analytically. We have verified that this way of computing $w_p(r_p)$ does not introduce any systematic errors in the modelling.

4.2 The group multiplicity function $n(N)$

Galaxy group statistics are naturally well suited to constraining halo models since groups are often systems of galaxies that occupy the same dark matter halo. Berlind & Weinberg (2002) proposed using the group multiplicity function, defined as the abundance of galaxy groups in the case of the $M_r < -19$ (left-hand panel) and $M_r < -21$ (right-hand panel) samples. Red points show measurements for SDSS galaxies with error bars estimated via jackknife resampling on the sky. Grey lines show individual results for each of 200 mock galaxy catalogues. Blue points and error bars show the mean and standard deviation of $w_p(r_p)$ for the mock catalogues. The bottom section of each panel shows $\chi$, which is the difference between the mock and SDSS measurements divided by the standard deviation of the mocks. Figure 1. Projected correlation function $w_p(r_p)$ measurements for both SDSS and mock galaxies in the case of the $M_r < -19$ (left-hand panel) and $M_r < -21$ (right-hand panel) samples. Red points show measurements for SDSS galaxies with error bars estimated via jackknife resampling on the sky. Grey lines show individual results for each of 200 mock galaxy catalogues. Blue points and error bars show the mean and standard deviation of $w_p(r_p)$ for the mock catalogues. The bottom section of each panel shows $\chi$, which is the difference between the mock and SDSS measurements divided by the standard deviation of the mocks.

$n(N) = \int_0^\infty \frac{dn}{dM} P(N|M) dM,$

where $dn/dM$ is the halo mass function. Unfortunately, group catalogues are plagued by severe systematic effects where single haloes are split into multiple groups and multiple haloes are merged into a single group (e.g. Berlind et al. 2006; Campbell et al. 2015). The measured group multiplicity function is thus very different from the halo multiplicity function given by equation (5). The only way to accurately use group statistics is with a fully numerical modelling procedure where groups are identified directly in mock catalogues and group errors thus affect both data and model equally. Hearin et al. (2013) used the group multiplicity function as measured in mock catalogues to test subhalo abundance matching (SHAM) models and demonstrated that it contains complementary information to the correlation function. However, the multiplicity function has not been used in a full-HOD modelling of galaxy survey data because it is computationally difficult to include mock catalogue construction and analysis within the MCMC modelling procedure. This work represents the first such analysis to date.

We use the Berlind et al. (2006) FoF algorithm for identifying groups in both SDSS and mock data. According to the algorithm, two galaxies are linked together if their projected and line-of-sight separations are both less than a corresponding linking length. A galaxy group then consists of all the galaxies that are linked to each other in this way. We adopt the Berlind et al. (2006) linking lengths of $b_\perp = 0.14$ and $b_l = 0.75$, which are given in units of the mean intergalaxy separation $n^{-1/3}$, where $n_p$ is the sample number density. These linking lengths were specifically optimized to produce a multiplicity function that is as unbiased as possible relative to the true halo multiplicity function. However, this is mostly irrelevant for our study because group finding errors are equally present in both data and model. From our perspective, the group multiplicity function is just a different clustering statistic and any set of linking lengths would work. Using the sample densities listed in Table 1, the comoving linking lengths for our two SDSS samples are $(r_\perp, r_l) = (0.57, 3.05) h^{-1}$Mpc for the $M_r < -19$ sample and $(r_\perp, r_l) = (1.32, 7.06) h^{-1}$Mpc for the $M_r < -21$ sample. The comoving linking lengths that we use on mock samples adjust according to the varying number density.

After running the group finder and generating a group catalogue, we measure $n(N)$ in bins of richness $N$. For the $M_r < -19$ sample, we adopt the following eight bins of $N$: (5–6), (7–9), (10–3), (14–19), (20–32), (33–52), (53–84), and (85–220). For the $M_r < -21$ sample, we adopt the following six bins of $N$: (5–6), (7–9), (10–13), (14–19), (20–27), and (28–40). $n(N)$ is then simply the comoving number density of all groups that have a number of members in the range given by each bin. The red points in Fig. 2 show our measurements of the group multiplicity function for the two SDSS samples. Fig. 2 also shows $n(N)$ for a set of mock catalogues (grey lines and blue points), which we describe in Section 5. Unlike in Fig. 1, the displayed error bars on the SDSS measurements are not
above terms scale with mass. The halo mass function scales as \( n \propto M \), where \( n \) is the number of haloes of a given mass, and \( M \) is the halo mass function, and \( M \) represents the mean number of galaxies per halo for that mass \( (N,M) \). In this discussion, we focus on small scales and we investigate what mass haloes dominate the number of galaxy pairs at each physical scale. We do not discuss the sensitivity of \( \xi \) to changes in the HOD, which has been presented in other studies (e.g. Berlind & Weinberg 2002; Watson et al. 2011).

The correlation function on small scales depends on the mean number of galaxy pairs per halo mass \( \langle N(N-1) \rangle_M \), as well as the spatial distribution of these pairs within haloes. Specifically, the one-halo term of \( \xi(r) \) is proportional to (Berlind & Weinberg 2002):

\[
r^2 \xi_{1h}(r) \propto \int_0^\infty \frac{dn}{dM} \frac{\langle N(N-1) \rangle_M}{M} \frac{F(r/R_{1h})}{R_{1h}(M)},
\]

where \( \frac{dn}{dM} \) is the halo mass function, \( R_{1h} \) is the virial radius of a halo, and \( F(r) \) is the fractional distribution of galaxy pairs separations within a halo. We can determine what mass regime dominates the correlation function by considering how each of the above terms scale with mass. The halo mass function scales as \( \sim M^{1/3} \) for \( M \ll M^* \) and \( \sim \exp(-M/M^*) \) for \( M \gg M^* \), where \( M^* \) is approximately equal to \( 2 \times 10^{14} h^{-1} M_\odot \). The mean number of galaxy pairs per halo scales as \( \langle N_{sat} \rangle^2 \sim M^2 \) for \( M > M^* \). The virial radius scales as \( \sim M^{1/3} \). Finally, the spatial distribution function \( F(r/R_{1h}) \) is relatively insensitive to mass. Combining these terms, we find that the total number of pairs contributed from a given mass scales as \( \sim M^{2/3} \) for \( M \ll M^* \) and \( \sim \exp(-M/M^*) \) for \( M \gg M^* \). As a result, haloes of mass \( M^* \) should dominate the signal in the one-halo term of the correlation function and this result should hold for any sample where \( M^* \) is less than \( M^\star \).

We next investigate this question in more depth using mock catalogues for our \( Mr < -19 \) and \( Mr < -21 \) galaxy samples. Fig. 3 shows the fraction of galaxy one-halo pairs as a function of mass and scale. In each pixel showing a bin of pair separation \( r \) and halo mass \( M \), the colour of the pixel represents the value of \( DD(r,M) \), the number of galaxy pairs in that bin, divided by \( DD(r) \), the total number of pairs at that scale. In other words, each vertical column of pixels shows the normalized fractional distribution of pair counts as a function of halo mass that contribute to a given scale. The top panel of the figure shows the correlation function \( \xi(r) \) with its one- and two-halo breakdown, and the right-hand panel shows the mean number of satellite galaxies \( \langle N_{sat}(M) \rangle \). Finally, the figure marks the halo virial radius at each mass, as well as the value of \( M_{1} \). Fig. 3 confirms that the majority of galaxy one-halo pairs come from the cluster regime and that this is true both when \( M_{1} \) is much lower than \( M^* \) (as in the \(-19 \) sample), and when \( M_{1} \) is of order \( M^* \) (as in the \(-21 \) sample). In the former case each of these clusters contains many satellites and hence several pairs, while in the latter case each cluster only typically contains a single pair. This will help explain the structure of the correlation matrix that we discuss in Section 5.2.

The halo masses that dominate the one-halo pairs naturally increase with scale since pairs must fit within the size of the halo.

### 5 Covariance Matrices

In order to perform accurate modelling of galaxy clustering measurements, it is not sufficient to have an accurate model; we must also have accurate estimates of the errors on the measurements, as...
while the right-hand panel shows the mean number of satellite galaxies $\langle N_{\text{sat}} \rangle$ separately. In the 2-halo regime, the ratio of the number of one-halo galaxy pairs in that bin $\Delta(r, M)$ divided by the number of all one-halo galaxy pairs at that scale $\Delta(r, M)$.

Accurate modelling of small-scale clustering

5.1 Methodology

To estimate the covariance matrix of each SDSS sample, we use the 200 independent mock catalogues described in Section 3. However, in order to construct the mock catalogues, we need to first choose an HOD model to construct 200 mock catalogues for each SDSS sample using jackknife resampling on the sky. We then use the Nelder & Mead (1965) downhill simplex algorithm to find the best-fitting HOD model to our measurements of the projected correlation function. Each time we need to evaluate $w_p(r_p)$ for a given set of parameters, we use the ‘minimum cosmic variance box’ as described in Section 4.1. We use these best-fitting HOD models to construct 200 mock catalogues for each SDSS sample using the methods detailed in Section 3.

Figs 1 and 2 show the projected correlation functions and group multiplicity functions for each of these mock catalogues as well as the average and standard deviation of the 200 mocks. The figures show that the two-point clustering of mock catalogues agrees well with the clustering of the SDSS samples. The group multiplicity function of the mocks roughly agrees with the SDSS, but the agreement is not perfect. This is not surprising given that the multiplicity function was not used in the fit that determined the fiducial mock HOD model. We emphasize that it is not essential that these mocks perfectly match the SDSS clustering; what matters is that the variance among the 200 mocks correctly captures the errors and covariances of our clustering statistics. For this purpose, it is sufficient to get the clustering approximately right. Fig. 1 shows that the errors estimated from the standard deviation of 200 mocks are approximately in agreement with the errors estimated from jackknife resampling of the SDSS samples on the sky.

Armed with measurements of the global number density $n_g$, $w_p(r_p)$, and $n(N)$ for each of 200 mocks, we can construct the joint covariance matrix for each of our two samples. The matrices have dimensions $23 \times 23$ and $21 \times 21$ for the $M_r < -19$ and $M_r < -21$ samples and they are calculated as

$$C_{ij} = \frac{1}{N-1} \sum_{l=1}^{N} (y_i - \bar{y}_i)(y_j - \bar{y}_j),$$

(7)
where the sum is over the \( N = 200 \) mocks, \( y_i \) and \( y_j \) are two of the measurements (e.g. the number density and one of the multiplicity function bins), and \( \bar{y}_i \) and \( \bar{y}_j \) are the mean measurements over the 200 mocks. We also compute the correlation matrices, which are simply the covariance matrices normalized by their diagonal elements

\[
R_{ij} = \frac{C_{ij}}{\sqrt{C_{ii}C_{jj}}}
\]

(8)

The correlation matrix has values between \(-1\) and \(1\), with the diagonal elements being equal to unity by definition.

5.2 The joint correlation matrix

Fig. 4 shows the joint correlation matrix for the \( M_r < -19 \) sample. The \( x \)- and \( y \)-axes represent the various bins of the statistics we use (\( n_{gal}, w_p(r_p) \), and \( n(N) \)). The square blocks of cells along the diagonal moving from the bottom left to the top right (\( 1 \times 1, 14 \times 14 \), and \( 8 \times 8 \)) show the correlation matrices of the individual statistics separately, while the rest of the matrix shows the correlations between different statistics. For example, the first column and bottom row are identical and show the correlation between \( n_{gal} \) and all the bins of the other statistics. Fig. 5 shows the same for the \( M_r < -21 \) sample and differs only in the dimension of the \( n(N) \) portion of the matrix, which is \( 6 \times 6 \). The colour of each cell in the matrix denotes the correlation coefficient \( R_{ij} \), as given by equation (8). Within all the non-diagonal cells, we show the actual measurements from the 200 mock catalogues as black dots. For example, in the second cell of the bottom row of the matrix, the black dots show the scatter plot between \( n_{gal} \) and the smallest scale bin of \( w_p(r_p) \). These mock data provide a more in-depth way to understand the correlation matrix. Cells where the mock values appear highly correlated (i.e. the black dots resemble a straight line) have \( R_{ij} \) values close to unity, while cells where the mock values seem to be distributed randomly have correlation coefficients close to zero. Since the correlation coefficient \( R_{ij} \) only measures linear trends, it is possible in theory to have a scenario where the two variables are highly correlated (e.g. if they are constrained to be on a circle), but the associated correlation coefficient is zero. However, this is not a concern in this case since the cells with low \( R_{ij} \) shown in Figs 4 and 5 do not show any obvious non-linear correlations.

Within the diagonal cells of the correlation matrices shown in Figs 4 and 5 we display the probability distribution function of each clustering measurement from the 200 mock values (black histograms), along with the best-fitting Gaussian function (white lines). We find that the distribution of mock values is well described by a Gaussian for all the bins in \( n_{gal}, w_p(r_p) \), and \( n(N) \). Since the error distributions are Gaussian, we can use the \( \chi^2 \) technique to evaluate the goodness of fit. Finally, in all the cells we have also displayed the SDSS measurements for comparison (white crosses). Since our correlation matrices were created with mocks that were optimized to fit \( n_{gal} \) and \( w_p(r_p) \), all of the SDSS values for these statistics are consistent with the mock values shown by the black dots. However, we did not optimize the mocks to fit \( n(N) \), hence in quite a few of the \( n(N) \) cells, the SDSS and mock measurements do not overlap.

Let us now examine the structure in the joint correlation matrices, focusing first on \( w_p(r_p) \) and then on \( n(N) \). The portions of the correlation matrices that correspond to \( w_p(r_p) \) reveal that neighbouring bins are very highly correlated. This is especially true for small scales in the \( M_r < -19 \) sample, where the first six bins contain almost no independent information (they have correlation coefficients greater than 0.9). On larger scales, correlations remain this high, but only for a couple neighbouring bins on each side of a given bin. In general, one must look very far to find bins that exhibit weak correlations. Only the smallest and largest scales of \( w_p(r_p) \) that we consider are relatively uncorrelated with each other. The overall degree of correlation is significantly less in the \( M_r < -21 \) sample, though even there neighbouring couple bins exhibit correlation coefficients higher than 0.8. The largest difference occurs at small scales, where the \( M_r < -21 \) sample displays much weaker correlations than the \( M_r < -19 \) sample. These results are in agreement with previous works. In particular, McBride et al. (2011) show their correlation matrices for both the redshift-space and the projected correlation function, revealing that much of the strong correlations on small scales are due to projection. This occurs because each projected scale includes pairs of galaxies from a wide range of physical scales, resulting in a high degree of scale mixing that causes different projected scales to contain very similar information. Even without projection, neighbouring scales in the correlation function are always correlated because they share the same underlying Fourier density modes. The \( M_r < -21 \) sample displays weaker correlations than the \( M_r < -19 \) sample for two main reasons. First, it is a lower density sample and thus shot noise, which is inherently uncorrelated, contributes more to the error budget. Secondly, as we discussed in Section 4.3, most one-halo pairs come from cluster-sized haloes in both samples. In the \( M_r < -21 \) sample, these haloes typically only contribute a single pair, while in the \( M_r < -19 \) sample they contribute many pairs. As the number of clusters fluctuates from mock to mock, in the low-luminosity sample these fluctuations will enhance or suppress pairs at all one-halo scales simultaneously, thus correlating the scales strongly. In the luminous sample, this will occur at a much lesser extent.

We next move to the portions of the correlation matrices that correspond to the group multiplicity function, \( n(N) \). Different multiplicity bins are not as correlated as different scales of \( w_p(r_p) \), but there are still significant correlations (coefficients in the range of 0.4–0.8) in the case of the \( M_r < -19 \) sample. In the \( M_r < -21 \) sample \( n(N) \) correlations are much weaker (coefficients in the range of 0.1–0.4). Some of the correlation between bins of \( n(N) \) is likely of an indirect nature, due to direct correlations between these bins and the overall number density \( n_{gal} \), which can be quite strong according to Figs 4 and 5. To uncover the intrinsic correlations between the \( n(N) \) bins, we subsample the 200 mocks (for both samples) to have identical number densities. We then create new correlation matrices out of these 200 \( n_{gal} \) matched mocks. In Fig. 6, we show the original (top panels) and subsampled (bottom panels) joint correlation matrices for the \( M_r < -19 \) (left-hand panels) and \( M_r < -21 \) (right-hand panels) samples. The correlation matrices of \( w_p(r_p) \) do not change because the correlation function is not affected by subsampling. However, in both samples we find that, when controlling for \( n_{gal} \), the correlation matrix of \( n(N) \) becomes much more diagonal. The matrix becomes completely diagonal in the \( M_r < -21 \) case, while in the \( M_r < -19 \) case there is some anticorrelation present between low- and high-multiplicity groups. This anticorrelation is a result of the nature of the group-finder since, in the case of constant density, an above average abundance of high-multiplicity groups in one mock catalogue must come at the expense of low-multiplicity groups. From this subsampling test, we thus learn that the correlations seen in the group multiplicity function are largely due to correlations with the overall number density. When the density is higher, the entire \( n(N) \) is boosted.

5.3 Noise in the correlation matrix

Each correlation matrix \( R \) that we have estimated from 200 mock catalogues contains some degree of noise due to the fact that the
number of mocks is limited. When we invert the matrix, this noise can amplify and affect the calculation of $\chi^2$ values in unpredictable ways. We deal with this problem using a singular value decomposition (SVD) approach (e.g. Scoccimarro 2000; Eisenstein & Zaldarriaga 2001; Gaztañaga & Scoccimarro 2005; Norberg et al. 2009). Specifically, we decompose the correlation matrix into its principle components by finding the eigenvectors $E_i$ and eigenvalues $\lambda_i$ that satisfy the equations

$$RE_i = \lambda_i E_i. \quad (9)$$

This rotates the space of our measurements into a basis where the eigenvectors are uncorrelated (i.e. where the correlation matrix is diagonal). We can then sort the eigenvectors by their eigenvalues and trim out those with low eigenvalues, which contain much of the noise, but little information. Following Gaztañaga & Scoccimarro (2005), we only keep eigenvectors for which $\lambda_i^2$ is approximately larger than the resolution with which $R$ is measured, which is

$$\lambda_i^2 \gtrsim \sqrt{\frac{2}{N_{\text{mocks}}}}. \quad (10)$$

where, in our case, $N_{\text{mocks}} = 200$. This procedure effectively reduces the number of data points that we use for fitting models. For the $M_r < -19$ correlation matrix, we trim 12 of the 23 eigenvectors and thus only keep 11 data points in the new orthogonal measurement space. For the $M_r < -21$ matrix, we only trim 5 of the 21 eigenvectors and thus keep 16 data points. The more drastic trimming for the low-luminosity sample is a direct result of the higher amount of
correlation present in its correlation matrix. For all the joint fits to \( n_{\text{gal}}, w_p(r_p), \) and \( n(N) \) that we perform and describe in the next section, we run two versions: one with the full correlation matrix and one after trimming out noisy eigenvectors (which we label as PCA in the text). The PCA fits throw away some of the signal present in the data, but the resulting \( \chi^2 \) values are more reliable and so these are the more conservative results.

6 MODEL FITTING

Our primary objective is to accurately model galaxy clustering statistics and thus test the standard ΛCDM + halo model. To do this, we explore the HOD parameter space using an MCMC method.

6.1 Computing the likelihood function

For each parameter combination we compute the likelihood function \( P(\text{data}|\text{model}) \). Since the error distributions are Gaussian, the likelihood function is proportional to \( \exp(-\chi^2/2) \). We compute the \( \chi^2 \) statistic as follows:

\[
\chi^2 = \sum_{ij} \chi_i \mathcal{R}_{ij}^{-1} \chi_j,
\]

where \( \mathcal{R}^{-1} \) is the inverted correlation matrix and

\[
\chi_i = \frac{D_i - M_i}{\sigma_i},
\]

where \( D_i \) is the SDSS measurement for data point \( i \), \( M_i \) is the model prediction for that measurement, and \( \sigma_i \) is the error in the SDSS measurement. When we use the PCA analysis to reduce noise introduced by inverting the correlation matrix, we calculate \( \chi^2 \) differently. Since the resulting eigenvectors are uncorrelated, \( \chi^2 \) is simply a sum of the \( \chi_i^2 \) contributions for the eigenvectors we have kept (where \( D_i \) and \( M_i \) are recomputed for the new orthogonal space).
Figure 6. The effect of number density correlations on the correlation matrix for the $M_r < -19$ (left-hand panels) and the $M_r < -21$ (right-hand panels) samples. Top panels show the fiducial correlation matrices as shown in Figs 4 and 5. Bottom panels show correlation matrices created from the same initial 200 mocks, but after randomly downsampling, the mocks to the same galaxy number density, $n_{\text{gal}}$ (the lowest amongst the 200 mocks). By matching the $n_{\text{gal}}$ in all the mocks, we remove the effect of two measurements showing a correlation due to their individual correlations with $n_{\text{gal}}$. Since the correlation function does not depend on density, randomly downsampling each of the mocks has almost no effect on the correlation coefficients in the $w_p(r_p)$ part of the matrix. However, the correlations in the $n(N)$ bins change drastically. Most of the positive correlations seen in the top panels vanish to reveal nearly diagonal $n(N)$ coefficients in the bottom panels.

Ideally, each time we moved to a new point in the HOD parameter space, we would compute $\chi^2$ using updated values of $\sigma_i$ and an updated matrix $R$ that were generated from the new model. In other words, we would have to create a new set of 200 mock catalogues. This would increase the computational requirement of the MCMC chain by two orders of magnitude and is currently unfeasible. We thus make the approximation that our fiducial HOD, which reproduces the $w_p(r_p)$ measured in the SDSS sample, is representative of the errors and correlation matrix in the parameter-space of interest. Specifically, we adopt the fractional errors given by the fiducial mock catalogues and scale them by the SDSS measurements $D_i$ to obtain

$$\sigma_i = \frac{\sigma_{\text{mock},i}}{M_{\text{mock},i}} \times D_i,$$

where $M_{\text{mock},i}$ and $\sigma_{\text{mock},i}$ are the mean and standard deviation of data point $i$, as measured from our 200 fiducial mock catalogues. $\sigma_i$ are then the scaled SDSS errors that we use when computing $\chi^2$ in equation (12).

Given that we fix $R$ and $\sigma_i$ using the fiducial mocks, the only ingredient we must compute for each parameter combination within the MCMC is the model prediction $M_i$. As was discussed in Section 3.1.2, we use a substantially larger volume to compute $M_i$ than the SDSS volume that was used to compute $D_i$. For $w_p(r_p)$, we use a single mock catalogue made from a whole simulation cube, which has approximately 13 times more volume than the corresponding SDSS sample. For $n(N)$, we use the mean of eight mock catalogues, each of which has the same volume as the corresponding SDSS sample. Furthermore, we reduce cosmic variance errors by an additional factor of $\sim 2$ by carefully selecting which simulation boxes we use to compute $M_i$. The uncertainties in the model prediction $M_i$ are thus subdominant compared to the uncertainties in $D_i$ and we can safely ignore them when we compute the model likelihood.
6.2 Running and analyzing the MCMC chains

For each new set of HOD parameters, we must perform the following operations: (1) populate the halo catalogues from two simulation boxes with galaxies according to the HOD; (2) compute \( w_p(r_p) \) on one of these boxes; (3) create eight SDSS-like mock catalogues from the two boxes; (4) run the FoF group finder on these eight mocks and measure the mean \( n(N) \). The codes used for these operations are heavily optimized and take approximately 15 s of wallclock operations: (1) populate the halo catalogues from two simulation samples, the ranges are \( \log M_\text{min} \), \( \log M_0 \), \( \alpha = 0.001 \), \( \log M_1 = 12 \), and \( \log M_2 = 6 \). We run a total of ten MCMC chains for each luminosity sample. Two of these are model fits to \( w_p(r_p) \) and \( n(N) \), while the remaining eight are for joint fits to the SDSS \( w_p(r_p) \). However, visual inspection can be a highly misleading way to assess goodness of fit when data points are highly correlated, as is the case here. We thus rely on values of \( \chi^2 \) and corresponding \( p \)-values. The best-fitting values of \( \chi^2 \) are 14.2 and 19.2 for the \( M_r < -19 \) and \( M_r < -21 \) samples, respectively. Both samples have 10 degrees of freedom (14 bins for \( w_p(r_p) \) and 5 for \( n_p = 5 \) free HOD parameters). The resulting \( p \)-value for the \( -19 \) sample is 0.162, which is acceptable. The \( p \)-value for the \( -21 \) sample is 0.038, which reveals a tension slightly larger than 2\( \sigma \), but certainly not enough to warrant ruling out the model. It is notable that the low-luminosity sample shows the best goodness-of-fit even though it looks like a worse fit by visual inspection of Fig. 7. This is due to the high degree of correlation between bins in the \( -19 \) measurement of \( w_p(r_p) \).

6.3 Fits to the correlation function \( w_p(r_p) \)

Before modelling the correlation function and the group multiplicity function jointly, we model each statistic alone in order to compare the constraining power of the statistics. First, we model the correlation function measurements together with the number density. This is comparable to what other authors have done in modelling the SDSS DR7 data using both analytic (Zehavi et al. 2011) and mock-based (Zentner et al. 2016) methods. For this analysis we adopt the LasDamas cosmological model and the virial halo definition (Mvir). We also use the original correlation matrices, rather than the PCA versions that have trimmed eigenvectors in order to reduce noise. Since we are only fitting a model to \( n_p \) and \( w_p(r_p) \), we only use the first \( 15 \times 15 \) portion of the correlation matrices shown in Figs 4 and 5.

Fig. 7 shows \( w_p(r_p) \) for the best-fitting model compared to the SDSS data for the two luminosity samples we consider in this work. The error bars in the plot show \( \sigma \), as estimated using equation (13). The best-fitting model appears by-eye to provide a decent match to the SDSS \( w_p(r_p) \). For the \( M_r < -19 \) sample is 0.162, which is acceptable. The \( p \)-value for the \( -21 \) sample is 0.038, which reveals a tension slightly larger than 2\( \sigma \), but certainly not enough to warrant ruling out the model. It is notable that the low-luminosity sample shows the best goodness-of-fit even though it looks like a worse fit by visual inspection of Fig. 7. This is due to the high degree of correlation between bins in the \( -19 \) measurement of \( w_p(r_p) \).

6.4 Fits to the group multiplicity function \( n(N) \)

We now model the group multiplicity function measurements together with the number density. This is the first time this statistic is being used for HOD modelling of SDSS data. As before, for this analysis we adopt the LasDamas cosmological model, the virial halo definition (Mvir), and the original non-PCA correlation matrices. Since we are only fitting a model to \( n_p \) and \( n(N) \), we only use a \( 9 \times 9 \) subset of the full correlation matrix shown in Fig. 4 and a \( 7 \times 7 \) subset of the matrix shown in Fig. 5.

Fig. 8 shows \( n(N) \) for the best-fitting model compared to the SDSS data for the two luminosity samples we consider in this work. The error bars in the plot show \( \sigma \), as estimated using equation (13). The best-fitting model appears by-eye to provide a perfect match to the SDSS \( n(N) \) and the \( p \)-values estimated from the \( \chi^2 \) values confirm this. The best-fitting values of \( \chi^2 \) are 1.6 and 0.1 for the \( M_r < -19 \) and \( M_r < -21 \) samples, respectively. The corresponding number of degrees of freedom for the two samples are 4 and 2 (8 or 6 bins for \( n(N) + 1 \) for \( n_p = 5 \) free HOD parameters). The resulting \( p \)-values are 0.81 and 0.95, which are large enough to suggest that the model has too much freedom. A simpler HOD form with fewer free parameters might be equally successful at reproducing the observed \( n(N) \).

Since this is the first time that the group multiplicity function is being used to constrain HOD parameters, it is worth comparing its constraining power to that of the correlation function. In the case of the \( M_r < -19 \) sample, fitting to \( n(N) \) yields similar marginalized uncertainties in \( \log M_\text{min} \) and \( \sigma_\log M \) compared to fitting to \( w_p(r_p) \), but a 20 per cent smaller uncertainty in \( \log M_\text{l} \) and a 35 per cent...
smaller uncertainty in $\alpha$. In the case of the $M_r < -21$ sample however, fitting to $n(N)$ leads to substantially weaker constraints, with uncertainties in $\log M_{\text{min}}$, $\sigma \log M$, $\log M_1$, and $\alpha$ blowing up by 45–100 per cent compared to $w_p(r_p)$. This difference is partly due to the more correlated nature of $w_p(r_p)$ in the low-luminosity sample, which reduces its statistical power compared to the high-luminosity sample. Since we are not proposing replacing $w_p(r_p)$ as a statistic to fit to, a more relevant way to assess the value of $n(N)$ is to see how HOD constraints improve when fitting to both statistics compared to $w_p(r_p)$ alone. We investigate this next.

### 6.5 Joint fits to both $w_p(r_p)$ and $n(N)$

In this section, we present results from the MCMC chain that simultaneously fits $w_p(r_p)$, $n(N)$, and the galaxy number density. Once again, for this analysis we adopt the LasDamas cosmological model, the virial halo definition ($M_{\text{vir}}$), and the original non-PCA correlation matrices (the full correlation matrices shown in Figs 4 and 5). In the next sections we will explore a different cosmology and halo definition, as well as the effect of reducing noise in the correlation matrices via PCA.

Fig. 9 shows both statistics for the best-fitting model compared to the SDSS data, for the two luminosity samples we consider in this work. It is clear from the figure that jointly fitting $w_p(r_p)$ and $n(N)$ is a significantly more challenging task than fitting each statistic individually. The best-fitting model for the $M_r < -19$ sample (top panels) reproduces $w_p(r_p)$ reasonably well, but fails at matching $n(N)$, while the reverse is true for the $M_r < -21$ sample (bottom panels). The corresponding values of $\chi^2$ are 39.6 and 34.6 and the number of degrees of freedom are 18 and 16, respectively. The
resulting $p$-values are 0.0024 and 0.0045, which suggest that the model is ruled out at the $\sim 3\sigma$ level when tested against either luminosity sample. The best-fitting model results for this and all subsequent MCMC chains are listed in Table 3.

To better understand the tension between fitting $w_p(r_p)$ and $n(N)$, it is helpful to study the posterior probability distributions for the HOD parameters. Fig. 10 shows joint distributions for central galaxy HOD parameters $\log M_{\text{min}}$ versus $\sigma_{\log M}$, and satellite parameters $\log M_1$ versus $\alpha$. We do not show results for $M_0$ because it is always very poorly constrained. In each case we compare results when fitting our model to $w_p(r_p)$ only (blue contours), $n(N)$ only (green contours), and both statistics jointly (orange contours). The galaxy number density is also used as a constraint in all cases.

First, let us examine the parameter constraints for the $w_p(r_p)$ and $n(N)$ only chains. In the case of the $M_r < -19$ sample, the $n(N)$ chain prefers lower values of $\log M_1$ and $\alpha$, and a higher value of $\log M_{\text{min}}$ compared to the $w_p(r_p)$ chain. To understand this, we can look back at Fig. 2, which shows $n(N)$ for the mock catalogues that we used to make the joint correlation matrices (and that matched the SDSS $w_p(r_p)$ by design). The top panel of that figure shows that the model predictions for $n(N)$ are lower than the SDSS $n(N)$ at all $N$ except for the largest $N$ bin. In other words, a HOD model that matches

![Figure 9](https://example.com/figure9.png)

**Figure 9.** Projected correlation function $w_p(r_p)$ (top panels) and group multiplicity function $n(N)$ (bottom panels) for the best-fitting HOD model when the model is jointly fit to $w_p(r_p)$, $n(N)$, and the galaxy number density. Blue points show the best-fitting model, while red points show the SDSS measurements in the case of the $M_r < -19$ (left-hand panels) and $M_r < -21$ (right-hand panels) samples. Error bars on the SDSS measurements are estimated from the dispersion among 200 mock catalogues (shown in Figs 1 and 2). The bottom section of each panel shows $\chi$, which is the difference between the best-fitting and SDSS measurements, divided by the standard deviation of the mocks. The $\chi^2$, degrees of freedom, and $p$-values are listed in the panels. The model shown here assumes the LasDamas cosmology and the virial halo definition ($M_{\text{vir}}$) and does not include PCA reduction.
the SDSS $w_p(r_p)$ underpredicts $n(N)$ by almost 2σ. To fit the SDSS $n(N)$, we would thus need to boost $n(N)$ at all $N$. This can be achieved primarily by reducing log $M_1$ (with an adjustment to $\alpha$). However, reducing log $M_1$ increases $n_{\text{gal}}$ because, in the $M_1 < -19$ sample, the satellite fraction is $\sim 0.3$ so changes in the satellite occupation can contribute a significant change in $n_{\text{gal}}$. This increase in $n_{\text{gal}}$ has to be compensated by increasing log $M_{\text{min}}$ as well. As a result, the parameter constraints for the $n(N)$ only chain have systematically lower log $M_1$ and higher log $M_{\text{min}}$ than for the $w_p(r_p)$ only chain.

In the case of the $M_1 < -21$ sample, the $n(N)$ chain prefers a somewhat lower value of $\alpha$, but similar values of the other three parameters, compared to the $w_p(r_p)$ chain. The bottom panel of Fig. 2 shows that the HOD model that matches the SDSS $w_p(r_p)$ predicts a $n(N)$ with a slope that is too shallow (it underpredicts the abundance of small-$N$ groups but overpredicts the abundance of large-$N$-groups). To fit the SDSS $n(N)$, we would thus primarily need to lower $\alpha$ (with a small adjustment to log $M_1$). This change to the satellite occupation does not significantly affect $n_{\text{gal}}$ because the satellite fraction in the $M_1 < -21$ sample is only $\sim 0.1$. Consequently, the central galaxy parameters do not need to change.

Fig. 10 shows that the constraints on HOD parameters when fitting jointly to $w_p(r_p)$ and $n(N)$ are significantly tighter than when fitting $w_p(r_p)$ alone. In the case of the $M_1 < -19$ sample, adding group statistics reduces the marginalized uncertainties in log $M_{\text{min}}$, $\sigma_{\log M}$, log $M_1$, and $\alpha$ by 40–60 per cent. In the case of the $M_1 < -21$ sample, while the uncertainty in log $M_{\text{min}}$ does not change and the uncertainty in log $M_1$ only improves by 10 per cent, the uncertainties in $\sigma_{\log M}$ and $\alpha$ improve by 30 per cent. The means and standard deviations of the marginalized posterior distributions for all HOD parameters in this and all subsequent MCMC chains are listed in Table 4. These improved parameter constraints, along with the lower best-fitting $p$-values, demonstrate the power of combining new clustering statistics with $w_p(r_p)$ into HOD modelling of the galaxy distribution.

6 Changing log $M_{\text{min}}$ only mildly affects $n(N)$ since the linking length changes adaptively as $n_{\text{gal}}^{-1/3}$. The effect is stochastic and affects mostly the small-$N$ groups.

### 6.6 Accounting for noise in the correlation matrix

Our results jointly fitting $w_p(r_p)$, $n(N)$, and the galaxy number density of $M_1 < -19$ and $-20$ galaxies with a LasDamas cosmological model, $M_{\text{vir}}$ haloes, and the standard five-parameter HOD, show that the model is ruled out at the $\sim 3\sigma$ level for both low- and high-luminosity samples. Before we conclude that our model assumptions are incorrect, we need to make sure that the best-fitting $p$-values are robust. As discussed in Section 5.3, noise in our estimated correlation matrices can bias our fit results in difficult to predict ways. We thus follow the PCA procedure outlined in that section to trim noisy eigenvectors from the correlation matrices and we rerun all of our MCMC chains.

Table 3 shows the best-fitting results for the PCA chains. In the case of the $M_1 < -19$ sample, the best-fitting $p$-value increases by a factor of $\sim 30$, from 0.0024 to 0.0751. In other words, the tension between model and data reduces from $3\sigma$ to 1.8σ and we no longer rule it out. For the $M_1 < -21$ sample, however, the results do not change significantly and the tension between model and data remains strong. The difference between the two samples is due to the much more correlated nature of the low luminosity measurements, as seen in their correlation matrices (Figs 4 and 5). Table 4 shows the effect that our PCA analysis has on the HOD parameter constraints. For the $M_1 < -19$ sample, the constraints degrade considerably, with $\sigma$ uncertainties in log $M_{\text{min}}$, $\sigma_{\log M}$, log $M_1$, and $\alpha$ increasing by factors of 4.3, 2.7, 1.6, and 1.7, respectively. Parameter uncertainties for the $M_1 < -21$ sample are unaffected. In summary, the PCA approach leads to no change for the high-luminosity sample, and weaker but more reliable constraints for the low luminosity sample. We adopt these as our main results moving forward.

### 6.7 Varying the halo definition and cosmology

In the halo model framework, the clustering of galaxies is determined by two main ingredients: the halo distribution and the way in which galaxies populate haloes, i.e. the HOD. The halo distribution depends on the assumed cosmological model and halo definition. The HOD is comprised of various assumptions, such as the functional form of the mean galaxy occupation, the scatter about the
mean, and the spatial and velocity distribution of galaxies within haloes. In this section, we probe the sensitivity of our results to the assumed halo definition and cosmological model. Specifically, we introduce a different halo definition (M200b) and different cosmological model (Planck), and we run MCMC chains for all the combinations of these choices. We can then study how sensitive the results are to these assumptions.

First, we consider a change in the halo definition. In principle, any reasonable halo definition should be able to successfully model the galaxy distribution. However, some halo definitions might require more complicated HOD parametrizations than others. For a given parametrization (like the 5-parameter model we adopt in this work), some halo definitions likely work better than others. This dependence of HOD modelling on halo definition is a research area that has not been previously explored so the test we perform here represents the first step in that direction.

Our fiducial halo definition, \( M_{\text{vir}} \), corresponds to values of the mean halo overdensity (with respect to the mean density) \( \Delta = 351 \) and 321 for the LasDamas cosmological model at the two median redshifts of the \( M_r < -19 \) and \( -21 \) samples (Bryan & Norman 1998). For the Planck cosmological model that we consider below, the corresponding values are \( \Delta = 315 \) and 292. Changing the halo definition to M200b, which corresponds to \( \Delta = 200 \), makes all haloes approximately 20 per cent larger in radius. As a result, satellite galaxies placed inside haloes will be more spatially extended. Moreover, some haloes that were previously classified as host haloes will now be classified as subhalos, thus changing the probability that they receive a galaxy. These changes will certainly alter \( w_p(r_p) \) and perhaps \( \eta (N) \) for a fixed HOD model, but the question is whether a different set of HOD parameters can compensate for this.

Fig. 11 shows posterior probability distributions for HOD parameters for the two halo definitions, when the model is fit jointly to measurements of \( w_p(r_p) \), \( n(N) \), and \( \eta (\text{gal}) \). The main effect of changing the halo definition is to shift \( \log M_{\text{min}} \) to larger values. This is required to preserve the number density since all haloes grow in mass under the M200b definition. The satellite occupation does

Figure 10. Posterior probability distributions for HOD parameters when the model is fit to SDSS measurements in the case of the \( M_r < -19 \) (left-hand panels) and \( M_r < -21 \) (right-hand panels) samples. Top panels show the joint distribution of central galaxy parameters \( \log M_{\text{min}} \) and \( \sigma_{\log M} \), while bottom panels show the joint distribution of satellite galaxy parameters \( \log M_1 \) and \( \alpha \). In each case, the probability distributions are marginalized over the remaining three HOD parameters. In each panel, blue, green and orange contours show results when the model is fit to measurements of \( w_p(r_p) \) only, \( n(N) \) only, and both statistics jointly (together with the galaxy number density in all cases). The contours show the regions of parameter space that contain 68 per cent and 95 per cent of the MCMC probability. The model shown here assumes the LasDamas cosmology and the virial halo definition (\( M_{\text{vir}} \)) and does not include PCA reduction.

\(^7\)This calculation assumes the Navarro, Frenk & White (1997, NFW) density profile.
not change, however. The best-fitting results in Table 3 show that adopting this lower $\Delta$ definition leads to similar quality fits for the $M_r < -19$ sample ($p$-value decreases from 0.0751 to 0.0653), but substantially worse fits for the $M_r < -21$ sample ($p$-value decreases from 0.0056 to 0.0017). Therefore, lowering $\Delta$ from the Mvir definition does not alleviate the tension between model and data that we find here.

Next, we consider a change in the cosmological model. Since cosmology and the HOD are not degenerate (Zheng & Weinberg 2007), it is possible that some models work better than others. We compare our LasDamas model to the Planck model. The primary difference between the two is that Planck has a higher value of $\Omega_m$ (0.302 versus 0.25), but there are also differences in $\Omega_b$ and $n_s$. Fig. 12 shows posterior probability distributions of HOD parameters for the two cosmologies, when the model is fit jointly to measurements of $w_p(r_p)$, $n(N)$, and $n_g$ (assuming the Mvir halo definition). The main effect of adopting the Planck cosmology is to shift the whole mean galaxy occupation to higher mass (i.e. increase both $\log M_{min}$ and $\log M_1$). This shift compensates for the higher halo masses due to the larger value of $\Omega_m$. Table 3 reveals that all the fits improve considerably with the Planck cosmology. In the case of

**Table 4.** Marginalized HOD parameter constraints. Similar to Table 3, except that listed non values show the median, and the upper and lower limits corresponding to the 84 and 16 percentiles of the parameter values from the MCMC chain.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Cosmology</th>
<th>Halo def</th>
<th>PCA</th>
<th>$\log M_{min}$</th>
<th>$\sigma_{\log M}$</th>
<th>$\log M_0$</th>
<th>$\log M_1$</th>
<th>$\alpha$</th>
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<tbody>
<tr>
<td>$M_r &lt; -19$</td>
<td>LasDamas</td>
<td>Mvir</td>
<td>No</td>
<td>11.53$^{+0.05}_{-0.15}$</td>
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<td>8.95$^{+2.05}_{-0.11}$</td>
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<td>Mvir</td>
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<td>0.60$^{+0.30}_{-0.41}$</td>
<td>10.78$^{+3.24}_{-0.26}$</td>
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<td>0.93$^{+0.06}_{-0.05}$</td>
</tr>
<tr>
<td></td>
<td>LasDamas</td>
<td>M200b</td>
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<td>11.59$^{+0.09}_{-0.05}$</td>
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<td>0.59$^{+0.41}_{-0.17}$</td>
<td>10.74$^{+3.22}_{-0.11}$</td>
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<td>0.94$^{+0.06}_{-0.05}$</td>
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<td></td>
<td>Planck</td>
<td>Mvir</td>
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<td>12.71$^{+0.09}_{-0.09}$</td>
<td>0.94$^{+0.06}_{-0.05}$</td>
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</table>

**Figure 11.** Posterior probability distributions for HOD parameters comparing the cases of two different halo definitions. Panel layout is identical to that in Fig. 10. In each panel, orange and green contours show results for the Mvir and M200b halo definitions, respectively. In both cases, the model shown assumes the LasDamas cosmology and does not include PCA reduction.

**Figure 12.** Posterior probability distributions for HOD parameters comparing the cases of two different cosmological models. Panel layout is identical to that in Fig. 10. In each panel, orange and green contours show results for the LasDamas and Planck cosmological models, respectively. In both cases, the model shown assumes the Mvir halo definition and does not include PCA reduction.
While changing the halo definition did not relieve the tension between model and data, changing the cosmological model does just that. The tension in the $M_\text{t} < -21$ sample when adopting the Planck model and the $M_{\text{vir}}$ definition reduces to the 2.3σ level, which is not high enough to warrant ruling the model out. This sensitivity of $\chi^2$ to the assumed cosmology illustrates the power of small-scale clustering to constrain cosmological models. Unfortunately, to properly probe the cosmological parameter space would require a very large number of simulations and is not a trivial exercise.

Looking at all the (PCA) chain results in Table 3, we see that the model that works best for both luminosity samples is the Planck $M_{\text{vir}}$ model. This model combined with the ‘vanilla’ $\text{HOD}$ model described in Section 3.2 predicts $w_p(r_p)$, $n(N)$, and $n_{\text{gal}}$ that are consistent with SDSS measurements for low luminosity galaxies, but show 2.3σ tension for high-luminosity galaxies. To further alleviate this tension we would likely have to relax some of the assumptions built into the $\text{HOD}$ model. However, the most important result of this paper is that we have achieved a sufficiently robust modelling methodology that we can now test halo models in a statistical sense.

7 FUTURE MODELLING IMPROVEMENTS

In this section, we discuss improvements to our modelling pipeline that we leave for future work. Improvements come in two types: changes that reduce systematic errors and thus make the modelling more accurate, and changes that add freedom to the model being tested. Regarding improvements to the model accuracy, we believe that our methodology represents the most accurate $\text{HOD}$ modelling to-date given that we compute statistics from realistic mock catalogues, we compute covariance matrices from a large ensemble of mocks, we account for noise in the matrices, etc. Nevertheless, there are three potential sources of systematic error that need further testing.

First, we do not include fibre collision incompleteness in our modelling pipeline. We apply the nearest neighbour correction to the SDSS data, but we do not attempt to model how this correction might fail. We have argued that, for the scales in $w_p(r_p)$ and multiplicities in $n(N)$ that we consider in this paper, this is sufficient. However, a proper modelling of fibre collisions may become necessary if we extend the analysis to different scales or statistics. Including fibre collisions in the mocks is non-trivial because doing it correctly requires mocks of the full flux-limited SDSS sample. However, it may be possible to approximate the effect with our existing simulations via semi-analytic methods.

Secondly, the resolution of the simulations that we use to construct the covariance matrix of the $M_\text{t} < -21$ sample is fairly low. We expect that this will not have a large impact on the scatter among many simulation realizations, but this needs to be tested more thoroughly. Ideally, we need a proper convergence test, but that would require a very large number of additional simulations (50 boxes per resolution tested). More realistically, we could use fewer mock catalogues of smaller volume each to get a feel for how the covariance matrix depends on resolution.

Thirdly, we construct the correlation matrix from a fiducial $\text{HOD}$ model and assume that it is fixed as we explore the $\text{HOD}$ parameter space. Ideally, to estimate $P(\text{data} | \text{model})$, we should be recomputing the correlation matrix at each new point in parameter space. This is prohibitively expensive because it would require construction and analysis of 200 mocks at each unique link of the MCMC chain, instead of the ~10 that we use now. If testing shows that varying the correlation matrix is necessary, we can explore ways to interpolate between a sparse set of matrices within the parameter space.

There are a number of extensions to the ‘vanilla’ $\text{HOD}$ model that we plan to implement for the purpose of giving the model more freedom and to probe interesting aspects of the galaxy–halo connection. For example, we can drop the assumption that satellite occupation follows Poisson statistics, which is not necessarily the case (Boylan-Kolchin et al. 2010; Mao et al. 2015). Another extension is to drop the assumption that the central galaxy is at rest at the halo centre and that satellite galaxies trace the spatial and velocity distributions of dark matter within haloes. Several studies, both theoretical (e.g. Berlind et al. 2003) and observational (e.g. van den Bosch et al. 2005), have shown that these are not good assumptions. An $\text{HOD}$ model with built-in velocity bias for centrals and satellites was recently used by Guo et al. (2015a). Finally, we can drop the assumption that galaxy occupation statistics only depend on halo mass and account for galaxy assembly bias (Croton, Gao & White 2007). Though there are theoretical reasons to expect some level of assembly bias for luminosity threshold samples (e.g. Zentner et al. 2005; Zehavi et al. 2017), there is not yet strong observational evidence for this. To incorporate assembly bias, we could adopt the ‘decorated $\text{HOD}$’ model of Hearin et al. (2016), which was recently used to model SDSS data by Zentner et al. (2016). Naturally, as we add freedom to the $\text{HOD}$ model, we can also add additional clustering statistics. A great advantage of our mock-based modelling methodology is that it can easily incorporate new statistics.

8 SUMMARY AND DISCUSSION

In this paper, we have developed an accurate mock-based $\text{HOD}$ modelling framework and have applied it to measurements of the projected correlation function $w_p(r_p)$, the group multiplicity function $n(N)$, and the galaxy number density $n_{\text{gal}}$ of two luminosity threshold samples in the SDSS DR7. Features of the modelling framework include (1) construction of realistic mock galaxy catalogues by populating dark matter haloes in cosmological $N$-body simulations with galaxies, and applying redshift distortions and survey selection functions; (2) calculation of model predictions by running the same analysis codes on mock catalogues as on the SDSS data, and averaging over enough mocks so that statistical errors in the model are negligible; (3) estimation of errors and covariances via 200 independent mock catalogues; (4) reduction of noise in the covariance matrices via an eigenmode analysis; (5) parameter search using the EMCEE MCMC code that includes ~5 × 10^4 evaluations of likelihood; (6) new blazing fast analysis codes that make this data-intensive approach feasible.

The specific model we have tested in this paper is the $\Lambda$CDM + ‘vanilla’ $\text{HOD}$ model, whereby the dark matter halo population is given by cosmological $N$-body simulations that contain only dark matter, and galaxies are placed within haloes according to a simple five-parameter $\text{HOD}$ model (Zheng et al. 2007) that contains no spatial or velocity bias of centrals or satellites and no galaxy assembly bias. We have tested two cosmological models (LaDamas and Planck) and two halo definitions ($M_{\text{vir}}$ and $M_{200b}$). Our main results are the following:

(i) The model is successful at fitting either $w_p(r_p)$ or $n(N)$ (plus $n_{\text{gal}}$ in both cases) for both luminosity samples. However, the regions of $\text{HOD}$ parameter space selected are different depending on which clustering statistic is used. In terms of constraining power,
Most importantly, we have demonstrated that it is possible to use galaxy clustering on small scales to perform sensitive statistical tests of cosmology + halo models. This is made possible by our fully numerical mock-based methodology combined with fast analysis codes and a careful treatment of systematic and statistical errors. Though most halo model analyses in the literature use analysis codes and a careful treatment of systematic and statistical errors, there are a few studies that have adopted mock-based methods like ours. The first such studies modelled the projected correlation function of red galaxies in the high redshift (White et al. 2011) and low-redshift (Parejko et al. 2013) samples of the BOSS survey. Zheng & Guo (2016) developed a numerical method in which halo pairs and particle pairs within haloes are measured in a N-body simulation and tabulated as a function of halo mass and separation. The correlation function of galaxies can then be computed accurately by appropriately weighting these functions by the IDOD. This method was applied to the projected and redshift space correlation functions of BOSS and SDSS galaxies in order to constrain the velocity bias of central and satellite galaxies (Guo et al. 2015a,b,c). The downside of this methodology is that it cannot be extended to other statistics beyond pair (or triple) counts, like the group multiplicity function. The most similar modelling methodology to the one we present in this paper was developed as part of the powerful HALOFLUE software package (Hearin et al. 2017). This was recently used by Zentner et al. (2016) to model the clustering of SDSS galaxies with the goal of constraining galaxy assembly bias. Our analysis improves on Zentner et al. (2016) in the following ways: (1) we include the group multiplicity function as a constraint; (2) we directly adopt the spatial and velocity distribution of particles within haloes to place satellite galaxies, while they assume a NFW profile and a Gaussian velocity distribution; (3) we use independent mock catalogues to estimate the covariance matrix, while they use Jackknife resampling; (4) we eliminate noise in the covance matrix via PCA; (5) they adopt a Poisson error on the galaxy number density, which ignores cosmic variance and is only \( \sim 0.33\% \), while the correct errors that we obtain from our 200 mock catalogues are 8 per cent and 2.5 per cent for the \( M_r < -19 \) and \( -18 \) samples, respectively. This work therefore represents the most accurate modelling of SDSS galaxies to date.

Our hope is that with these and future improvements to the accuracy of modelling together with an optimal set of statistics, galaxy clustering on small scales will definitively measure the galaxy–halo connection, including second-order features like assembly bias. Moreover, small-scale clustering has the potential to become a standard test of cosmological models. Whether constraints on cosmology can compete with other probes remains to be seen.

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REFERENCES

Blanton M. R. et al., 2003b, AJ, 125, 2348

\( n(N) \) yields tighter HOD constraints for low luminosity galaxies, while \( w_p(r_p) \) is better at constraining parameters for high-luminosity galaxies. When all three statistics are used jointly, HOD constraints tighten significantly compared to the common case of only using \( w_p(r_p) \) and \( n_{gal} \).

(ii) The model struggles to jointly fit \( w_p(r_p) \), \( n(N) \), and \( n_{gal} \), demonstrating the power of combining multiple clustering statistics for ruling out models. Adopting a different halo definition does not make a big difference (though \( M_{vir} \) is slightly preferred over \( M_{200b} \)), but changing the cosmological model does. When adopting the LasDamas cosmology, the model is ruled out at the \( 3 \sigma \) level when tested against either luminosity sample. However, when adopting the Planck cosmology, the model is consistent with the clustering of low-luminosity galaxies and exhibits \( 2.3 \sigma \) tension with the clustering of high-luminosity galaxies.