Image and Video Dehazing by Regularized Optimization

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This dissertation is dedicated to my loving parents Zhenglong He and Shunping Zhu.
Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Jiaxi He
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Abstract

This thesis investigates the image and video dehazing problem. Images and videos captured in hazy weather often yield low contrast and offer limited visibility due to the presence of haze in the atmosphere. Hazed images and videos, which suffer from biased colour contrast and poor visibility, unavoidably degrade the performance of various computer vision applications that require robust detection of image and video features, such as photometric analysis, object recognition and target tracking. Dehazing is a process of restoring the true appearance, i.e. recovering what the scene should have looked like on a clear day, by enhancing the colour contrast and sharpening the details.

This thesis presents a novel and systematic regularized optimization method for the image dehazing problem. The optimization is based on the optical model for haze images which is inherently bilinear in terms of the unknown haze-free image and light transmission distribution. By combining the unknown bilinear term as a single optimization variable, the dehazing problem is formulated into a linear optimization problem. The regularization functions for the optimization problem are further formulated, taking into account the image contrast dependency on the light transmission distribution and the piecewise smoothness of the light transmission distribution. To speed up the dehazing processing and based on the low frequency characteristic of the light transmission distribution, the discrete Haar wavelet transform is applied to derive a low-pass sub-band hazed image model with considerably reduced model dimension. Sub-band models can result in reduction of computational
workload of the optimization, leading to fast dehazing processing. As a result, the proposed regularized optimization algorithms for dehazing can be efficiently implemented and executed to provide fast and high quality image dehazing. Computational experiments have been carried out to demonstrate the advantages of our proposed algorithms over current state-of-the-art dehazing algorithms in haze removal and image reconstruction, as well as in computational efficiency.

The proposed regularized optimization method for static image dehazing problem is further developed and modified for video dehazing problems. A digital video consists of a sequence of image frames and is ordinarily characterized by strong temporal correlation between its adjacent image frames. In this thesis, the temporal coherence is enforced by yielding highly correlated transmission function values and highly consistent atmospheric light values between adjacent image frames of the video sequence. The correlation of light transmission distribution between adjacent frames is enhanced by introducing a coherence cost function into the regularization functions, while the consistency of the atmospheric light intensity is achieved by implementing an adaptive technique. In addition, multi-level discrete Haar wavelet transform is applied to reduce the computational workload and to achieve real-time dehazing processing. Experimental results demonstrate that the proposed dehazing method can efficiently restore a hazed video, without causing flickering artifacts, at low computational complexity.

To summarize, a novel and fast regularized convex optimization method for image and video dehazing is proposed in this work. Our dehazing algorithm outperforms current state-of-the-art algorithms for improved visual quality of the dehazed images and much faster processing speed.
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Nomenclature

Roman Symbols

0 A matrix of approximate dimension with all 0 as entries.

1 A matrix of appropriate dimension with all 1 as entries.

Greek Symbols

λ The wavelength of light.

ϕ The azimuth angle between the incident ray of light and the emanating ray of light, range \([0, 2\pi]\).

θ The elevation angle between the incident ray of light and the emanating ray of light, range \([0, \pi]\).

Superscripts

\(a, v, h, d\) Indicate the approximation (LL), horizontal (LH), vertical (HL) and diagonal (HH) sub-band blocks of the DHWT, respectively.

Subscripts
\( c = 1, 2, 3 \) \hspace{1cm} The colour index for RBG channels, where \( c = 1 \) represents the red channel, \( c = 2 \) represents the green channel, and \( c = 3 \) represents the red channel.

**Other Symbols**

\( \beta \) \hspace{1cm} The scattering coefficient represents the ability of the unit volume scattering medium to scatter flux of all wavelengths in all directions.

\( \beta_{sc}(\theta, \phi, \lambda) \) \hspace{1cm} The angular scattering function depending on the wavelength \( \lambda \) of incident light and the angles \((\theta, \phi)\) between the incident ray of light and the emanating ray of light.

\( \hat{I}_c, c = 1, 2, 3 \) \hspace{1cm} The single-level DHWT of \( \hat{I}_c \).

\( \hat{I}_{c,K}, c = 1, 2, 3 \) \hspace{1cm} The K-level DHWT of \( \hat{I}_c \).

\( \hat{J}_c, c = 1, 2, 3 \) \hspace{1cm} The single-level DHWT of \( \hat{J}_c \).

\( \hat{J}_{c,K}, c = 1, 2, 3 \) \hspace{1cm} The K-level DHWT of \( \hat{J}_c \).

\( \hat{t}_K \) \hspace{1cm} The low-pass sub-band distribution of the K-level DHWT of \( t \).

\( d \) \hspace{1cm} The distance map from the scene to the camera.

\( E \) \hspace{1cm} An identity matrix with appropriate dimensions

\( I_c, c = 1, 2, 3 \) \hspace{1cm} The RGB matrices of the hazed colour image.

\( J_c, c = 1, 2, 3 \) \hspace{1cm} The RGB matrices of the haze-free digital colour image.

\( P(\lambda) \) \hspace{1cm} The radiant flux/power of the incident light of wavelength \( \lambda \).

\( P_s(\theta, \phi, \lambda) \) \hspace{1cm} The radiant power of the light, of wavelength \( \lambda \), scattered from a unit volume medium in direction \((\theta, \phi)\).
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<td>$t$</td>
<td>The transmission distribution representing the portion of the light, not being scattered, illuminating on camera sensors.</td>
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<td>$W$</td>
<td>The orthogonal Haar matrix.</td>
</tr>
<tr>
<td>$A$</td>
<td>The intensity of environment illumination.</td>
</tr>
<tr>
<td>$a_c$, $c = 1, 2, 3$</td>
<td>The atmospheric light constant of the corresponding colour channel.</td>
</tr>
<tr>
<td>$d$</td>
<td>The distance from the object to the observer.</td>
</tr>
<tr>
<td>$I_{Ar}$</td>
<td>The intensity of the airlight.</td>
</tr>
<tr>
<td>$J$</td>
<td>The intensity of reflected light from the object.</td>
</tr>
<tr>
<td>$J_r$</td>
<td>The attenuated light intensity of the object received by the camera.</td>
</tr>
<tr>
<td>$P$</td>
<td>The total radiant power of the visible incident light.</td>
</tr>
<tr>
<td>$P(\theta)$</td>
<td>The radiant power of light scattered from a unit volume haze medium of all wavelengths in direction $\theta$.</td>
</tr>
<tr>
<td>$P_0$</td>
<td>The initial radiant power of the light source.</td>
</tr>
<tr>
<td>$P_A$</td>
<td>The radiant power of the environment illumination.</td>
</tr>
<tr>
<td>$P_d$</td>
<td>The radiant power of light after travelling a distance $d$ in the haze scattering medium.</td>
</tr>
<tr>
<td>$P_r$</td>
<td>The radiant power of light beam received by the image sensor.</td>
</tr>
<tr>
<td>$P_s$</td>
<td>The total radiant power of light scattered from the unit volume scattering medium in all directions.</td>
</tr>
<tr>
<td>$P_{As}$</td>
<td>The radiant power of the environment illumination scattered after a unit volume haze medium.</td>
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Acronyms / Abbreviations

ADMM The alternating direction method of multipliers.

CO The proposed dehazing method by regularized convex optimization without the DHWT.

CO-DHWT\(_K\) The proposed dehazing method by regularized convex optimization with multiple \(K\)–level DHWT.

COV Convex optimization for video dehazing.

COV-DHWT\(_K\) Convex optimization for video dehazing with \(K\)-level discrete Haar wavelet transform.

DCP Dark Channel Prior.

DHWT Discrete Haar Wavelet Transform.

MAE Mean absolute error.

MRF Markov Random Field.
Chapter 1

Introduction

1.1 Motivation and Objectives

When aerosol particles, such as dust, smoke, mist or fine water droplets, accumulate in the air, a haze phenomenon occurs. In hazy weather, many aerosol particles of significant size are suspended in the atmospheric medium. In recent decades, the haze phenomenon has occurred with increasing frequency and increasingly involves larger areas. Haze can be categorized into "dry haze" and "wet haze". Photographs of dry haze and wet haze respectively are presented in Fig.1.1 and Fig.1.2. The term wet haze is used to describe visibility-reducing aerosols of the wet type, such as mist, fog, and fine water droplets. Therefore, wet haze tends to be primarily a warm-season phenomenon. Dry haze is often caused by excessive amounts of pollutants resulting from combustion, for example, the large-scale usage of fire to clear land, and the uncontrolled expansion of palm oil and paper production is one of the main drivers of the dry haze phenomenon (see Fig.1.3).
1.1 Motivation and Objectives

Fig. 1.1 Image of dry haze.

Fig. 1.2 Image of wet haze.
1.1 Motivation and Objectives

Fig. 1.3 Causes of dry haze.

Due to the presence of haze particles, the quality of our daily photographs and videos is easily undermined. The haze particles not only scatter and attenuate the reflected radiance of the scene, but also scatter and add atmospheric light from the hazy medium to the camera sensor. The scattered atmospheric light, called airlight, veils the reflected radiance of the scene and leads to colour shifts in hazed images and videos. Therefore, haze in images and videos signifies a combination of two scattering effects: direct attenuation of the scene radiance and undesired airlight.

$$\text{Haze} = \text{Attenuation} + \text{Airlight}$$

Fig. 1.4 gives a comparison of a hazed photograph taken in bad weather and a haze-free photograph taken on a clear day. The scene shown in the hazed image is distorted due to the attenuation of the scene radiance and the addition of the airlight by suspended haze particles. It is also observed that, in the hazed photograph, both the colour contrast and the visibility are drastically altered or degraded.
1.1 Motivation and Objectives

Since the incident radiance of the scene to the camera is attenuated and the airlight is introduced, hazed images and videos often lack visual vividness, exhibit low colour contrast and consequently offer limited visibility of the scene contents. As a result, the objects or obstacles in the image and video are difficult to recognize and detect. This has become a key problem in many computer vision applications that require robust detection of image or video features, such as photometric analysis, terrain classification, object recognition, surveillance, target tracking, autonomous navigation, and camera-based driving assistance system (DAS). For instance, video recorded by a surveillance system in hazy conditions yields limited visibility, which would be a key problem for police when investigating a crime. In addition, the degraded visibility and lack of luminance in hazy weather pose a serious threat to the safety of drivers. These conditions increase the danger of vehicle collisions and are a major cause of injuries and fatalities on roads affected by haze. Therefore, camera-based DAS is one of the core technology trends in intelligent vehicles. Furthermore, in photographs
of our daily life, the presence of haze is an annoyance, because it significantly reduces the contrast and impairs the clarity of the photograph. Moreover, in satellite remote sensing and aerial photography, owing to the remote distance between the object and the camera sensor, photographs are much more easily degraded by hazy conditions, which may cause problems in post-aerial works. Therefore, recovering the visibility of images and videos taken in hazy weather is an area of great interest for researchers.

In respect of the hardware aspect of camera sensors, many super-telephoto lenses are designed to incorporate specific filters or coatings so as to enhance the colour contrast. However, these super-telephoto lenses are very expensive and bulky, and are not applicable in daily life. Therefore, the restoration of hazed images and videos has attracted increasing attention in the last few years. Dehazing is a process of restoring the true appearance, i.e. recovering how the scene would have looked on a clear day, by regulating the colour contrast and sharpening the details. However, dehazing from a single observation made in hazy weather remains challenging, due to the inherently bilinear ambiguity of the scene’s intensity and the light transmission function in the underlying image and video formation process.

To sum up, images and videos taken in hazy conditions exhibit poor visibility. Overcoming this visibility problem is important in a multitude of applications, such as surveillance systems, intelligent vehicles, satellite imaging, and outdoor object recognition systems.

### 1.2 Review of Image Dehazing

The general consideration in image dehazing is to regulate the contrast and sharpen details of the image, to restore the visibility of the image. To date, various algorithms have been proposed for dehazing.
At an early stage, image dehazing was carried out by using additional information. Schechner et al. proposed an approach to remove haze by taking two images through one polarizer at different orientations for the same scene [79, 84, 78]. Narasimhan and Nayar presented a dehazing technique using multiple images taken of the same scene under different weather conditions [67–69]. These algorithms could remove haze effectively, but required multiple images of the same scene under different conditions, and therefore had limited application to actual case. Other techniques [48, 65] did not require multiple input images, however, they demanded 3-D rough geometrical information, which limited their application. Later, the near-infrared technique was applied in image dehazing due to its stronger penetration capability than visible light [77, 30], but it requires a special camera and cannot cope with normal existing digital images. The applications of these methods are limited by their requirement of additional information.

Image dehazing methods were further developed to resolve the additional information requirements, and use a single hazed image as the only input for dehazing. The success of these methods lie in strong a priori assumptions. One of the most successful methods is the Dark Channel Prior (DCP) method by He et al. [39]. He et al. observed that the light transmission function can be approximately inferred from the image’s dark channel. Then, the Matting technique [50] was used to refine the light transmission function. Later, He et al. improved the algorithm by replacing the Matting technique with the Guided filter [38]. Although the dehazing theory of the DCP method was simple and robust, it involved low efficient pixel computing and over-correction when estimating the approximate transmission function. The DCP technique has been further adopted and improved in many studies [96, 55, 32, 19], but these methods inherited the low efficiency and over-correction drawbacks of the DCP algorithm.
1.2 Review of Image Dehazing

The Markov Random Field (MRF) approach is another widely used single image dehazing method and has made significant progress recently [29, 86, 71, 16, 91, 34, 64]. The MRF approach is based on probability analysis of the spatial and contextual dependency of physical phenomena [51]. In an image dehazing problem, the MRF technique is employed to model the hazy scene. Under the assumption that the local contrast of the haze-free image is much higher than that of the hazed image in general, Tan [86] developed a cost function within the framework of MRF to enhance the colour contrast of the hazed image. Although Tan’s algorithm recovered the details and structures of the hazy image, it yielded halo artifacts and over-saturation. Fattal [29] proposed a dehazing algorithm based on the assumption that the surface shading and the light transmission function are two uncorrelated fields. However, this assumption was not reliable when handling densely hazed images, and thus a Gaussian-MRF extrapolation technique was demanded to smooth the transmission function. Kartz et al. [71] modelled a hazy scene with a factorial MRF in which the scene albedo and depth are assumed to be statically independent. Then, an expectation maximum algorithm was implemented to jointly estimate the scene albedo and depth. Their method could generate quite compelling results, but suffered from distorted colours and significant halos. In the work [16] by Cataffa et al, a MRF model was proposed to obtain better results of road images with the flat road assumption. Wang et al. [91] incorporated MRF with a fusion technique to achieve sharp details at finer granularity, however, the time efficiency was low because of the application of the low efficient alternative optimization algorithm; in addition, their results were not visually satisfactory. To summarise, although some of these methods could produce compelling results with physically plausible MRF models, they suffered from considerable computational burden; some methods generated distorted colours and obvious halos due to their deficient MRF model.

Moreover, certain image dehazing methods restore visibility by focusing on manipulating the image’s colour contrast but taking no account of the formation mechanism for hazed
1.2 Review of Image Dehazing

images. These methods include ones which utilized the histogram equalization [53], Retinex [95, 100, 57] or image fusion technique [1, 2]. Liu et al. in [53] carried out colour contrast enhancement by implementing histogram equalization. Although the histogram equalization technique enhanced the colour contrast to some degree, since most of the additive airlight component remained unchanged, it is not suitable for the dehazing problem. Yang et al [95] proposed a variable filter Retinex algorithm for hazed image restoration. Their results exhibited enhanced colour contrast, but, lost colour fidelity. Ancuti et al. [1, 2] restored image visibility by a Laplacian pyramid fusion-based method which considered the pyramid contrast, contrast, saliency, and exposure features using a white-balanced image and a colour-corrected image. Then, an exposure fusion algorithm was utilized to obtain the final result. However, their results showed colour shifts as a result of the exposure processing. In general, these non-physics-based methods achieved enhanced colour contract, but generated colour distortion.

In addition, some image dehazing works studies have been on specific computer vision applications, such as an autonomous robot/vehicle vision system [35], video surveillance systems [54, 82, 99, 55], and aerial and remote sensing for applications such as land cover classification [56, 103] and even underwater dehazing [15, 19, 31, 81].

Recently, researchers have focused on fast image dehazing [87, 102, 46, 52, 49, 105]. Some algorithms [102, 52, 49] apply the DCP technique, in which inefficient pixel-operation is required, and as a result, they can hardly be regarded as fast dehazing methods. Tarel et al.’s method [87], based on the median filter, is not suitable for all hazed images because such a strong assumption would be violated in some cases. It is also computationally intensive when dealing with images with large dimensions. Moreover, it is not adaptive, as too many parameters need to be controlled in the approach. Zhu et al. [105, 104] proposed a fast dehazing algorithm by creating a linear model for modeling the scene depth of the hazed
1.3 Review of Video Dehazing

A digital video is composed of a sequence of image frames and is generally characterized by strong temporal coherence between its adjacent image frames. Significant progress has also been made in the video dehazing field.

Initially, fast image dehazing methods were naturally extended to video dehazing, and each frame in the hazed video was processed separately [98, 49, 107]. Yeh et al. [98] used the pixel-based DCP technique and bilateral filter to estimate the light transmission function, which could not achieve real-time processing due to its computing complexity. The video dehazing method by Kumari et al. [49] increased the processing speed at the cost of dehazing quality and its results were over-corrected, making the dehazing meaningless. Apart from the above drawbacks, these video dehazing methods broke temporal coherence when applying the dehazing method to each image frame in the video independently. However, according to [90], the human visualization system (HVS) is very sensitive to temporal inconsistencies presented in the video sequence, and flickering artifacts occur if the temporal coherence is disrupted.
Later, the video dehazing algorithms took into account the temporal coherence between adjacent image frames in the video. Kim et al. [45, 46] proposed a real-time video dehazing method based on the assumption that the adjacent image frames are temporally coherent when their transmission function values are similar. Zhang et al. [102] improved the spatial and temporal coherence between adjacent frames by establishing a MRF model on the light transmission function. However, their algorithm required high complexity in computing, because it used the information in at least three frames to estimate the transmission map of a frame, and therefore was not applicable for real-time video dehazing. In [83], Shin et al. raised the phenomenon that flickering artifacts occur not only by the change of transmission maps but also by a frequent change of atmospheric light values calculated from each frame in a video. Therefore, they proposed an adaptive temporal averaging approach to calculate the atmospheric light values. Most other video dehazing methods prevent the change of atmospheric light values by fixing the value throughout the video sequence [45, 46, 102, 18]. However, colour shifts and colour distortion may occur when the environment illumination changes dramatically.

1.4 Main Contributions

This thesis presents a novel and systematic regularized optimization method for the single image dehazing problem, which has advantages for both image reconstruction quality and processing speed. The formulation of the optimization problem is based on the optical image model. Because the optical image model contains bilinearly coupled haze-free image and light transmission distribution, with both coupled items unknown, the image dehazing problem is essentially a non-convex problem. Our proposed dehazing method is motivated by the work in [97], where the non-convex problem is transformed into a convex problem by formulating the bilinearly coupled term as a single term. This resolves the non-convex difficulty of the
original problem. Its solution can be effectively computed by standard convex optimization and is sufficient for straightforward reconstruction of a haze-free image. The regularization functions for the reformulated convex optimization problem are determined by taking into account the image contrast dependency on the light transmission function and the piecewise smoothness of the transmission function. To further improve computational efficiency and based on the a priori assumption that the haze spectrum in an image is relatively concentrated in the low frequency sub-band, the discrete Haar wavelet transform (DHWT) is applied to derive a low-pass sub-band hazed image model with considerably reduced dimensions. Based on the low-pass and smoothness characteristics of the light transmission distribution, a piecewise constant assumption of the light transmission distribution is introduced. Using this assumption, it is shown that solving the dehazing problem of the low-pass sub-band hazed image model is sufficient for the solution of the original dehazing problem. The DHWT low-pass sub-band image model results in reduction of the computational workload of dehazing optimization, leading to fast dehazing processing.

The proposed regularized optimization method for static image dehazing is then further developed and modified for video dehazing. The temporal coherence of the light transmission distribution between every two adjacent video frames is enforced by introducing a coherence cost function into the regularization functions, and the consistency of the atmospheric light intensity is enforced by employing an improved adaptive technique. Multi-level DHWT is also applied to increase the computing speed so as to achieve real-time processing.

Our proposed regularized optimization method has several advantages over existing image and video dehazing methods reported in the literature, which can be summarised as follows:
1.5 Outline of Thesis

- The model-based formulation of the regularized optimization is a systematic and deterministic approach to a feasible solution for the reconstruction of haze-free images and videos which meets the optical image model.

- The optimization problem is computational efficient and can be readily implemented with standard computational procedures and software.

- The regularization terms of the optimization formulation provide flexibility in computing the dehazing solution by incorporating a priori knowledge of the hazed image or video, which guides the computation to a meaningful solution.

- The application of the regularized optimization to the proposed DHWT low-pass sub-band hazed image model enables significant reductions in problem dimensions and computational workload.

These advantages result in high quality image and video reconstruction and fast dehazing processing, which are demonstrated by the computational dehazing results of a number of images and videos.

Notations used in this paper are as follows. Bold-faced letters are reserved for matrices and vectors. The $|\cdot|$ represents the $l_1$ norm and $\|\cdot\|^2_F$ represents the Frobenius norm. $\odot$ is the Hadamard (elementwise) product, $\otimes$ is the Kronecker product and $\oslash$ is the elementwise quotient.

## 1.5 Outline of Thesis

The remainder of the thesis is organized as follows:
In Chapter 2, the optical model for the formation of a hazed image is given. We start from the optical light scattering model of the haze particle. Then, the direct attenuation model, which formulates the scattering of the scene’s radiant intensity, and the airlight model, which represents the scattering of the atmospheric light, i.e. environment illumination, are given. Accordingly, the optical model for hazed images can be formulated by combining the direct attenuation model and the airlight model.

In Chapter 3, the foundation of convex optimization is laid. First, the general mathematical models for the convex optimization and the regularized optimization problems are described, followed by some specific optimization algorithms.

In Chapter 4, the discrete Haar wavelet transform (DHWT) is studied with the purpose of decomposing the hazed image into several sub-bands of different frequencies and increasing calculation speed. Graphic analysis of the DHWT of the hazed image is given, which demonstrates that the haze spectrum is concentrated in the low frequency sub-band. By applying the DHWT to the optical hazed image model, and based on the piecewise smoothness characteristic of the light transmission distribution, a low-pass sub-band hazed image model is derived, which leads to significant reduction of computational workload in the dehazing process.

In Chapter 5, a novel regularized optimization method to solve the image dehazing problem is proposed. The formulation of the optimization problem is based on the optical image model. The transformation of the bilinearly coupled unknown image and light transmission terms into one single variable leads to a linear and convex model, which provides an efficient means of optimization. The regularization functions for the convex optimization problem are formulated considering the image contrast dependency on the light transmission distribution and the piecewise smoothness of the light transmission distribution. The algorithm for the proposed regularized optimization is then presented. To verify the
effectiveness and practicability of the proposed dehazing method, we test it on various real world hazy images and synthetic images. Experimental results have demonstrated that our proposed algorithm is capable of removing haze effectively and it outperforms other state-of-the-art algorithms. In addition, quantitative assessments, which compute the newly visible edges after resotation, the mean ratios of the gradients at visible edges and the percentages of saturation colour pixels, are given.

In Chapter 6, a regularized optimization method to solve the video dehazing problem is proposed. It is observed that, without taking the temporal coherence of the video sequence into account, severe flickering artifacts may occur. A coherence cost is added to our regularization functions to enforce the temporal coherence. In addition, the atmospheric light value is estimated adaptively in order to further enhance the temporal coherence and eliminate the flickering artifacts. Moreover, multi-level discrete Haar wavelet transform is implemented to improve efficiency. Comparative experiments are made with other video dehazing algorithms and the experimental results show that our proposed algorithm can achieve high quality real-time dehazed video without flickering artifacts.

In Chapter 7, we conclude the thesis and highlight some potential future research directions.
Chapter 2

Optical Model for Hazed Images

Computer vision is a discipline that provides innovative technologies for a wide spectrum of applications, including feature detection, surveillance, target tracking and telecommunications, by making the best use of visual data, i.e. images and videos. Nowadays, cameras are ubiquitous and the number of images and videos generated is overwhelming [66]. In recent years, automatic image and video processing has attracted extensive research interest.

Although computer vision systems have achieved great success in controlled and structured indoor environments, they have limitations when deployed outdoors, especially in hazy weather, because most computer vision systems are designed for clear weather images and videos and they assume the input is the unaltered scene radiance. Suspended haze particles in the atmosphere can scatter, refract and absorb light, and consequently lead to poor visibility, low contrast and colour offset in the images and videos captured in hazy conditions (see Fig.2.1(a)). In order to successfully deploy computer vision systems outdoors, a robust dehazing process for hazed images and videos is essential. Fig.2.1(b) gives an example of the dehazed result for the hazed image in Fig.2.1(a). The first step in dehazing process is to investigate and model the physical process that generates a hazed image and video.
2.1 Fundamental Mechanisms of Light Scattering

In this chapter, the optical model for the formation of hazed images is proposed. First, the fundamentals of light scattering are studied based on several simplifying assumptions. Next, the direct attenuation model and the airlight model, which are based upon the light scattering mechanism, are derived. According to the linear characteristic of light propagation, the light intensity received by a camera is the result of the addition of direct attenuation and airlight. Consequently, a mathematical model describing the optical formation of hazed digital RGB colour images in computer vision is established.

2.1 Fundamental Mechanisms of Light Scattering

Light scattering occurs when light interacts with suspended particles (haze aerosol) in the atmosphere. As light propagates from a scene point to the camera sensor through a hazy medium, its key characteristics, i.e. light power and intensity, are altered, principally due to scattering by suspended haze particles in the medium. Generally, the exact nature of light scattering is highly complex and depends on the wavelengths and directions of the incident
2.1 Fundamental Mechanisms of Light Scattering

light, as well as the distributions and sizes of the particles that constitute the hazy medium [41, 17]. This thesis is based on the following four assumptions:

**Assumption 1:** The haze particles are approximately viewed as independent scatterers whose scattered intensities do not interfere with each other.

Independent scattering denotes the light scattering by independent scatters and it is also termed incoherent scattering. Independent scattering means that the total radiative properties can be obtained by using a summation procedure based on the individual/single particle properties.

This independent scattering approximation is acceptable, provided that the distances between particles are large compared to the particle dimensions [43]. According to [59], in haze and dense haze, the separation between scattering particles is large and is usually more than three times longer than the radius of the particle. Therefore, the haze scattering can be regarded as independent.

According to [26, 47, 23, 66], the portion of the radiant flux/power of light that is independently scattered from a unit haze medium can be represented by a positive angular scattering function \( \beta_{sc}(\theta, \phi, \lambda) \in \mathbb{R}_+ \). The variable \( \theta \) is the elevation angle between the incident ray of light and the emanating ray of light, ranging from 0 to \( \pi \); \( \phi \) is the azimuth angle between the incident ray and emanating ray, ranging from 0 to \( 2\pi \); and \( \lambda \) is the wavelength of the incident light. A diagram of the independent scattering from a unit volume haze medium and the geometry for angles \( \theta, \phi \) are shown in Fig.2.2. The transparent rectangular solid, with a unit length and a unit cross-section area, illustrates a unit volume of randomly-oriented suspended haze particles. The radiant power of the incident light of wavelength \( \lambda \) on the unit volume scattering medium is denoted by vector \( \mathbf{P}(\lambda) \), then the radiant power of the light, of wavelength \( \lambda \), scattered from the unit volume haze medium in
2.1 Fundamental Mechanisms of Light Scattering

Fig. 2.2 Scene radiance passes through a unit volume of randomly-oriented suspended haze particles.

\[ P_s(\theta, \phi, \lambda) = \beta_{sc}(\theta, \phi, \lambda)P(\lambda) \]  

Assumption 2: The angular scattering function \( \beta_{sc}(\theta, \phi, \lambda) \) is symmetric with respect to the incident ray of light.

Based on [47, 23, 63], the light is symmetrically scattered with respect to incident rays of light when the particles in the atmosphere are spherical or small, which means that the angular scattering function \( \beta_{sc}(\theta, \phi, \lambda) \) is symmetric about the line of propagation of the incident light and the \( \beta_{sc}(\theta, \phi, \lambda) \) is independent of the azimuth angle \( \phi \). This assumption is feasible for small particles, such as air molecules and fine water vapour, and spherical
particles, like haze, fog and small raindrops. Non-spherical particles, large with respect to \( \lambda \), are found in dust storms and in heavily polluted environments \[47\]. Accordingly, in our dehazing problem, the spherical characteristics of the suspended haze particles, that make up the hazy medium, make \textbf{Assumption 2} reasonable.

Therefore, it is sufficient to write \( \beta_{sc}(\theta, \lambda) \) as the \( \beta_{sc}(\theta, \phi, \lambda) \) in our study and thus further gives:

\[
P_s(\theta, \lambda) = \beta_{sc}(\theta, \lambda)P(\lambda).
\] (2.2)

\textbf{Assumption 3:} The angular scattering function \( \beta_{sc}(\theta, \lambda) \) is independent of the wavelength \( \lambda \).

Generally, light of different wavelengths is scattered differently by particles of different sizes. The wavelength selective behaviour of the light scattering is the cause of many beautiful effects. Among these light scattering effects are the blue of the sky and ocean, the colours of the sunset, and the appearance of a scene in haze.

To date, substantial theories have been developed for the scattering functions \( \beta_{sc}(\lambda, \theta) \) and their relations to the size of scattering particles \[41, 17, 22\]. Fig.2.3 shows several light scattering models of particles with different sizes. It is observed that the light scattering model is closely related to the ratio of particle size to the wavelength of the incident light. As the ratio of particle size to the light wavelength increases, the radiant power of the scattered light, in the forward direction, is augmented. Table 2.1 adapted from \[59, 70\] presents several different atmospheric particles and their associated radii. The scattering model for single particles of sizes much less than the wavelength \( \lambda \) is termed Rayleigh scattering and the scattering model by single particles of a size that is comparable to the wavelength (rather than much smaller or much larger) of the light is termed Mie scattering.
2.1 Fundamental Mechanisms of Light Scattering

Fig. 2.3 Light scattering patterns from haze particles with different sizes
2.1 Fundamental Mechanisms of Light Scattering

<table>
<thead>
<tr>
<th>Condition</th>
<th>Particle Type</th>
<th>Radius (µm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIR</td>
<td>Molecule</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>HAZE</td>
<td>Aerosol</td>
<td>$10^{-2.1}$</td>
</tr>
<tr>
<td>FOG</td>
<td>Water Droplet</td>
<td>1-10</td>
</tr>
<tr>
<td>CLOUD</td>
<td>Water Droplet</td>
<td>1-10</td>
</tr>
<tr>
<td>RAIN</td>
<td>Water Drop</td>
<td>$10^2$-$10^4$</td>
</tr>
</tbody>
</table>

Table 2.1 Weather conditions and associated particle types and sizes, adapted from [59, 70].

According to [59, 85], the angular scattering function for Rayleigh scattering would be:

$$\beta_{sc}(\theta, \lambda) = \left(\frac{2\pi}{\lambda}\right)^4 \frac{1 + \cos^2 \theta}{2}$$  \hspace{1cm} (2.3)

Rayleigh scattering refers to the light scattering from air molecules, and can be extended to scattering from particles up to about a tenth of the wavelength of the light. It can be seen from the above Equation (2.3) that Rayleigh scattering is strongly dependent upon the wavelengths

$$\beta_{sc}(\lambda) \propto \frac{1}{\lambda^4}. \hspace{1cm} (2.4)$$

The intensity of the Rayleigh scattered light increases rapidly as the wavelength decreases. This strong wavelength dependence enhances the short wavelengths and gives us the blue sky. However, the Rayleigh scattering model breaks down when the ratio of particle size to the wavelength of the incident light is larger than 10%.

In the case of particles with dimensions greater than one tenth of the light’s wavelength, the Mie scattering model can be employed to model the intensity of the scattered radiation from these particles [9]. The Mie angular scatter function written in normalized form is as below [42, 40]:

$$\beta_{sc}(\theta, \lambda) = \frac{1 - g^2}{4\pi(1 + g^2 - 2g \cos \theta)^{3/2}}$$  \hspace{1cm} (2.5)
where $g$ is the parameter decided by the property of the suspended particles and is varied to determine the majority of light to be scattered either forward or backward. If $g > 0$, Equation (2.5) is used to generate a primarily forward scattering function; if $g < 0$, Equation (2.5) would present a primarily backward scattering function; if $g = 0$, Equation (2.5) is an isotropic scattering function. In general, for haze aerosols, $g$ is set as 0.76, and the haze particles contribute forward scattering and the intensity of scattered light is larger in the forward direction than in the reverse direction.

It can be shown that the Mie scattering differs from Rayleigh scattering in several aspects: the first is that the Mie scattering is strongly directional and the scattered light prefers to go close to its original direction; the second is that the Mie scattering is roughly independent of wavelength, which means the Mie scattering is of equal strength for all wavelengths. In this limit, all wavelengths of visible light are scattered approximately identically in Mie scattering, and as a result, the colour of the Mie-scattered light appears to be whitish. For instance, the cloud, which is composed of fine water droplets that are of relatively larger size than the wavelengths in visible light, appears to be white or grey (thick cloud). The Mie scattering also gives us the bright white halos around the light source. Fig. 2.4 presents a comparison of Rayleigh and Mie scattering, where the left sky appearance is dominated by Mie scattering and the right sky appearance is dominated by Rayleigh scattering. The whitish colour of the sky is caused by light being scattered by atmospheric particles that are of comparable size to the light wavelength; while the blue colour of the sky is caused by light being scattered by air molecules which are much smaller than the light wavelength.

According to Table 2.1, as the haze aerosols are of comparable size to the wavelengths of visible light [390 – 700] nm, the light scattering of haze obeys the Mie scattering model rather than the Rayleigh model [74, 72, 13]. As a result, the haze in the image and video appears whitish. Since the visible light is scattered equally for all wavelengths by the haze aerosols,
the angular scatter function $\beta_{sc}(\theta, \lambda)$ is independent of the wavelength $\lambda$, and Equation (2.2) becomes:

$$P(\theta) = \beta_{sc}(\theta)P$$  \hspace{1cm} (2.6)

where, the scalar $P$ is the total radiant power of the visible light and $P(\theta)$ is the radiant power of light scattered from a unit volume haze medium of all wavelengths in direction $\theta$.

The total flux scattered in all directions by a unit volume of hazy medium is obtained by integrating the entire sphere:

$$\int P_s(\theta)d\Omega = \int \beta_{sc}(\theta)Pd\Omega$$

$$P_s(\theta) = \int_0^\pi \int_0^{2\pi} \beta_{sc}(\theta)P \sin \theta d\theta d\phi$$

$$P_s = 2\pi \int_0^\pi \beta_{sc}(\theta) \sin \theta d\theta P$$

$$P_s = \beta P$$  \hspace{1cm} (2.7)
where,

\[ \beta = 2\pi \int_0^\pi \beta_{sc}(\theta) \sin \theta \, d\theta \]  

(2.8)

is called the scattering coefficient and represents the ability of the unit volume to scatter flux of all wavelengths in all directions, and \( P_s \) is the total radiant power of light scattered from the volume.

**Assumption 4:** The scattering coefficient \( \beta \) is constant.

To satisfy this premise, we restrict the haze medium to being homogeneous. We also assume the observer is at or close to the ground level and is interested in objects at the ground level. In addition, the range of distance is limited to a few kilometers [70].

### 2.2 Optical Formation Process of Hazed Images

As mentioned in Chapter 1, the optical formation of hazy scenes is a combination of two components: direct attenuation and airlight. Both of the two components are caused by the light scattering mechanism. Direct attenuation is the result of the scattering of the scene’s reflected light, while airlight is caused by the scattering of atmospheric light, i.e., environment illumination.

#### 2.2.1 Direct Attenuation

We start from the first component, i.e. direct attenuation. The attenuation model describes the way that light is attenuated as it traverses from a scene point to the observer. Due to scattering by suspended aerosols, a fraction of light flux is removed from the incident beam
2.2 Optical Formation Process of Hazed Images

Fig. 2.5 Attenuation of a beam of light by suspended haze particles, adapted from [70].

...of light as it travels through the atmosphere. The direct attenuation causes the radiant power of a scene point to decrease as its distance from the camera increases.

Fig. 2.5 illustrates a beam of light projected on the haze medium. Assume that the beam of light has a unit cross-sectional area and the distance between the object and the camera is $d$. As the incident beam of light interacts with the suspended haze particles, the radiant power conveyed by the light beam decreases. If the haze medium has the scattering coefficient $\beta$ for a unit volume (a unit cross-section area and a unit length), then in a differential segment of the beam of length $dx$, where the beam radiant power is $P$, the power depletion from the beam is presented as [62]:

$$dP = -\beta P dx$$ (2.9)
By integrating through the whole path, i.e. from \( x = 0 \) to \( x = d \), the incident light beam with original radiant power \( P_0 \) after travelling a distance \( d \) through the medium is found as:

\[
\int_{P_0}^{P_d} \frac{dP}{P} = \beta \int_0^d dx
\]

\[
P_d = P_0 \exp^{-\beta d}
\]  (2.10)

Since we are interested in the intensity value of an image taken by the camera, the relationship between the radiant power and intensity is investigated. The light intensity or irradiance can be measured by the camera sensor and produces the intensity values as the image. Fig.2.6 gives a two-dimensional diagram of a camera lens forming an image of a distant object. From the diagram, the object’s reflected light depicted in brown colour passes the haze medium to the camera lens, and then an image is formed accordingly on the image sensor. Assume the radius of the lens is \( R_{\text{lens}} \). Suppose a small patch \( \delta_{\text{object}} \) of the object is imaged into the patch \( \delta_{\text{image}} \) on the image sensor. Both patches \( \delta_{\text{object}} \) and \( \delta_{\text{image}} \) are depicted in black colour and the area for them are respectively represented by \( A_{\text{object}} \) and \( A_{\text{image}} \). Consider the purple line from the centre of patch \( \delta_{\text{object}} \) to patch \( \delta_{\text{image}} \); \( \alpha \) is the angle between this purple line and the optical axis of the lens and \( \psi \) is the angle between the purple line and the normal of \( \delta_{\text{object}} \). The following mathematical analysis is adapted from the work by Cozman et al. [23] and is placed here for the integrity and completeness of the thesis.

Due to optical magnification and foreshortening, the area of the object patch \( A_{\text{object}} \) and the corresponding image patch \( A_{\text{image}} \), will not in general be equal. The ratio of \( A_{\text{object}} \) to \( A_{\text{image}} \) is determined by the distances of these patches from the lens. According to the characteristics of the convex lens in the camera, the solid angle of the cone of rays leading to the patch on the object from the lens is equal to the solid angle of the cone of rays leading to
the corresponding patch on the image sensor from the lens [92]. The two solid angles are calculated by the definition of the solid angle as the ratio of area to the square of the distance:

\[
\Omega_{\text{object}} = \frac{A_{\text{object}} \cos \phi}{(d_{\text{object}} / \cos \alpha)^2}
\]

\[
\Omega_{\text{image}} = \frac{A_{\text{image}} \cos \alpha}{(d_{\text{image}} / \cos \alpha)^2}
\]  

(2.11)

where, \(\Omega_{\text{object}}\) and \(\Omega_{\text{image}}\) are respectively the solid angles of patches \(\delta_{\text{object}}\) and \(\delta_{\text{image}}\) as seen from the lens. Since these two solid angles are equal, we can equate and solve for the ratio of their areas:

\[
\frac{A_{\text{object}} \cos \phi}{(d_{\text{object}} / \cos \alpha)^2} = \frac{A_{\text{image}} \cos \alpha}{(d_{\text{image}} / \cos \alpha)^2}.
\]  

(2.12)

We have:

\[
\frac{A_{\text{object}}}{A_{\text{image}}} = \frac{\cos \alpha}{\cos \psi} \left( \frac{d_{\text{object}}}{d_{\text{image}}} \right)^2
\]  

(2.13)
Similarly, the solid angles of the lens as seen from the patches $\delta_{\text{object}}$ and $\delta_{\text{image}}$ are written as:

\[
\Omega_{\text{lens-object}} = \frac{\pi R_{\text{lens}}^2 \cos \alpha}{(d_{\text{object}} / \cos \alpha)^2}
\]

\[
\Omega_{\text{lens-image}} = \frac{\pi R_{\text{lens}}^2 \cos \alpha}{(d_{\text{image}} / \cos \alpha)^2}
\]

(2.14)

Assume $P_0$ is the initial radiant power of reflected light from the patch $\delta_{\text{object}}$ and $P_{\text{len}}$ is the radiant power of light that is not scattered and passes through the lens. Based on Equation (2.10), we have the following relationship:

\[
P_{\text{len}} = P_0 \exp(-\beta d_{\text{object}}) \cos \psi \Omega_{\text{lens-object}}
\]

\[
= P_0 \exp(-\beta d_{\text{object}}) \cos \psi \frac{\pi R_{\text{lens}}^2 \cos \alpha}{(d_{\text{object}} / \cos \alpha)^2}
\]

(2.15)

In addition, assuming $P_r$ is the radiant power of light arriving on the image sensor, as seen from the image sensor side, the radiant power of light through the lens $P_{\text{len}}$ is:

\[
P_{\text{len}} = P_r \cos \alpha \Omega_{\text{lens-image}}
\]

\[
= P_r \cos \alpha \frac{\pi R_{\text{lens}}^2 \cos \alpha}{(d_{\text{image}} / \cos \alpha)^2}
\]

(2.16)

Suppose there is no power loss in the camera lens. Solving Equations (2.15) and (2.16), we obtain:

\[
P_0 \exp(-\beta d_{\text{object}}) \cos \psi \frac{\pi R_{\text{lens}}^2 \cos \alpha}{(d_{\text{object}} / \cos \alpha)^2} = P \cos \alpha \frac{\pi R_{\text{lens}}^2 \cos \alpha}{(d_{\text{image}} / \cos \alpha)^2}
\]

(2.17)
Moreover, it is noted that we have the following relationship between the light power and light intensity:

\[
\begin{align*}
P_0 &= J_{A_{\text{object}}} \\
Pr &= J_r A_{\text{image}}
\end{align*}
\]  

(2.18)

where, \(J\) and \(J_r\) are respectively the initial light intensity of the object and the light intensity measured from the image sensor.

Taking Equation (2.18) into Equation (2.17), we have:

\[
J_{A_{\text{object}}} \exp(-\beta d_{\text{object}}) \cos \psi \frac{\pi R_{\text{lens}}^2 \cos \alpha}{(d_{\text{object}} / \cos \alpha)^2} = J_{r A_{\text{image}}} \cos \alpha \frac{\pi R_{\text{lens}}^2 \cos \alpha}{(d_{\text{image}} / \cos \alpha)^2}
\]

(2.19)

Combining Equation (2.13) and (2.19), we obtain:

\[
J_r = J \exp^{-\beta d_{\text{object}}}
\]

(2.20)

where, \(\beta\) is the positive scattering coefficient, \(d_{\text{object}}\) is the distance from the object to the camera, and the expression \(\{\exp^{-\beta d_{\text{object}}}\}\) is called the light transmission function and represents the portion of the light that is not scattered and reaches the camera.

Consequently, the direct attenuation model is derived and the light intensity is attenuated exponentially with the distance.

### 2.2.2 Airlight

The airlight is caused by the scattering of the atmospheric light, i.e. environment illumination. That is, the illuminant environment behaves as a source of light, and this effect increases the power of the incident beam of light.
Fig. 2.7 Airlight formation from point light source.

Fig. 2.7 gives an example of the airlight that is formed from an isotropic point light source. As shown in Fig. 2.7, light rays may (a) directly transmit from the source; (b) be scattered by the suspended haze particles from source to viewer, leading to glows around the sources; (c) scatter from the source to the object’s surfact point, leading to reflected surface radiance [40]. The contribution of airlight is depicted in (b). However, outdoors, the environment illumination can have infinite sources, including direct sunlight, diffuse skylight and light reflected by the ground, and therefore cannot be viewed as an isotropic point light source. In this thesis, we make the simplifying assumption that the illumination received from the environment is uniform and constant. This assumption is reasonable for the general outdoors environment and can possibly be approximated in large indoor environments [23, 66].

Consider the example of airlight formation, as shown in Fig. 2.8. A ray of the environment illumination passes through a unit volume haze medium at distance $x$ from the camera. The black dot represents a unit volume of suspended haze particles in the atmosphere. $\gamma$ is the angle between the ray from the environment illumination and the perpendicular line; $\omega$ is the
angle between the scattered ray and the optical axis of the camera lens. Therefore, the angle $\theta$ between the incident ray of environment illumination on the unit volume and its emanating ray from the unit volume is:

$$\theta = \frac{\pi}{2} - \omega + \gamma.$$  \hfill (2.21)

Suppose that the radiant power of environment illumination in the atmosphere is represented by the constant $P_A$. Therefore, according to Equation (2.6), the power of the scattered ray from the unit volume in the direction $\theta$ at distance $x$ to the camera would be:

$$P_{As}(\theta,x) = P_A \beta_{sc}(\theta)$$ \hfill (2.22)
2.2 Optical Formation Process of Hazed Images

Then, based on Equation (2.20), after travelling the distance \( x \), the power of this ray received by the camera \( P_{Ar} \) is:

\[
P_{Ar}(\theta, x) = P_{As}(\theta, x) \exp^{-\beta x} = P_A \beta_{sc}(\theta) \exp^{-\beta x}
\]  

(2.23)

where, \( P_{Ar}(\theta, x) \) contributes the airlight. Therefore, the power of airlight received from a distance \( x \) to the camera is obtained by integrating over the solid angle \( d\Omega \):

\[
P_{Ar}(x) = \int P_A \beta_{sc}(\theta) \exp^{-\beta x} \, d\Omega
\]

\[
= P_A \exp^{-\beta x} \int \beta_{sc}(\theta) \, d\omega
\]  

(2.24)

Using the relationship in Equation (2.8), we have:

\[
P_{Ar}(x) = P_A \exp^{-\beta x} \beta
\]  

(2.25)

Then, the total airlight is the integral of environment illumination from the camera to the object point, as shown in Fig.2.9. By integrating the distance from \( x = 0 \) to \( x = d \), the total airlight can be achieved:

\[
P_{Ar} = \int_0^d P_A \exp^{-\beta x} \beta \, dx
\]

\[
= P_A (1 - \exp^{-\beta d}).
\]  

(2.26)

This shows that the power of airlight increases with the pathlength. Similar to direct attenuation, the relationship between the radiant power and intensity of the airlight was studied, and the airlight model is as follows:

\[
I_{Ar} = A(1 - \exp^{-\beta d})
\]  

(2.27)
2.2 Optical Formation Process of Hazed Images

Fig. 2.9 Airlight formation by integral of environment illumination from camera to object scene.

where $A$ is constant and represents the intensity of atmospheric light, and $I_{Ar}$ is the intensity of the received airlight.

2.2.3 Formation of Hazed Images

Due to the linear characteristics of light propagation, these two light scattering effects are additive. In this way, we obtain the well-known Koschmieder’s model [62, 37] by adding Equations (2.20) and (2.27):

$$I = J \exp^{-\beta d} + A(1 - \exp^{-\beta d})$$

(2.28)
where, $I$ is the received light intensity by the camera, $J$ is the scene’s reflected light intensity, i.e. the true radiance of the scene, $A$ is the intensity of the atmospheric light, and $\exp^{-\beta d}$ is the transmission value.

A diagram of the formation of a hazy scene is presented in Fig.2.10. The object’s direct transmission light is attenuated along its original course and is distributed to other directions when it passes through the hazy medium. In addition, airlight scattered from all other directions is added. While the scene radiance decreases with pathlength $d$ and airlight increases with pathlength, it is easily observed that a distant object tends to vanish by the effect of the airlight.
2.3 Optical Image Model

In computer vision and computer graphics, however, the optical model used in Equation (2.28) cannot precisely describe the model for a hazed digital RGB colour image. Therefore, in this thesis, the optical model of hazed image is formulated as below.

Let \( J_c, I_c \in \mathbb{R}_+^{M \times N} \) denote matrices of haze-free and hazed digital RGB colour images, with non-negative entries denoted by \( J_c(m,n) \) and \( I_c(m,n) \), respectively, where \( c = 1, 2, 3 \) is the colour index for the RGB channels, \( M \) and \( N \) are positive integers, and \( m, n \in \Omega \) are indices of the 2-dimensional \( M \times N \) index set \( \Omega \). Let \( t \in \mathbb{R}_+^{M \times N} \) denote the matrix of the light transmission function with non-negative entries denoted by \( t(m,n) \), and it has the following expression:

\[
t = \exp^{-\beta d},
\]

(2.29)

where, \( d \in \mathbb{R}_+^{M \times N} \) is the distance map from the objectives to the camera with non-negative entries denoted by \( d(m,n) \), and \( \beta \) is the scattering coefficient depending on the hazy medium. Both \( d \) and \( \beta \) are positive, which implies that \( t \) is bounded elementwise by 0 and 1 matrices, i.e.

\[
0 \prec t \preceq 1,
\]

(2.30)

where, \( \prec (\preceq) \) denotes the elementwise operation of \( < (\leq) \) on matrices, \( 0 \) and \( 1 \) are respectively the matrices of appropriate dimensions with all 0 and all 1 as entries.

Then the optical image model used in this thesis is formulated as:

\[
I_c = J_c \odot t + a_c(1 - t), \quad c = 1, 2, 3,
\]

(2.31)

where, each \( a_c \) is the atmospheric light constant of the corresponding colour channel, \( t \in \mathbb{R}_+^{M \times N} \) is the transmission distribution representing the portion of the light not being scattered,
illuminating the camera sensors, and $\odot$ denotes the elementwise multiplication operation. The hazy scene appearance $I_c$ is the result of the attenuated image intensity $J_c$ through the scattering transmission path, together with the scattered transmission atmospheric light. Given $I_c$ and with $J_c$, $a_c$, $c = 1, 2, 3$, and $t$ generally unknown, the objective of image dehazing is to estimate $J_c$, as well as $a_c$ and $t$, $c = 1, 2, 3$, in order to reconstruct the composed haze-free colour image $J = J_1 \oplus J_2 \oplus J_3$. Since the video is composed of a sequence of image frames, the above model is also applicable to the formation of hazed videos.

Based on Equation (2.29), once the light transmission function $t$ is achieved, the distance map $d$ is available as a byproduct. The scene distance information $d$ can be utilized for other applications, such as image refocusing and novel view synthesis[21, 31].

\section*{2.4 Conclusion}

In this chapter, the optical formation of hazed images has been investigated. The fundamental model for light scattering has been given based on several simplifying assumptions. Thereupon, the direct attenuation model, which formulates the scattering of scene radiance, and the airlight model, which denotes the scattering of atmospheric light, were formulated. It has been demonstrated that direct attenuation causes scene radiance to decrease with pathlength and the airlight model causes the airlight intensity to increase with pathlength. By adding these two models, the light intensity received by the observer is achieved. Finally, the mathematical model for the optical formation of hazed digital RGB colour images has been derived.
Chapter 3

Fundamentals of Convex Optimization

3.1 Introduction

This chapter discusses the convex optimization theory, which is the backbone theoretical approach we use to achieve our dehazing objectives. Mathematical optimization is usually regarded as the selection of the best solution from all feasible solutions. Convex optimization, a sub-field of optimization, studies the problem of minimizing convex functions over convex constraints. The convexity property can make optimization much easier than the general case, in the sense that, any locally optimal point is globally optimal. Furthermore, with recent improvements in optimization theory and numerical algorithms, such as ADMM and Split Bregman, convex optimization is nearly as straightforward as linear programming, and accordingly, the globally optimal solution can be achieved efficiently and reliably. Based on [11], convex optimization can also be used to solve non-convex problems by either casting or converting the non-convex problem into a convex one. The concept of recognizing or formulating the non-convex problem into a convex one, is the main idea of our dehazing method.
In this chapter, we start with the basic mathematical models for convex optimization and regularized convex optimization, followed by the optimality criterion. Next, algorithms for convex optimization are described, and the alternating direction method of multipliers (ADMM) algorithm and the Split Bregman method for regularized convex optimization are discussed in detail.

3.2 Mathematical Model

The standard form of an optimization problem is:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0, \quad i = 1, 2, \ldots, m. \\
& \quad h_i(x) = 0, \quad i = 1, 2, \ldots, p. \quad (3.1)
\end{align*}
\]

where, vector \( x \in \mathbb{R}^n \) is the optimization variable of the problem; function \( f : \mathbb{R}^n \to \mathbb{R} \) is known as the objective function or cost function; \( g_i : \mathbb{R}^n \to \mathbb{R}, \quad i = 1, \ldots, m, \) are the inequality constraint functions, and \( h_i : \mathbb{R}^n \to \mathbb{R}, \quad i = 1, 2, \ldots, p, \) are the equality constraints. If there is at least one feasible value that meets all the constraints, the problem is then called feasible. All of the feasible values constitute the feasible set. A point \( x^* \) is said to be optimal or a possible solution if the objective \( f(x^*) \) gives the minimum value among the constraint set. If there are no constraint functions \( g_i, h_i, \) (i.e., \( m = p = 0 \)), we say the problem is an unconstrained optimization problem.

A convex optimization problem is one in which the objective and constraint functions \( f, g, h \) are convex. To recognise whether the objective or constraints functions are convex, knowledge of convex set is required.
3.2 Mathematical Model

Fig. 3.1 Comparison of convex set and non-convex set.

3.2.1 Convex Set

A convex set is the region such that, for every two points within the region, the straight line that joins the two points is also completely within the region. Therefore, the boundary of a convex set is always a convex curve. Fig.3.1 illustrates a comparison between a convex set and a non-convex set. Objective functions and constraints, that operate over the convex set, have convexity. In other words, a function \( f : \mathbb{R}^n \to \mathbb{R} \) is defined as convex if the domain of \( f \) is a convex set and if for all \( x, y \in \mathbb{R}^n \) in the domain of \( f \) satisfy the following inequality equation:

\[
f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \quad 0 \leq \theta \leq 1 \tag{3.2}
\]

Geometrically speaking, this inequality means that a function is convex if the line joining any two points \((x, f(x))\) and \((y, f(y))\) on the graph of the function lies above the graph of \( f \), as shown in Fig.3.2. This property allows us to tell whether a function is convex by a line.
3.2 Mathematical Model

Fig. 3.2 Graph of convex function, and line segment between any two points on the graph of the function lies above or on the graph.

3.2.2 Convex Optimization

A convex optimization problem is an optimization problem in which the objective function and the inequality constraints are strictly convex, and the equality constraint functions are linear [11]. The standard form of a optimization problem is presented as:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0, \; i = 1, 2, \ldots, m, \quad h_i(x) = 0, \; i = 1, 2, \ldots, p. \\
\end{align*}
\]

(3.3)

where \( x \in \mathbb{R}^n \) is the optimization variable, and the objective functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), inequality constraint functions \( g_i : \mathbb{R}^n \rightarrow \mathbb{R} \), and equality constraint functions \( h_i : \mathbb{R}^n \rightarrow \mathbb{R} \) are convex and satisfy:

\[
\begin{align*}
& f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \\
& g_i(\alpha x + \beta y) \leq \alpha g_i(x) + \beta g_i(y) \\
& h_i(\alpha x + \beta y) \leq \alpha h_i(x) + \beta h_i(y)
\end{align*}
\]

(3.4)
for all $x, y \in \mathbb{R}^n$ and all $\alpha, \beta$ with $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$. The feasible set of the convex optimization problem $\mathcal{C}$, is defined as the intersection of the domain of all the constraint functions and the objective function:

$$
\mathcal{C} = \bigcap_{i=1}^{m} \text{dom}g_i \cap \bigcap_{i=1}^{p} \text{dom}h_i \cap \text{dom}f
$$

which is also a convex set.

$x^*$ is called optimal, or a solution of the problem (3.3), if it has the smallest objective value $f(x^*)$ among the feasible set $\mathcal{C}$.

### 3.2.3 Regularized Optimization

In the fields of computer vision and computer graphics, regularized optimization refers to a process of solving an ill-posed problem by introducing a regularization, i.e. additional information or constraints.

Consider a bi-criterion problem, the goal of which is to make the function $f_1(x)$ small, and minimize the function $f_2(y)$, with a linear constraint function $Ax + By = b$. This problem can be described as an optimization problem with two objectives, $f_1(x)$ and $f_2(y)$, as below:

$$
\begin{align*}
\text{minimize} & \quad (f_1(x), f_2(y)) \\
\text{subject to} & \quad Ax + By = b
\end{align*}
$$

where $x, y \in \mathbb{R}^n$ are the optimization variables, $A, B \in \mathbb{R}^{m \times n}$ are known matrices, $b \in \mathbb{R}^m$ is a known vector, and $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are objective functions.
The idea of regularized optimization is to solve the above bi-criterion problem by minimizing the weighted sum of the objectives and it gives the following optimization form:

\[
\begin{align*}
\text{minimize} & \quad f_1(x) + \lambda f_2(y) \\
\text{subject to} & \quad Ax + By = b
\end{align*}
\] (3.6)

where, \( \lambda > 0 \) is the weighting parameter and the functions \( f_1(x) \) and \( f_2(y) \) are known as the regularization functions or cost functions. As \( \lambda \) varies over \([0, \infty]\), we achieve a trade-off between the two regularizations. The regularization functions in Equation (3.6) penalize the large \( f_1(x) \) and \( f_2(y) \) as our prior knowledge and solve the bi-criterion problem of making both \( f_1(x) \) and \( f_2(y) \) small. Moreover, the problem (3.6) is a regularized convex optimization problem if both \( f_1 \) and \( f_2 \) are convex.

### 3.3 Optimization Algorithms

Computational algorithms, which are used to find the globally optimal values of convex optimization problems, are widely developed [7, 6, 14]. However, to date, there has been no such optimization algorithm that produces the best performance for all types of applications, and the choice of optimization methods depends on the particular application. According to [27], basic optimization algorithms include gradient descent [73], conjugate gradient [24], Newton’s algorithm [11] (quasi-Newton [44], Gauss-Newton [73], etc.), and interior-point methods [94]. In practice, the gradient descent method is a first order optimization algorithm and is used to deal with unconstrained optimization problems; and it can be extended to handle constraint functions by including a projection onto the set of constraints. In Newton’s method, solving an unconstrained or equality-constrained problem is reduced to solving a sequence of quadratic problems. Moreover, the interior-point is designed to solve the
3.3 Optimization Algorithms

inequality-constrained problems. In addition, many commercial optimization solvers and computational tools have been developed for different programming platforms for convex problems, such as CVX [12] in Matlab, CVXOPT [3] in Python, OPT++ [61] in C++, and JOptimizer in Java.

### 3.3.1 Optimality Criterion

Consider the convex optimization problem in Equation (3.3). Assume the convex objective function $f$ is continuously differentiable and assume the problem is feasible. Let $\mathcal{C}$ denote the feasible set. Therefore, for all $x, y \in \mathcal{C}$, we can derive the following inequality from (3.4):

$$
f(y) \geq f(x) + \nabla f(x)^T (y - x) \tag{3.7}
$$

where $\nabla f(x)$ is the gradient of function $f(x)$. Then, according to [11], $x^*$ is optimal if and only if $x^* \in \mathcal{C}$ and

$$
\nabla f(x^*)^T (y - x^*) \geq 0, \text{ for all } y \in \mathcal{C}. \tag{3.8}
$$

### 3.3.2 Unconstrained Convex Optimization Algorithm

Unconstrained convex optimization considers the problem of minimizing convex objective function without any restrictions, and the standard form for unconstrained convex optimization problem is as given below:

$$
\text{minimize } f(x) \tag{3.9}
$$

where $x \in \mathbb{R}^n$ is the optimization variable and the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex. Suppose the objective function $f$ is differentiable, and let $\mathcal{C}$ denote the feasible set, then $x^*$ is
optimal if and only if $x^* \in \mathcal{C}$ and

$$\nabla f(x^*) = 0. \quad (3.10)$$

**Proof:** Since $f$ is differentiable as our prior knowledge, so all $y$ sufficiently close to $x^*$ are feasible. Let us define $y = x^* - t \nabla f(x^*)$, where $t \in \mathbb{R}$ is a small and positive parameter. Then according to (3.8), we have

$$\nabla f(x)^T (y - x) = -t \| \nabla f(x) \|^2 \geq 0 \quad (3.11)$$

Therefore, we conclude that $\nabla f(x^*) = 0$ is the necessary and sufficient condition for $x^*$ to be optimal. Thus, solving the unconstrained minimization problem in (3.9) is the same as finding a solution of (3.10).

In a few cases, we can find a solution to the unconstrained convex problem by analytically solving the optimality equation (3.10). However, generally, the unconstrained problem should be solved by an iterative algorithm [20]. The iterative algorithm is a mathematical procedure that generates a sequence of improving approximate solutions for optimization problems. By this we mean an algorithm that computes a sequence of solutions $x^{(0)}, x^{(1)}, \ldots x^{(k)} \in \mathcal{C}$, with $x^{(k)} \to x^*$ as $k \to \infty$. The iterative algorithm generates the solution sequence $\{x^{(k)} \}$ according to

$$x^{(k+1)} = G^{(k)}(x^{(k)}), \quad k = 0, \ldots, n \quad (3.12)$$

where $G^{(k)} : \mathbb{R}^n \to \mathbb{R}^n$ is a certain function that may depend on $k$, and $x^{(0)}$ is the starting point. Then, the convergence of the generated sequence $\{x^{(k)} \}, k = 0, \ldots, n$, to some desirable point is of interest. A stationary iterative algorithm is obtained when $G^{(k)}$ does not depend on $k$, i.e.,

$$x^{(k+1)} = G(x^{(k)}). \quad (3.13)$$
This iterative algorithm is called descent method, for the reason that it produces a minimizing sequence in objective function

\[ f(x^{(k+1)}) < f(x^{(k)}), \] (3.14)

except when \( x^{(k)} \) is optimal. The general format of the iterative descent method is described as:

\[ x^{(k+1)} = x^{(k)} + \alpha^k d^k \] (3.15)

where \( \alpha^k \) is a positive stepsize used to ensure that the iteration makes progress towards the solution set of the corresponding problem and \( d^k \) is the descent direction.

A classical descent method is the gradient descent iteration [58, 93], which has the following form:

\[ x^{(k+1)} = x^{(k)} + \alpha^k d^k = x^{(k)} - \alpha^k \nabla f(x^{(k)}), \] (3.16)

where the \( d^k = -\nabla f(x^{(k)}) \) is the descent direction. The algorithm will eventually converge when the gradient is zero \( \nabla f(x) = 0 \), which corresponds to the optimality criterion. The convergence rate for the gradient descent algorithm is at least linear [11].

Another widely used iterative descent optimization algorithm is Newton’s method, in which the descent direction is formulated as:

\[ d^k = - (\nabla^2 f(x^{(k)}))^{-1} \nabla f(x^{(k)}), \] (3.17)

provided that \( \nabla^2 f(x^{(k)}) \) exists and is positive definite, and the iteration takes the form

\[ x^{(k+1)} = x^{(k)} - \alpha^k (\nabla^2 f(x^{(k)}))^{-1} \nabla f(x^{(k)}). \] (3.18)
Newton’s method requires much more information, and makes much better updates, and thus converges in fewer iterations. The convergence rate for Newton’s descent algorithm is quadratic.

### 3.3.3 Constrained Optimization Algorithm

Generally speaking, in optimization problems, the presence of constraint functions complicates the algorithmic solution, and limits the range of available algorithms. The most common way to solve the constrained problem is to try to eliminate constraints by using approximations of the corresponding indicator functions. In particular, the so-called penalty function method is used to replace the constraints by prescribing a high cost for their violation [20, 4, 25].

#### Penalty Function Method

The penalty function method resolves the constrained problem by approximating the constraints into an unconstrained problem structure. The key technique is to add penalty term, which produces a high cost for violation of constraints, to the objective function. The penalty function method can deal with both equality and inequality constraints. Consider the problem in (3.3):

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0, \quad i = 1, \ldots, p \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, m. \\
\end{align*}
\]  
(3.19)
3.3 Optimization Algorithms

The penalty function method reformulates the constraints as part of the objective function. The reformulated unconstrained objective function is described as follows:

\[
\text{minimize } P(x, \rho, \beta), \tag{3.20}
\]

where

\[
P(x, \rho, \beta) = f(x) + \sum_{i=1}^{m} \rho_i h_i^2(x) + \sum_{i=1}^{p} \beta_i g_i^2(x).
\]

The reformulated terms \( \sum_{i=1}^{m} \rho_i h_i^2(x) \) and \( \sum_{i=1}^{p} \beta_i g_i^2(x) \) are called the quadratic penalty functions or cost functions and the vectors \( \rho = (\rho_1, \ldots, \rho_m), \beta = (\beta_1, \ldots, \beta_m) \) are known as the penalty parameters with:

\[
\begin{align*}
\rho_i &> 0, & i = 1, \ldots, m \\
\beta_i &= 0, & \text{if } g_i(x) \leq 0 \\
\beta_i &> 0, & \text{if } g_i(x) > 0
\end{align*} \tag{3.21}
\]

In this manner, the problem is formulated as an unconstrained optimization problem. If the value of the penalty parameters \( \rho, \beta \) are made suitably large, the penalty functions will result in a heavy cost for any constraint violation; therefore, minimizing the reformulated objective function \( P(x, \rho, \beta) \) in (3.20) can yield a feasible solution for (3.19). Accordingly, the iterative algorithms for unconstrained minimization, as described in Section 3.3.2, can be employed. However, based on [25], large values of penalty parameters can cause instability and inefficiency when deriving the sequential solution. Thereby, the penalty parameters are set small initially and are incrementally increased as we derive the solution sequence \( \{x^{(k)}\}, k = 0, \ldots, n \), of the unconstrained problem \( P(x, \rho, \beta) \).

The disadvantage of the penalty function method is that the penalty parameters are technically hard to control and may cause instability when used. In addition, the penalty
function method only forces the constraints to be satisfied approximately but not completely. Alternatively, some computationally more robust methods for constrained optimization, such as the Augmented Lagrangian Method [8], have been developed based on the penalty function method.

Augmented Lagrange Method

The Augmented Lagrangian method is designed to solve convex optimization problems with equality constraint functions. Consider the following constrained optimization problem:

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad h_i(x) = 0, \quad i = 1, \ldots, m.
\end{align*}$$

(3.22)

where \( x \in \mathbb{R}^n \) is the optimization variable, and objective function \( f: \mathbb{R}^n \to \mathbb{R} \), and equality constraint functions \( h_i: \mathbb{R}^n \to \mathbb{R} \) are convex.

The augmented Lagrangian method formulates an unconstrained objective as follows:

$$\begin{align*}
\text{minimize} & \quad L_A(x, \lambda, \rho) = f(x) - \lambda^T h(x) + \sum_{i=1}^{m} \rho_i h_i^2(x)
\end{align*}$$

(3.23)

where \( \lambda \in \mathbb{R}^m \) is called the Lagrangian multiplier and \( \rho \in \mathbb{R}^m \) is the penalty parameter. Hence, algorithms for unconstrained convex optimization can be applied to the problem in (3.23). In addition, for the purpose of the algorithm’s stability and efficiency, parameters \( \lambda \) and \( \rho \) are small initially and are updated according to the following rule for each iteration.
3.3 Optimization Algorithms

[25]:

\[
\begin{align*}
\lambda_i^{k+1} & \leftarrow \lambda_i - 2\rho_i h_i(x_k) \\
\rho_i^{k+1} & \leftarrow 2\rho_i, \quad \text{if } \|\lambda_i^k - \lambda_i^{k-1}\|_2^2 < 0.5
\end{align*}
\] (3.24)

3.3.4 Regularized Convex Optimization

Since our proposed dehazing method is based on regularized convex optimization, the alternating direction method of multipliers (ADMM) [10] and the Split Bregman method [33], that solve regularized optimization problems by breaking or splitting them into smaller pieces, each of which are then easier to handle, are investigated.

Regularized Convex Optimization using ADMM

Consider the regularized optimization problem of two sets of variables \(x, y\), with separable objectives in (3.6):

\[
\begin{align*}
\text{minimize} & \quad f_1(x) + \lambda f_2(y) \\
\text{subject to} & \quad Ax + By = b
\end{align*}
\] (3.25)

where \(x, y \in \mathbb{R}^n\) are optimization variables, \(A, B \in \mathbb{R}^{m \times n}\) are known matrices, and \(b \in \mathbb{R}^m\) is a known vector. Suppose the cost functions \(f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R}\) are convex. Then, its augmented Lagrangian function is written as:

\[
L_A(x, y, \lambda) = f_1(x) + f_2(y) - \lambda^T (Ax + By - b) + \rho \|Ax + By - b\|_2^2
\] (3.26)

where \(\lambda \in \mathbb{R}^l\) is the Lagrangian multiplier and \(\rho > 0\) is the penalty parameter.
In contrast to the conventional Lagrangian method, ADMM takes advantage of the separable form of the objective function and minimizes $L_A(x, y, \lambda)$ with respect to $x$ and $y$ separately via a Gauss-Seidel type iteration. The ADMM iterative is given as follows:

$$
\begin{align*}
    x^{(k+1)} &\leftarrow \min_x L_A(x, y^{(k)}, \lambda^{(k)}) \\
y^{(k+1)} &\leftarrow \min_y L_A(x^{(k+1)}, y, \lambda^{(k)}) \\
\lambda^{(k+1)} &\leftarrow \lambda^{(k)} - 2\rho(Ax^{(k+1)} + By^{(k+1)} - b) \\
\rho^{(k+1)} &\leftarrow 2\rho^{(k)}, \quad \text{if } \|\lambda^{k} - \lambda^{k-1}\|_2^2 < 0.5
\end{align*}
$$

In the ADMM technique, the optimization problem is solved approximately by first solving for $x$ with $y$ fixed, and then solving for $y$ with $x$ fixed. The augmented Lagrangian method and the ADMM method are employed to solve our dehazing model.

### $l_1$-norm Regularized Convex Optimization using Split Bregman Method

The Split Bregman method has recently been an efficient tool to solve non-differentiable $l_1$ regularized convex problems. Assume an unconstrained optimization problem with the $l_1$ regularization

$$
\text{minimize } |f_1(x)| + f_2(x) \tag{3.27}
$$

where $x \in \mathbb{R}^n$ is the optimization variable, the $| \cdot |$ represents the $l_1$ norm, and the regularizations $|f_1(x)| : \mathbb{R}^n \to \mathbb{R}$ with $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $f_2(x) : \mathbb{R}^n \to \mathbb{R}$ are convex. The idea of the Split Bregman method is to decouple the $l_1$ term by introducing a splitting variable $z \in \mathbb{R}^n$, as shown in the following model:

$$
\begin{align*}
    &\text{minimize } |z| + f_2(x) \\
    &\text{subject to } z = f_1(x) \tag{3.28}
\end{align*}
$$
Then, the problem can be converted into an unconstrained function $L_B(x, z, b)$ by the Split Bregman method:

$$L_B(x, z, b) = |z| + f_2(x) + \frac{\mu}{2} ||z - f_1(x) - b||_F^2,$$

where $\mu$ is the penalty parameter and the $b \in \mathbb{R}^n$ is called the Bregman parameter, the value of which is updated through the Split Bregman iterative as follows:

$$\begin{align*}
x^{(k+1)} &= \min_x L_B(x, z^{(k)}, b^{(k)}) \\
z^{(k+1)} &= \min_z L_B(x^{(k+1)}, z, b^{(k)}) \\
b^{(k+1)} &= b^{(k)} + (f_1(x^{(k+1)}) - z^{(k+1)})
\end{align*}$$

The Split Bregman method converges very quickly when applied to certain types of regularized convex optimization problem, especially for problems containing an $l_1$ regularization term [33]. In our dehazing model, the $l_1$ regularization functions are processed using the Split Bregman method.

### 3.4 Conclusion

In this chapter, the optimization problem has been investigated. We started from the concept of convex set. Next, the mathematical models for convex optimization and regularized optimization were presented. The optimality criterion was then given and mainstream optimization algorithms were described. Finally, the ADMM and Split Bregman algorithms for regularized convex optimization algorithms were discussed in detail.
Chapter 4

Discrete Wavelet Transform of Digital Hazed Images

Wavelet transform is similar to Fourier transform in that it allows a target function over an interval to be represented in terms of an orthonormal function basis. However, the key advantage it has over Fourier transforms is the temporal resolution: it captures both frequency and location information.

The first reference to the wavelet transform goes back to the early twentieth century, and it is the Haar wavelet proposed by mathematician Alfred Haar [89]. To date, in the wavelet family, many other wavelets have been constructed and developed, such as Deubechies wavelets, Mexican hat wavelets and Morlet wavelets. Compared to other wavelets, the Haar wavelet is conceptually simpler and computationally cheaper. In addition, the Haar wavelet is exactly reversible without the edge effects which are a problem with many other wavelet transforms [28]. Owing to these advantages, the Haar wavelet is chosen in this thesis.
This chapter first investigates the fundamentals of the discrete Haar wavelet transform (DHWT) and further verifies the application of DHWT in image dehazing through graphic analysis. Then, based on the low-pass and smoothness characteristics of the light transmission distribution, a piecewise constant assumption for the light transmission distribution is introduced. Using this assumption, a low-pass sub-band hazed image model with considerably reduced dimension is derived by implementing the DHWT in the optical hazed image model. Moreover, it is mathematically demonstrated that solving the dehazing problem of the derived DHWT low-pass sub-band image model is sufficient for the solution of the original dehazing problem. The DHWT low-pass sub-band hazed model can result in reduction of the computational workload of the dehazing optimization and yield fast dehazing processing.

4.1 Fundamentals of Discrete Haar Wavelet Transform

In the discrete wavelet transform, every wavelet is implemented through their wavelet filters, low-pass and high-pass. That is, filters of different cut-off frequencies are used to analyse the signal. The signal passes through a series of high-pass filters to analyse its high frequencies, and it passes through a series of low-pass filters to analyse its low frequencies.

4.1.1 The 1-D Discrete Haar Wavelet Transform

The DHWT decomposes the signal into two sub-signals of half its length, where half is the running average by the low-pass filter and half is the running difference by the high-pass filter. Given two numbers \( a \) and \( b \), its DHWT is:

\[
(a, b) \xrightarrow{\text{DHWT}} ((b + a)/2, (b - a)/2)
\]  (4.1)
4.1 Fundamentals of Discrete Haar Wavelet Transform

The low-pass averaging filter $\tilde{h}$ and high-pass difference filter $\tilde{g}$ for DHWT can be respectively represented as:

$$\tilde{h} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \tilde{g} = \left(-\frac{1}{2}, \frac{1}{2}\right)$$  \hspace{1cm} (4.2)

Then, the so-called $2 \times 2$ Haar matrix associated with the DHWT is constructed as:

$$\tilde{W}_2 = \begin{bmatrix} \tilde{h} \\ \tilde{g} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \hspace{1cm} (4.3)$$

where the subscript 2 for $\tilde{W}$ indicates the dimension of the Haar matrix, in the context $\tilde{W}_2 \in \mathbb{R}^{2 \times 2}$.

For a signal $v = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^N$, where $N = 2n$, with $n$ being positive integers, a Haar matrix $\tilde{W}_N \in \mathbb{R}^{N \times N}$ is built to compute the DHWT of $v$ as follows:

$$\hat{v} = \tilde{W}_N v = \begin{bmatrix} \tilde{h}_N \\ \tilde{g}_N \end{bmatrix} = \begin{bmatrix} E \otimes \tilde{h} \\ E \otimes \tilde{g} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} (v_1 + v_2)/2 \\ (v_3 + v_4)/2 \\ \vdots \\ (v_{n-1} + v_n)/2 \\ (v_n - v_{n-1})/2 \end{bmatrix}. \hspace{1cm} (4.4)$$
where \( \hat{v} \in \mathbb{R}^n \) is the DHWT of \( v \), \( E \) is an identity matrix with proper dimensions and \( \otimes \) is the Kronecker product.

Accordingly, the inverse process from \( \hat{v} \) to recover \( v \) can also be written as a matrix product:

\[
v = \tilde{W}_N^{-1} \hat{v} = \begin{bmatrix}
1 & 0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & \ldots & \ldots \\
0 & 1 & 0 & 0 & -1 & 0 & \ldots & \ldots \\
0 & 1 & 0 & 0 & 1 & 0 & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots & -1 \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
(v_1 + v_2)/2 \\
(v_3 + v_4)/2 \\
\vdots \\
(v_{n-1} + v_n)/2 \\
(v_2 - v_1)/2 \\
(v_4 - v_3)/2 \\
\vdots \\
(v_n - v_{n-1})/2
\end{bmatrix} = \begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
v_8
\end{bmatrix}
\]

(4.5)

Based on [80], an orthogonal matrix \( U \) satisfies the following property:

\[
U^{-1} = U^T.
\]

(4.6)

In addition, it is observed from Equations (4.4) and (4.5) that,

\[
\tilde{W}_N^{-1} = 2 \tilde{W}_N^T,
\]

(4.7)

with \( \tilde{W}_N \) very close being an orthogonal matrix.
If we multiply $\hat{W}_N$ by $\sqrt{2}$, we then obtain an orthogonal matrix $W_N$:

$$W_N = \begin{bmatrix} H_N & E \otimes h \\ G_N & E \otimes g \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ldots & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ldots & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

where the orthogonal matrix $W_N$ is called the DHWT matrix with $W_N^{-1} = W_N^T$, and $h = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $g = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ being respectively the low-pass Haar filter and high-pass Haar filter. Note that the orthogonal matrix $W_N$ is used as the Haar matrix for the DHWT in this thesis.

The significance of having $W_N$ as an orthogonal matrix is that orthogonal matrices have condition number 1. Therefore, multiplying and dividing by them is a numerical operation that does not increase the (norm-wise) errors. Practically, in computer vision, processing a digital image with an orthogonal Haar matrix means that the energy or the total colour intensity contained in the image will not change.

### 4.1.2 The 2-D Discrete Haar Wavelet Transform

In the field of image processing, two-dimensional DHWT is required. The 2-D DHWT basis functions are separable and can be computed by the row-column method. Let $B \in \mathbb{R}_{+}^{M \times N}$ represent the matrix of a grayscale digital image, where $M = 2i$ and $N = 2j$, with $i, j$ being positive integers. The first step of image wavelet transform is to compute the 1-D DHWT of
each column of the image by

\[ W_M B, \]  

(4.9)

with \( W_M \in \mathbb{R}^{M \times M} \) being the DHWT matrix. In this manner, we obtain a new array with a structure taking the low frequency information in the top half while having the detailed information (i.e. high frequency information) in the bottom half. As the example presented in Fig.4.1, the well-known benchmark image "Lena" is shown on the left. Fig.4.1(b) interprets the result of the 1-D DHWT of each column of the image "Lena" and is composed of a coarse and scaled version of the original and details. In Fig.4.1(c), the structure pattern of Fig.4.1(b) is illustrated as a low frequency block and a high frequency block, denoted by L and H, respectively. Note that most of the high frequency coefficients are shown in grey colour, which corresponds to small values around zero.

Then, the 2-D DHWT of the digital image \( B \) is achieved by applying the 1-D DHWT to each row of the already vertically transformed image:

\[ \hat{B} = W_M B W_N^T \]  

(4.10)

where, \( \hat{B} \in \mathbb{R}^{M \times N} \) is the 2-D DHWT of the image \( B \), and \( W_N \in \mathbb{R}^{N \times N} \) is the DHWT matrix. Like the example shown in Fig.4.1, the image in Fig.4.1(d) is the 2-D DHWT of the input image "Lena" and its corresponding structure with four sub-band images is illustrated in Fig.4.1(e). The interpretations of the four sub-band blocks LL, HL, LH, HH are given as follows:

- **LL**: The approximation sub-band, also denoted by (A). Coefficients in this sub-band block are obtained by filtering the original image by the analysis low-pass filter \( h \) along the columns and then filtering along the corresponding rows with the low-pass filter \( h \) again. This sub-band block image, with one fourth the size of the input image,
4.1 Fundamentals of Discrete Haar Wavelet Transform

(a) Image "Lena", with size $512 \times 512$

(b) 1-D DHWT applied to the columns of "Lena"

(c) Structure diagram of the image in (b)

(d) 2-D DHWT applied to "Lena"

(e) Structure diagram of the image in (d)

Fig. 4.1 Discrete Haar wavelet transform of digital image "Lena"
represents the approximated/coarse version of the input image at half the resolution. In addition, most of the energy is concentrated in this low resolution version of the original image.

- **LH**: The horizontal sub-band, also denoted by (H). This sub-band block image is derived by filtering the original image along the columns with low-pass filter \( h \) followed by high-pass filter \( g \) along the corresponding rows. This sub-band image contains the detailed information of the input image in horizontal directions.

- **LV**: The vertical sub-band, also denoted by (V). This sub-band block image is derived by filtering the original image first along the columns with high-pass filter \( g \) followed by low-pass filter \( h \) along the corresponding rows. The detailed information of the input image in vertical directions can be found in this sub-band image.

- **HH**: The diagonal sub-band, also denoted by (D). This sub-band block image is achieved with the use of the analysis high-pass filter \( g \) along both the columns and rows. We can interpret this sub-band image as the area where we find edges of the input image in diagonal direction.

The 2-D DHWT can be further applied to the approximation sub-band image LL, recursively, in order to achieve a lower resolution sub-band image and further de-correlate neighbouring pixels of the input image. In Fig. 4.2, the images and their structure patterns after two, three, four levels of 2-D DHWT are presented. The figure shows that the multi-level 2-D DHWT obeys a pyramidal scheme and the sub-band blocks in higher decomposition levels have smaller size.

At a resolution depth of \( K \), an input image is decomposed into \( 3K + 1 \) sub-band blocks including \( LL_K, LH_K, HL_K, HH_K, ..., LH_1, HL_1, \) and \( HH_1 \), where \( LH_1 = LH, HL_1 = HL, \) and \( HH_1 = HH \), respectively. Among these sub-band blocks, the \( LL_K \) is known as the low-
Fig. 4.2 Diagram of multi-level 2-D DHWT
pass sub-band while the others are counted as high-pass sub-bands. The sub-band blocks $LL_K$, $LH_K$, $HL_K$, $HH_K$, contain the finest scale coefficients when $K = 1$, and the coefficients become coarser as $K$ increases. The $LL_K$ contains the coarsest coefficients.

Note that, in the remainders of the thesis, the term DHWT is used for two-dimensional DHWT.

### 4.2 Graphical Analysis of the DHWT of Hazed Images

Generally, the haze spectrum in an image is relatively concentrated in the low frequency sub-band because the haze caused by light scattering is evenly distributed in the atmosphere [96, 101, 106].

Fig.4.3 presents a comparison between the DHWT of a synthetic clear image and a synthetic hazed image. These synthetic images are from the database FRIDA [88], and they are constructed according to the optical hazed image model. Fig.4.3 (c) and Fig.4.3(d) respectively present the single-level DHWT of the clear image and the hazed image. A comparison of the approximation sub-band block LL of Fig.4.3 (c) and Fig.4.3(d) easily shown that the sub-band image in the LL sub-band block of the DHWT of the hazed image is much brighter than that of the clear image. The difference is the result of the presence of the haze. The low frequency characteristic of the haze causes it to concentrate in the LL sub-band block. Furthermore, by comparing the three high frequency sub-band blocks LH, HL and HH of the DHWT of the clear image and that of the hazed image, it is observed that the coefficients in the sub-band blocks LH, HL and HH of the DHWT of the hazed image are much smaller than those of the clear image, which means that the details in the hazed image are degraded and removed by the effect of haze.
Fig. 4.3 Comparison of DHWTs of synthetic clear image and synthetic hazed image
We further investigate the two-level DHWT of the clear image and the hazed image in Fig.4.3 (e) and Fig.4.3(f), respectively. The coefficients in the block LL\(^2\) of the two-level DHWT of the hazed image are much larger than the corresponding sub-band block of the clear image, while the coefficients in the high frequency sub-band blocks LH\(^2\), HL\(^2\) and HH\(^2\) of the two-level DHWT of the hazed image are decreased compared to those of the clear image. Therefore, in the two-level DHWT of the hazed image, the haze information is further compressed and intensively presented in the sub-band block LL\(^2\).

Inductively, at a resolution depth of \(K\), the haze spectrum is held in the coarsest sub-band block LL\(^K\). As a consequence, for an image with high resolution, we can alleviate the computational complexity by performing \(K\)-level 2-D DHWT. Then, the dehazing problem of the sub-band hazed image, with reduced dimension, in block LL\(^K\), is solved to remove the haze and enhance the colour contrast, while the coefficients in the high frequency sub-band blocks LH\(^K\), HL\(^K\), HH\(^K\), \ldots, LH\(^1\), HL\(^1\), and HH\(^1\) are enhanced to restore the edge information and sharpen the image.

Although we conclude that the frequency response of haze in an image is distributed within the low frequency sub-band, we cannot conclude that the model for the sub-band hazed image in the approximation block LL\(^K\) is the same as in the input hazed image. The models for the high frequency sub-band blocks LH\(^K\), HL\(^K\), HH\(^K\), \ldots, LL\(^1\), LH\(^1\), HL\(^1\), and HH\(^1\) are also required so that we can enhance the high frequency coefficients soundly.

### 4.3 The DHWT Low-pass Sub-band Image Model

In this section, we analyse the application of DHWT in dehazing problems from the mathematical perspective by applying the DHWT to the optical hazed image model and derive a low-pass sub-band image model to improve the processing speed.
4.3 The DHWT Low-pass Sub-band Image Model

Recall the optical model for hazed digital RGB colour image in Equation (2.31):

\[ I_c = J_c \odot t + a_c (1 - t), \quad c = 1, 2, 3, \]  

(4.11)

where \( c = 1, 2, 3 \) is the colour index for the RGB channels, \( J_c, I_c \in \mathbb{R}^{M \times N}_+ \) are the matrices of the haze-free and hazed digital RGB colour images, with non-negative entries denoted by \( J_c(m, n) \) and \( I_c(m, n) \), respectively, \( M \) and \( N \) being positive integers, and \( m, n \in \Omega \) are indices of the 2-dimensional \( M \times N \) index set, and \( a_c \) is the atmospheric light constant of the corresponding colour channel, and \( t \in \mathbb{R}^{M \times N}_+ \) is the matrix of transmission distribution with non-negative entries \( t(m, n) \).

4.3.1 Single-level DHWT Low-pass Sub-band Image Model

We start from the single-level DHWT. To simplify the complexity in mathematical derivation, we assume that \( M \) and \( N \) are multiples of 2, \( M = 2i \) and \( N = 2j \), with \( i, j \) being positive integers.

Specifically, \( t \) in homogeneous atmosphere, is represented by

\[ t = e^{-\beta d}, \]  

(4.12)

where \( d \in \mathbb{R}^{M \times N} \) is the distance map from the object to the camera and \( \beta \) is the scattering coefficient depending on the hazy medium. In general, pixels in each geometric local image patch share a similar depth value to constitute a region or object, while abrupt depth jumps of pixel values constitute the objects’ edges or regions’ boundaries. Therefore, it is reasonable to consider that the distance map \( d \) is piecewise constant for most images. Since \( t \) is a continuous map of \( d \) in (4.12), it is also piecewise constant. With this consideration, it is
assumed that the $M \times N$ dimensional distribution $t$ is 2-patch piecewise constant in the sense that
\[ t(2m+i, 2n+j) = t(2m, 2n), \] (4.13)
for $m \in [0, M/2)$, $n \in [0, N/2)$ and $i, j = 0, 1$. Using this assumption and the DHWT, the sub-band image model is presented in the following.

Let $W$ be the well-known DHWT matrix. The single-level DHWT of $I_c$ and $J_c$, $c = 1, 2, 3$, results in their transformed matrices with four $\frac{M}{2} \times \frac{N}{2}$ dimensional sub-band blocks, i.e.,
\[
\hat{I}_c = W^M \times I_c \times (W^N)^T = \begin{bmatrix} \hat{I}^a_c & \hat{I}^h_c \\ \hat{I}^v_c & \hat{I}^d_c \end{bmatrix}, \tag{4.14}
\]
\[
\hat{J}_c = W^M \times J_c \times (W^N)^T = \begin{bmatrix} \hat{J}^a_c & \hat{J}^h_c \\ \hat{J}^v_c & \hat{J}^d_c \end{bmatrix}, \tag{4.15}
\]
where $\times$ is the matrix multiplication and the superscripts $a$, $h$, $v$, and $d$, respectively, indicate the approximation (LL), horizontal (LH), vertical (HL) and diagonal (HH) sub-band. In addition, the superscripts $M$ and $N$ indicate the size dimension of matrix $W$, in the context that $W^M \in \mathbb{R}^{M \times M}$ and $W^N \in \mathbb{R}^{N \times N}$. Moreover, matrix $W$ has the following form:
\[
W^M = \begin{bmatrix} H^M \\ G^M \end{bmatrix}
\]
4.3 The DHWT Low-pass Sub-band Image Model

with,

\[ H^M = \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & \cdots & 0 & 0 \\ & \ddots & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{2} & \sqrt{2} \end{bmatrix}_{M \times M} \]

and

\[ G^M = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} & 0 & \cdots & 0 & 0 \\ & \ddots & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}_{M \times M} \]

If the light transmission distribution \( t \) is 2-patch piecewise constant, it can be verified that its single-level DHWT is

\[ \hat{t} = W^M \times t \times (W^N)^T = \begin{bmatrix} 2\hat{t}_1 & 0 \\ 0 & 0 \end{bmatrix}, \]

(4.16)

where \( \hat{t}_1 \) is the low-pass sub-band distribution of the DHWT of \( t \) with its non-negative entries satisfying

\[ t_1(m,n) = t(2m,2n), \quad m,n \in \Omega^a, \]

with \( \Omega^a \) being the 2-dimensional \( M_2 \times N_2 \) index set of the low-pass sub-band transmission distribution \( \hat{t}_1 \).
Using Equations (4.14), (4.15) and (4.16), the application of single-level DHWT to the optical model of the hazed image in (4.11) results in:

\[
\begin{bmatrix}
\hat{I}_a^c \\
\hat{I}_b^c \\
\hat{I}_v^c \\
\hat{I}_d^c
\end{bmatrix} =
\begin{bmatrix}
\hat{J}_a^c \odot \hat{t}_1 + 2a_c(1 - \hat{t}_1) & \hat{J}_b^c \odot \hat{t}_1 \\
\hat{J}_v^c \cdot \hat{t}_1 & \hat{J}_d^c \odot \hat{t}_1
\end{bmatrix}
\]

(4.17)

The low-pass sub-band block of the matrix equation in (4.17) presents a DHWT low-pass sub-band image model, with the reduced dimension \(\frac{M}{2} \times \frac{N}{2}\), as follows:

\[
\hat{I}_a^c = \hat{J}_a^c \odot \hat{t}_1 + a_c,1(1 - \hat{t}_1),
\hat{I}_b^c = 2a_c, 1 = 2a_c,
\hat{I}_v^c = \hat{J}_v^c \odot \hat{t}_1
\]

(4.18)

which is of exactly the same form as that of the original optical image in (4.11). The coefficients in the high frequency sub-band blocks HL, LH, and HH of the DHWT of the hazed image are degraded by the low-pass sub-band transmission distribution \(\hat{t}_1\) as:

\[
\hat{I}_a^c = \hat{J}_a^c \odot \hat{t}_1
\]

\[
\hat{I}_b^c = \hat{J}_b^c \odot \hat{t}_1
\]

(4.19)

\[
\hat{I}_d^c = \hat{J}_d^c \odot \hat{t}_1
\]

**Proof:** Application of the single-level DHWT to the optical model of a hazed image \(I_c\) leads to:

\[
\hat{I}_c = W^M \times L_c \times (W^N)^T = \begin{bmatrix} H^M \\ G^M \end{bmatrix} \times [J_c \odot \mathbf{t} + a_c(1 - \mathbf{t})] \times \begin{bmatrix} (H^N)^T \\ (G^N)^T \end{bmatrix}
\]

(4.20)
4.3 The DHWT Low-pass Sub-band Image Model

Considering the Haar wavelet basis matrix $H^M$ and $G^M$ as well as the assumption that $t$ is 2-patch piecewise constant, it can be easily obtained that:

$$
\begin{bmatrix}
H^M \\
G^M
\end{bmatrix} \times a_c \cdot (1 - t) \times \begin{bmatrix}
(H^N)^T \\
(G^N)^T
\end{bmatrix} = a_c \cdot \begin{bmatrix}
2 \cdot (1_{\frac{M}{2} \times \frac{N}{2}} - \hat{t}_1) \\
0_{\frac{M}{2} \times \frac{N}{2}} \\
0_{\frac{M}{2} \times \frac{N}{2}} \\
0_{\frac{M}{2} \times \frac{N}{2}}
\end{bmatrix}
$$

(4.21)

On the other hand, the assumption that $t$ is 2-patch piecewise constant can also lead to

$$
H^M (J_c \odot t)(H^N)^T (m, n) = \left( \frac{1}{4} \sum_{p,q \in \{0,1\}} J_c(2m - p, 2n - q) \right) \cdot t(2m, 2n)
$$

(4.22)

Note the fact that after DHWT, $\hat{J}_a^p$ is the blur or low-frequency component of image $J_c$, i.e.

$$
\hat{J}_a^p(m, n) := H^M_1 J_c (H^N_1)^T (m, n) = \frac{1}{4} \sum_{p,q \in \{0,1\}} J_c(2m - p, 2n - q).
$$

(4.23)

Therefore, by comparing the above two Equations (4.22) and (4.23), we have:

$$
H^M (J_c \odot t)(H^N)^T (m, n) = \hat{J}_a^p(m, n) \odot \hat{t}_1(m, n)
$$

i.e.

$$
H^M (J_c \odot t)(H^N)^T = \hat{J}_a^p \odot \hat{t}_1
$$

(4.24)
Similarly, there exist:

\[
\begin{align*}
H^M(J_c \odot t)(G^N)^T(m,n) &= \left( \frac{1}{4} \sum_{p,q \in \{0,1\}} (-1)^q J_c(2m - p, 2n - q) \right) \odot t(2m, 2n) \\
&= \hat{J}_c^v \odot \hat{t}_1(m,n) \tag{4.25}
\end{align*}
\]

\[
\begin{align*}
G^M(J_c \odot t)(H^N)^T(m,n) &= \left( \frac{1}{4} \sum_{p,q \in \{0,1\}} (-1)^p J_c(2m - p, 2n - q) \right) \odot t(2m, 2n) \\
&= \hat{J}_c^h \cdot \hat{t}_1(m,n) \tag{4.26}
\end{align*}
\]

\[
\begin{align*}
G^M(J_c \odot t)(G^N)^T(m,n) &= \left( \frac{1}{4} \sum_{p,q \in \{0,1\}} (-1)^{p+q} J_c(2m - p, 2n - q) \right) \odot t(2m, 2n) \\
&= \hat{J}_c^d \odot \hat{t}_1(m,n) \tag{4.27}
\end{align*}
\]

Obviously, Equations (4.21), (4.24), (4.25), (4.26) and (4.27) constitute the Equation (4.17). This completes the proof.

The DHWT low-pass sub-band image model (4.18) contains the information on the low-pass sub-band transmission distribution \( \hat{t}_1 \) and can be used for its estimation. As a consequence, by solving the dehazing problem of the low-pass sub-band hazed image model (4.18) with reduced dimensions and enhancing the coefficients of high frequency sub-bands following from (4.19), the DHWT of the dehazed image can be obtained. After an inverse DHWT, the solution for the original dehazing problem can be achieved. Further, the low-pass
4.3 The DHWT Low-pass Sub-band Image Model

Sub-band image model with reduced dimensions can result in significant reduction of the computational workload in dehazing processing.

4.3.2 Multi-level DHWT Low-pass Sub-band Image Model

With an extension of the $2^l$-patch piecewise constant assumption on the light transmission distribution $t$ to $2^l$-patch piecewise constant for $l \geq 2$, a multiple $l$-level sub-band image model can be derived by the $l$-level DHWT, providing further reduced model dimensions and further reduced computational workload for the dehazing process. The following Theorem presents the general formula for the multi-level DHWT on an optical hazed image model.

**Theorem:** Given a hazed digital RGB image $I_c \in \mathbb{R}^{M \times N}$, where $c = 1, 2, 3$, $M = 2^K i$, $N = 2^K j$, and $i, j, K \in \mathbb{Z}^+$, assume that the transmission function $t \in \mathbb{R}^{M \times N}$ is $2^K$-patch piecewise constant, in the sense that $t(2^K m, 2^K n) = t(2^K m - p, 2^K n - q)$, for all $m \in \{1, 2, \cdots, \frac{M}{2^K}\}$, $n \in \{1, 2, \cdots, \frac{N}{2^K}\}$, and $p, q \in \{1, \cdots, 2^K - 1\}$. Let $W_l^M \in \mathbb{R}^{M \times M}$ be the $l$-th level DHWT matrix, $l \in \{1, 2, \cdots, K\}$. The $K$-level DHWT of $I_c$ results in the following transformed
4.3 The DHWT Low-pass Sub-band Image Model

matrix with $K$-level sub-band blocks:

\[
\hat{I}_{c,K} := W^M_K \cdots W^M_2 W^M_1 I_c(W^N_1)^T (W^N_2)^T \cdots (W^N_K)^T
\]

\[
= \begin{bmatrix}
\hat{I}^a_{c,K} & \hat{I}^v_{c,K} \\
\hat{I}^h_{c,K} & \hat{I}^d_{c,K} \\
\hat{I}^h_{c,2} & \hat{I}^d_{c,2} \\
\hat{I}^h_{c,1} & \hat{I}^d_{c,1} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\hat{J}^a_{c,K} \odot \hat{i}_K + 2^K a_c (1 - \hat{i}_K) & \hat{J}^v_{c,K} \odot \hat{i}_K \\
\hat{J}^h_{c,K} \odot \hat{i}_K & \hat{J}^d_{c,K} \odot \hat{i}_K \\
\hat{J}^h_{c,2} \odot \hat{i}_2 & \hat{J}^d_{c,2} \odot \hat{i}_2 \\
\hat{J}^h_{c,1} \odot \hat{i}_1 & \hat{J}^d_{c,1} \odot \hat{i}_1 \\
\end{bmatrix}
\]

where \( \hat{I}^a_{c,K}, \hat{I}^a_{c,K} \) respectively, represent the low-pass sub-band blocks $LL_K$ of the DHWT of the hazed image and dehazed image; \( \hat{I}^h_{c,l}, \hat{I}^h_{c,l}, l = 1, \ldots, K \), respectively, represent the $l$-level horizontal sub-band blocks $LH_l$ of the DHWT of the hazed image and dehazed image; \( \hat{I}^v_{c,l}, \hat{I}^v_{c,l}, l = 1, \ldots, K \), respectively, represent the $l$-level vertical sub-band blocks $HL_l$ of the DHWT of the hazed image and dehazed image; and \( \hat{I}^d_{c,l}, \hat{I}^d_{c,l}, l = 1, \ldots, K \), respectively, represent the $l$-level diagonal sub-band blocks $HH_l$ of the DHWT of the hazed image and
4.3 The DHWT Low-pass Sub-band Image Model

dehazed image. Also, we have

\[
W^M_l = \begin{bmatrix}
H^M_l & 0_{M \times (2^{l-1} - 1)M} \\
G^M_l & E_{(2^{l-1} - 1)M \times (2^{l-1} - 1)M}
\end{bmatrix},
\]

(4.29)

with \(E\) being an identity matrix and its subscripts denoting the dimension, and

\[
H^M_l = \begin{bmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \cdots & 0 & 0 \\
& & \ddots & & & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{bmatrix}_{M \times M},
\]

(4.30)

and

\[
G^M_l = \begin{bmatrix}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \cdots & 0 & 0 \\
& & \ddots & & & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{bmatrix}_{M \times M},
\]

(4.31)

Moreover,

\[
\hat{J}^{a}_{c,l} = \begin{cases}
H^M_l J_c (H^N_1)^T, & l = 1 \\
H^M_l \hat{J}^{a}_{c,l-1} (H^N_1)^T, & l = 2, \cdots, K
\end{cases}
\]

(4.32)
4.3 The DHWT Low-pass Sub-band Image Model

\( \hat{J}^c_{e,l} = \begin{cases} 
H^M_1 J_e (G^N_1)^T, & l = 1 \\
H^M_1 \hat{J}^u_{e,l-1} (G^N_1)^T, & l = 2, \cdots, K 
\end{cases} \)  
(4.33)

\( \hat{J}^h_{e,l} = \begin{cases} 
G^M_1 J_e (H^N_1)^T, & l = 1 \\
G^M_1 \hat{J}^u_{e,l-1} (H^N_1)^T, & l = 2, \cdots, K 
\end{cases} \)  
(4.34)

\( \hat{J}^d_{e,l} = \begin{cases} 
G^M_1 J_e (G^N_1)^T, & l = 1 \\
G^M_1 \hat{J}^u_{e,l-1} (G^N_1)^T, & l = 2, \cdots, K 
\end{cases} \)  
(4.35)

and \( \hat{t}_l \in \mathbb{R}^{M^2 \times N^2} \), \( l = 1, \ldots, K \) is the \( l \)-level sub-band transmission distribution, with its non-negative entries:

\[ \hat{t}_l(m,n) = t(2^l m, 2^l n), \quad m, n \in \Omega^a_l, \quad l = 1, \ldots, K \]  
(4.36)

and \( \Omega^a_l \) being the 2-dimensional \( \frac{M}{2^l} \times \frac{N}{2^l} \) index set of the sub-band transmission distribution \( \hat{t}_l \).

**Proof:** Similarly to Equation (4.20), for higher level Haar wavelet transform \( l \in \{2, 3, \cdots, K\} \), there exists:

\[
\left[ \begin{array}{c} H^M_l \\ G^M_l \end{array} \right] \times 2^{l-1} a_c \cdot (1 - \hat{t}_{l-1}) \times \left[ \begin{array}{c} (H^N_l)^T \\ (G^N_l)^T \end{array} \right] = a_c \cdot 
\left[ \begin{array}{c} 2^l \cdot (1_{\frac{M}{2^l} \times \frac{N}{2^l}} - \hat{t}_l) \\ 0_{\frac{M}{2^l} \times \frac{N}{2^l}} \end{array} \right]
\]

\( \hat{t}_l \in \mathbb{R}^{M^2 \times N^2}, l = 1, \ldots, K \) is the \( l \)-level sub-band transmission distribution, with its non-negative entries:

\[ \hat{t}_l(m,n) = t(2^l m, 2^l n), \quad m, n \in \Omega^a_l, \quad l = 1, \ldots, K \]  
(4.36)

and \( \Omega^a_l \) being the 2-dimensional \( \frac{M}{2^l} \times \frac{N}{2^l} \) index set of the sub-band transmission distribution \( \hat{t}_l \).
4.3 The DHWT Low-pass Sub-band Image Model

Based on Equation (4.22), as well as the assumption that $t$ is $2^K$-patch piecewise constant, it can be derived that $\hat{J}_{ac,l-1}$, as the low-frequency components of $\hat{J}_{ac,l-1}$, satisfies the following equation:

$$H_M(l)(\hat{J}_{ac,l-1} \cdot \hat{t}_{l-1}) = \hat{J}_{ac,l-1} \cdot \hat{t}_l, \quad l = 2, \ldots, K$$  \tag{4.38}

and by induction, $\hat{J}_{ac,l}$, $\hat{J}_{ac,l}$, and $\hat{J}_{ac,l}$ are the vertical differences, the horizontal differences, and diagonal differences of matrix $J_{ac,l-1}$ for $l = 2, 3, \ldots, K$, respectively, which also satisfy

$$\begin{bmatrix} H_M(l) & G_M(l) \\ H_N(l) & G_N(l) \end{bmatrix} \begin{bmatrix} \hat{J}_{ac,l-1} \circ \hat{t}_{l-1} \\ \hat{J}_{ac,l-1} \end{bmatrix}^T = \begin{bmatrix} \hat{J}_{ac,l} \cdot \hat{t}_l \\ \hat{J}_{ac,l} \cdot \hat{t}_l \\ \hat{J}_{ac,l} \cdot \hat{t}_l \\ \hat{J}_{ac,l} \cdot \hat{t}_l \end{bmatrix}, l = 2, \ldots, K. \tag{4.39}$$

It is obvious that Equations (4.37) and (4.39) constitute Equation (4.28). This completes the proof.

To sum up, the multi-level DHWT can lead to a low-pass sub-band hazed image model with further reduced dimensions as follows:

$$\hat{I}_{ac,K} = \hat{J}_{ac,K} \circ \hat{t}_K + a_{c,K}(1 - \hat{t}_K), \quad a_{c,K} = 2^K a_c, \quad c = 1, 2, 3, \quad K \geq 1,$$  \tag{4.40}

with $\hat{J}_{ac,K}, \hat{J}_{ac,K}, \hat{t}_K \in \mathbb{R}^{M \times N}$. Moreover, the K-level DHWT sub-band hazed image model (4.40) is of exactly the same form as that of the original optical image in (4.11). Therefore, the proposed sub-band image model (4.40) allows us to find the solution $(\hat{J}_{ac,K}, \hat{t}_K), c = 1, 2, 3$ with $4^K$ times reduction of the computational workload of the original problem (4.11).

The $\hat{t}_K$, solved from the model (4.40), can be used to estimate other sub-band transmission distributions $\hat{t}_l$, $l = 1, \ldots, K$ through resizing. Fig.4.4 illustrates the resizing process for
4.3 The DHWT Low-pass Sub-band Image Model

(a) The resizing of \( \hat{t}_K \) to obtain \( \hat{t}_{K-1}, K > 1 \)

(b) The resizing of \( \hat{t}_K \) to obtain \( \hat{t}_l, K > l \geq 1 \)

(c) The resizing of \( \hat{t}_K \) to obtain original transmission function \( t, K \geq 1 \)

Fig. 4.4 Diagram of resizing procedures for sub-band transmission functions \( \hat{t}_l \).
4.4 Conclusion

In this chapter, the discrete Haar wavelet transform (DHWT) was studied. Graphical analysis of the DHWT of the hazed image showed that the coefficients in the low frequency sub-band have a high value due to the concentration of haze in the low frequency, while the coefficients containing details in the high frequency sub-band are degraded and weakened by the effect of haze. By applying the DHWT to the optical hazed image model and with the piecewise smoothness characteristics of the light transmission distribution, a DHWT low-pass sub-band hazed image model with reduced dimensions was derived. It was further verified that solving the dehazing problem of the sub-band hazed image model is sufficient for the solution of the original dehazing problem (4.28).
original dehazing problem, which results in significant reduction of computational workload of the dehazing process, leading to fast dehazing processing.
Chapter 5

Static Image Dehazing by Regularized Optimization

This chapter proposes a novel regularized optimization method to solve the image dehazing problem, which has advantages in both the image reconstruction quality and the processing speed.

Fig. 5.1 presents the block diagram of the proposed image dehazing method. Initially, the $K$-level DHWT is proposed to decompose the input hazed image $I_c, c = 1, 2, 3$ into a low-pass sub-band hazed image $\hat{I}_{c,K}^a$ and several high-pass sub-band blocks, $\hat{I}_{c,l}^v, \hat{I}_{c,l}^h$ and $\hat{I}_{c,l}^d, l = 1, \ldots, K$. As explained in Chapter 4, solving the dehazing problem of the low-pass sub-band hazed image $\hat{I}_{c,K}^a$ with reduced dimension is sufficient for the solution of the original dehazing problem. Therefore, with the known $\hat{I}_{c,K}^a$, a regularized optimization is proposed and formulated based on the DHWT low-pass sub-band image model in (4.40).

To solve the proposed regularized optimization problem, the atmospheric light values $a_c, c = 1, 2, 3$ are estimated in the first place. Because the sub-band image model (4.40) con-
Fig. 5.1 Block diagram of proposed static image dehazing algorithm

Input hazed image $I_{c}, c = 1, 2, 3$

K-th level DHWT

Low-pass sub-band hazy image $I_{c,K}^{a}$

Atmospheric light estimation

Linear formulation of the DHWT low-pass sub-band hazy image model in (4.39)

Formulate the regularization functions and construct our regularized optimization problem

Fast Optimization Algorithm

The low-pass sub-band transmission distributions $\hat{t}_{c,l}, l = 1, \ldots, K$

The low-pass sub-band block of the DHWT of the dehazed image $\hat{J}_{c,K}^{a}$

Achieve the high-pass sub-band matrices $\hat{J}_{c,l}^{v}, \hat{J}_{c,l}^{b}, \hat{J}_{c,l}^{d}$ of the DHWT of the dehazed image by the quotient of $\hat{J}_{c,l}^{a}, \hat{I}_{c,l}^{a}, \hat{I}_{c,l}^{d}$ and $\hat{t}_{c,l}, l = 1, \ldots, K$

Inverse DHWT

Output dehazed image $\hat{J}_{c}, c = 1, 2, 3$
5.1 Atmospheric Light Estimation

Contains a term of bilinearly coupled low-pass sub-band hazed image $\hat{J}^a_{c,K}$ and low-pass sub-band light transmission distribution $\hat{t}_K$, with both coupled items unknown, the dehazing problem of the sub-band image model (4.40) is essentially a non-convex problem. Our proposed dehazing method is motivated by the work in [97] where the non-convex problem is transformed into a convex problem by formulating the bilinearly coupled term as a whole single term. This resolves the non-convex difficulty of the original problem. The regularization functions for the proposed optimization problem are further formulated, considering the image contrast dependency on the light transmission distribution and the piecewise smoothness of the light transmission distribution. The dehazing solution $(\hat{J}^a_{c,K}, \hat{t}_K)$ for the low-pass sub-band image model can then be effectively computed by standard convex optimization, where $\hat{t}_K$ can be further used to estimate the other sub-band transmission distributions $\hat{t}_l$, $l = 1, \ldots, K$ by resizing. Then, the high-pass sub-band blocks $\hat{J}^v_{c,l}$, $\hat{J}^h_{c,l}$ and $\hat{J}^d_{c,l}$, of the DHWT of the dehazed image can be estimated following Equation (4.41). In consequence, the haze-free RGB colour image $J_c$, $c = 1, 2, 3$, can be recovered by implementing the inverse DHWT on $\hat{J}_c = \{\hat{J}^a_{c,k}, \hat{J}^v_{c,l}, \hat{J}^h_{c,l}, \hat{J}^d_{c,l}\}$, $l = 1, \ldots, K$.

5.1 Atmospheric Light Estimation

As we have discussed in Chapter 2, the light scattering of haze particles obeys Mie scattering, that is all wavelengths of visible light scatter approximately identically. Therefore, the haze presented in the image has a whitish appearance (see Fig.5.2). Based on this fact, it is common in existing image dehazing studies [87, 39, 68], for the atmospheric light constants $a_c$, $c = 1, 2, 3$, to be estimated in the most haze-opaque region.

In this thesis, the atmospheric light is estimated by selecting the minimal pixel values among all local patches $\omega$ of all pixels $(m, n)$ within the two-dimensional $M \times N$ index set.
5.2 Linear Formulation of the Sub-band Image Model

Let \( \hat{a}_c \) denote the estimate of \( a_c \). Through the following process, the brightest pixel is considered to be the estimate of \( a_c \), i.e.

\[
\hat{a}_c = \max_{m,n \in \Omega} \min_{k,l \in \omega(m,n)} I_c(k,l), \quad c = 1, 2, 3, \tag{5.1}
\]

where \( \omega(m,n) \) is a \( 3 \times 3 \) local window centered at \( (m,n) \). With this result, it is assumed in the rest of the thesis, that the estimates of \( a_c, c = 1, 2, 3 \), have been obtained and are used in the haze image models for further estimation of \( J_c \) and \( t \).

5.2 Linear Formulation of the Sub-band Image Model

The proposed regularized optimization for dehazing is based on the low dimensional DHWT sub-band model in (4.40) using the known low-pass sub-band block \( \hat{I}_{c,K} \) and the estimated
\[ \hat{a}_{c,K} = 2^K \hat{a}_c, \quad c = 1, 2, 3. \]
For the purpose of readability, rewrite the low-pass sub-band model:

\[ \hat{I}^a_{c,K} = \hat{J}^a_{c,K} \otimes \hat{t}_K + \hat{a}_{c,K} (1 - \hat{t}_K). \] (5.2)

With both \( \hat{J}^a_{c,K} \) and \( \hat{t}_K \) unknown and bilinearly coupled in (5.2), the solution for \( \hat{J}^a_{c,K} \) and \( \hat{t}_K \) is a typical non-convex problem. However, the full benefits of convex optimization, in contrast, only come when the problem is known ahead of time to be convex.

It is however observed that the bilinearly coupled \( \hat{J}^a_{c,K} \otimes \hat{t}_K \), as whole single term, is linear in the matrix equation (5.2). We introduce a variable:

\[ \hat{Q}^a_{c,K} = \hat{J}^a_{c,K} \otimes \hat{t}_K, \quad c = 1, 2, 3, \] (5.3)

where \( \hat{Q}^a_{c,K} \in \mathbb{R}^{M_{2K} \times N_{2K}}_+ \) with entries denoted by \( \hat{Q}^a_{c,K}(m,n), \ m, n \in \Omega^2_K \) and \( \Omega^2_K \) being an index set of dimensions \( \frac{M}{2^K} \times \frac{N}{2^K} \). In addition, as a matter of convenience, we define a known variable:

\[ \hat{Y}^a_c = \hat{I}^a_c - \hat{a}_{c,K} \hat{1}, \quad c = 1, 2, 3. \] (5.4)

Then, the sub-band image model (5.2) can be rewritten as:

\[ \hat{Y}^a_{c,K} = \hat{Q}^a_{c,K} - \hat{a}_{c,K} \hat{t}_K, \quad c = 1, 2, 3. \] (5.5)

The problems in (5.2) and (5.5) are equivalent in the sense that, if \((\hat{J}^{a*}_{c,K}, \hat{t}^*_K)\) is an solution to problem (5.2), then there exists \( \hat{Q}^{a*}_{c,K} = \hat{J}^{a*}_{c,K} \otimes \hat{t}^*_K \), such that \((\hat{Q}^{a*}_{c,K}, \hat{t}^*_K)\) is an solution to (5.5). Vice versa, if \((\hat{Q}^{a*}_{c,K}, \hat{t}^*_K)\) is an solution to (5.5), then there exists \( \hat{J}^{a*}_{c,K} \) with \( \hat{J}^{a*}_{c,K}(m,n) = \hat{Q}^{a*}_{c,K}(m,n)/\hat{t}^*_K(m,n) \) such that \((\hat{J}^{a*}_{c,K}, \hat{t}^*_K)\) is an solution to problem (5.2).

With the known \( \hat{Y}^a_{c,K} \) and the estimate \( \hat{a}_{c,K} \), to solve \( \hat{Q}^a_{c,K} \) and \( \hat{t}_K \) from (5.5) is a linear and, hence, convex problem. The solutions for \( \hat{Q}^a_{c,K} \) and \( \hat{t}_K \) are sufficient for further estimation.
of the high-pass wavelet transformed sub-band image blocks. If \( \hat{Q}_{c,K}^a \) and \( \hat{t}_K \) have been obtained, it follows from Equations (4.41), (5.3) and \( \hat{t}_K \succ 0 \), that the wavelet transformed sub-band image blocks can be estimated by:

\[
\hat{J}_{c,K}^a = \hat{Q}_{c,K}^a \odot \hat{t}_K, \\
\hat{J}_{c,l}^h = \hat{I}_{c,l}^h \odot \hat{t}_l, \\
\hat{J}_{c,l}^v = \hat{I}_{c,l}^v \odot \hat{t}_l, \\
\hat{J}_{c,l}^d = \hat{I}_{c,l}^d \odot \hat{t}_l,
\]

where \( c = 1, 2, 3 \), \( l = 1, \ldots, K \) and \( \odot \) denotes the elementwise division operation on matrices.

Then, the reconstruction of the haze-free image matrices \( J_c, c = 1, 2, 3 \), can be further solved by the inverse DHWT of \( \hat{J}_c \).

### 5.3 Regularized Optimization for Dehazing

It is further observed that the wavelet transformed sub-band image model (5.5), with linear matrix variables \( \hat{Q}_{c,K}^a \) and \( \hat{t}_K \), is highly under-determined that, for a given \( \hat{Y}_{c,K}^a \), there are infinitely many solutions for \( \hat{Q}_{c,K}^a \) and \( \hat{t}_K \). In order to find feasible and meaningful solutions for \( \hat{Q}_{c,K}^a \) and \( \hat{t}_K \), it is necessary and important to incorporate available knowledge and information about the image and haze conditions into appropriate constraints on \( \hat{Q}_{c,K}^a \) and \( \hat{t}_K \), in order to regulate their solutions to satisfactory values.
Based on the sub-band image model (5.5) which is linear in $\hat{t}^a_{c,k}$ and $\hat{Q}^a_{c,k}$, the general formulation of the proposed regularized convex optimization for dehazing is written as:

$$\min_{\hat{Q}^a_{c,k}, \hat{t}^a_{c,k}} R(\hat{t}^a_{c,k}, \hat{Q}^a_{c,k}, c = 1, 2, 3), \quad (5.10)$$

subject to

$$\hat{Y}^a_{c,k} - \hat{Q}^a_{c,k} + a_{c,k} \hat{t}^a_{c,k} = 0,$$

$$0 < \hat{t}^a_{c,k} \leq 1, \quad 0 \leq \hat{Q}^a_{c,k}, c = 1, 2, 3,$$

where $R(\hat{t}^a_{c,k}, \hat{Q}^a_{c,k}, c = 1, 2, 3)$ denotes a convex regularization function of $\hat{Q}^a_{c,k}$ and $\hat{t}^a_{c,k}$ to be selected.

For selection of the regularization function, it is noted that the mean squared contrast of $\hat{J}^a_{c,K}$ is given by [46]:

$$C_{ms} = \sum_{c=1,2,3; (m,n) \in \Omega_K^c} \frac{(\hat{J}^a_{c,K}(m,n) - \bar{J}^a_{c,K})^2}{N_{\Omega_K^c}} = \sum_{c=1,2,3; (m,n) \in \Omega_K^c} \frac{(\hat{I}^a_{c,K}(m,n) - \bar{I}^a_{c,K})^2}{(\hat{I}(m,n))^2 N_{\Omega_K^c}}, \quad (5.11)$$

where $\bar{J}^a_{c,K}, \bar{I}^a_{c,K}$ are the average pixel values of $\hat{J}^a_{c,K}$ and $\hat{I}^a_{c,K}$, respectively, and $N_{\Omega_K^c}$ is the total pixel number. Since the haze effect reduces the degree of contrast in images, the general idea of the image dehazing process is to enhance the level of image contrast. A higher value of $C_{ms}$ indicates the higher contrast of the image. The above equation (5.11) implies that the image contrast $C_{ms}$ is inversely proportional to the square of the transmission function $\hat{t}^a_{c,K}(m,n)$. Therefore, reducing the value of $(\hat{I}(m,n))^2$ can improve the contrast of the image. To implement this consideration, a $\|\hat{t}^a_{c,K}\|_F^2$ term is introduced into the regularization function to penalize the values of $(\hat{I}(m,n))^2$. 
5.3 Regularized Optimization for Dehazing

The sub-band image model (5.5) used for the proposed regularized optimization is primarily an elementwise equation of image pixels. Straightforward elementwise operations based on this image model may not well represent and reconstruct the dependency and connectivity information of image pixels with their adjacent neighbourhoods. As a result, it is important and necessary that the regularization function takes into account the dependency and connectivity properties of pixels. It has been earlier described that the light transmission distribution $t$ has low-pass and piecewise constant characteristics. Within a reasonable range, such characteristics can be further extended to it’s transformed sub-band distribution $\hat{t}_K$. To promote the low-pass and piecewise constant characteristics of $\hat{t}_K$, a popularly known total variation function term $\|\hat{t}_K\|_{TV}$ [76] is introduced into the regularization function, where $\|\cdot\|_{TV}$ denotes the total variation norm.

With the above considerations, the regularization function $R(\hat{t}_K, \hat{Q}_{a,c,K}, c = 1,2,3)$ is specified as:

$$R(\hat{t}_K) = \|\hat{t}_K\|_F^2 + \alpha \|\hat{t}_K\|_{TV},$$

(5.12)

where $\alpha$ is a positive weighting parameter. It is to be adjusted to balance the penalty weights on regularization terms to guide the optimization solution to satisfactory values.

As a result, the regularized convex optimization for image dehazing is formulated as:

$$\min_{\hat{t}_K} \quad \|\hat{t}_K\|_F^2 + \alpha \|\hat{t}_K\|_{TV}$$

s.t. $\hat{Y}_{c,K}^a - \hat{Q}_{c,K}^a + \hat{a}_{c,K} \hat{t}_K = 0,$

$0 < \hat{t}_K \leq 1, \quad 0 \leq \hat{Q}_{c,K}^a, \quad c = 1,2,3,$

Fig.5.3 illustrates the feasible domain for our proposed regularized optimization (5.13) and an optimal solution can be found on the blue line. The optimization solution $\hat{t}_K$ and $\hat{Q}_{c,K}^a$, $c = 1,2,3$, can enable direct computation of $\hat{J}_{c,K}^a$ and consequently $\hat{J}_c$, $c = 1,2,3$, following
5.3 Regularized Optimization for Dehazing

Fig. 5.3 Feasible domain of proposed regularized optimization for dehazing from (5.7)-(5.9). The reconstruction of the haze-free image matrices $J_c, c = 1, 2, 3$, can be further obtained by applying the inverse DHWT.

**Remark 1:** This $K$-level DHWT-based convex optimization formulation for image dehazing is called CO-DHWT$_K$ (convex optimization - $K$-level discrete Haar wavelet transform) in the remainder of this paper.

**Remark 2:** The conditions for the proposed CO-DHWT$_K$ for the transformed sub-band image model (5.2) are all directly applicable to the original hazed image model (2.31), which can result in the following convex optimization for image dehazing, without the DHWT but with larger image dimension:

$$\min_t \|t\|^2_F + \alpha \|t\|_{TV} \quad (5.14)$$

s.t. $Y_c - Q_c + a_c t = 0$, $0 < t \preceq 1$, $0 \preceq Q_c$, $c = 1, 2, 3$.

where $Q_c = J_c \odot t$ and $Y_c = Q_c - a_c 1$, $c = 1, 2, 3$. This convex optimization without the DHWT is called CO in the remainder of this paper.
5.4 Iterative Algorithm for CO

In this section, the algorithm for the proposed regularized optimization problem in (5.14) is given. The ADMM and Split Bregman technique, described in Chapter 3, are employed in our algorithm.

In order to improve the computational speed, the $t$ is extracted from the linear constraints $Y_c - Q_c + a_c t = 0, c = 1, 2, 3$, by dividing the constraints with their corresponding atmospheric light values $a_c, c = 1, 2, 3$, which formulates the following problem:

$$\minimize \|t\|_F^2 + \alpha \|t\|_{TV}$$
$$\text{subject to } t = \frac{Q_c}{a_c} - \frac{Y_c}{a_c}, \; c = 1, 2, 3. \quad (5.15)$$

Then, its augmented Lagrangian function is written as:

$$L_A(t, Q, \lambda, \mu) = \|t\|_F^2 + \alpha \|t\|_{TV} - \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda(m,n)(t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}) + \frac{\mu}{2} \|t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}\|_F^2,$$

where, $\lambda \in \mathbb{R}^{M \times N}$, with entries $\lambda(m,n)$, is the Lagrangian multiplier and $\mu > 0$ is the penalty parameter.

In this thesis, the anisotropic Total Variation norm [76] is employed, and the following unconstrained problem is constituted:

$$L_A(t, Q_c, \lambda, \mu) = \|t\|_F^2 + \alpha |\nabla_x t| + \alpha |\nabla_y t| - \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda(m,n)(t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c})$$
$$\quad + \frac{\mu}{2} \|t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}\|_F^2, \quad (5.17)$$
where, \( |\nabla_x| \) and \( |\nabla_y| \), respectively, denote the horizontal and vertical partial derivative operators, with \( \nabla = [\nabla_x, \nabla_y] \) being the gradient operator in the discrete setting. The \( l_1 \)-norms of the partial derivatives of transmission distribution, \( |\nabla_x t| \) and \( |\nabla_y t| \), which consist the anisotropic Total Variation, are dealt by the Split Bregman method. To apply the Split Bregman method, we first replace the derivatives \( \nabla_x t \) and \( \nabla_y t \), respectively.

This yields the constrained problem as:

\[
\text{minimize } \|t\|_F^2 + \alpha |d_x| + \alpha |d_y| - \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda(m,n)(t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}) + \frac{\mu}{2} \|t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}\|_F^2 \\
\text{subject to } d_x = \nabla_x t, \quad d_y = \nabla_y t
\]

Then, the constraints \( d_x = \nabla_x t \) and \( d_y = \nabla_y t \) are strictly enforced by applying the Split Bregman Method:

\[
\text{minimize}_{t, Q_c} \quad L(t, Q_c, \lambda, \mu, d_x, d_y, b_x, b_y) = \|t\|_F^2 + \alpha |d_x| + \alpha |d_y| \\
- \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda(m,n)(t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}) + \frac{\mu}{2} \|t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}\|_F^2 \\
+ \frac{\gamma}{2} \|d_x - \nabla_x t - b_x\|_F^2 + \frac{\gamma}{2} \|d_y - \nabla_y t - b_y\|_F^2 \quad (5.18)
\]

where, \( \gamma \) is the penalty parameter, and \( b_x \) and \( b_y \) are Bregman parameters.

To solve the problem (5.18), we use the ADMM technique and split (5.18) into smaller pieces of variables \( t \) and \( Q_c \), each of which are easier to minimize as follows:

\[
t^{(k+1)} = \min_t \quad L(t, Q_c^{(k)}, \lambda^{(k)}, \mu^{(k)}, d_x^{(k)}, d_y^{(k)}, b_x^{(k)}, b_y^{(k)}) \quad (5.19)
\]

\[
Q_c^{(k+1)} = \min_{Q_c} \quad L(t^{(k+1)}, Q_c, \lambda^{(k)}, \mu^{(k)}, d_x^{(k)}, d_y^{(k)}, b_x^{(k)}, b_y^{(k)}) \quad (5.20)
\]
The detailed sub-problem for (5.19) is,

\[ t^{(k+1)} = \min_{t} \|t\|_{F}^{2} - \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda(m,n)(t - \frac{Q_{c}^{(k)}}{a_{c}} + \frac{Y_{c}}{a_{c}}) + \frac{\mu}{2} \|t - \frac{Q_{c}^{(k)}}{a_{c}} + \frac{Y_{c}}{a_{c}}\|_{F}^{2} \]

\[ + \frac{\gamma}{2} \|d_{x}^{(k)} - \nabla x t - b_{x}^{(k)}\|_{F}^{2} + \frac{\gamma}{2} \|d_{y}^{(k)} - \nabla y t - b_{y}^{(k)}\|_{F}^{2} \]  

(5.21)

because \( t \) is positively defined, therefore we have the optimality criterion for it as follows:

\[ (2 - 2\lambda^{(k)} + \mu^{(k)} - \gamma\Delta)t^{(k+1)} = \]

\[ -\lambda^{(k)} \odot \left( \frac{Q_{c}^{(k)}}{a_{c}} - \frac{Y_{c}}{a_{c}} \right) + \mu \left( \frac{Q_{c}^{(k)}}{a_{c}} - \frac{Y_{c}}{a_{c}} \right) + \gamma \nabla x^{T}(d_{x}^{(k)} - b_{x}^{(k)}) + \gamma \nabla y^{T}(d_{y}^{(k)} - b_{y}^{(k)}) \].

(5.22)

where, \( \Delta = -(\nabla x^{T} + \nabla y^{T}) \). Since the above system is strictly diagonally dominant, the most natural choice is the Gauss-Seidel method [33]. The Gauss-Seidel solution to this problem can be written component-wise as:

\[ t^{(k+1)}(m,n) = \frac{\gamma}{2 - 2\lambda^{(k)} + \mu^{(k)} - \gamma\Delta}(t^{(k)}(m+1,n)) + t^{(k)}(m-1,n) + t^{(k)}(m,n+1) \]

\[ + t^{(k)}(m,n-1) + d_{x}^{(k)}(m-1,n) - d_{x}^{(k)}(m,n) + d_{y}^{(k)}(m,n-1) - d_{y}^{(k)}(m,n) \]

\[ - b_{x}^{(k)}(m-1,n) + b_{x}^{(k)}(m,n) - b_{y}^{(k)}(m,n-1) + b_{y}^{(k)}(m,n) \]

\[ + \frac{1}{2 - 2\lambda^{(k)} + \mu^{(k)} - \gamma\Delta}(-\lambda^{(k)} \odot \left( \frac{Q_{c}^{(k)}}{a_{c}} - \frac{Y_{c}}{a_{c}} \right) + \mu \left( \frac{Q_{c}^{(k)}}{a_{c}} - \frac{Y_{c}}{a_{c}} \right)) \]

(5.23)
The model in (5.20) requires us to solve the sub-problem as below:

\[
Q_c^{(k+1)} = \min_{Q_c} - \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda^{(k)}(m,n)(t^{(k+1)} - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}) \\
+ \frac{\mu}{2} \|t^{(k+1)} - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}\|_F^2
\]  
(5.24)

Then the condition for the solution of \(Q_c\) is:

\[
(\frac{\lambda}{a_c} - \frac{\mu}{a_c})Q_c^{(k+1)} = -\frac{\lambda}{a_c}(t^{(k+1)} + \frac{Y_c}{a_c}) + \frac{\mu}{a_c}(t^{(k+1)} + \frac{Y_c}{a_c})
\]  
(5.25)

Using the solvers in (5.25) and (5.23), our iterative algorithm is derived and is written as follows:

**Iterative Algorithm for Our Proposed Regularized Convex Optimization**

Initial: \(t^0 = 1\), \(Q_c^0 = I_c\), \(c = 1, 2, 3\), and \(d_x^0 = d_y^0 = b_x^0 = b_y^0 = 0\)

While \(\|t^{(k)} - t^{(k-1)}\|_F^2 \geq tol\)

\[
\begin{align*}
\text{Update } t^{(k+1)} &\text{ by (5.25)} \\
\text{Update } Q_c^{(k+1)} &\text{ by (5.23)} \\
\lambda^{(k+1)} &\text{ = } \lambda^{(k)} - \mu(t - \frac{Q_c}{a_c} + \frac{Y_c}{a_c}) \\
\mu^{(k+1)} &\text{ = } 2\mu^{(k)} \\
d_x^{(k+1)} &\text{ = shrink}(\nabla_x t^{(t+1)} + b_x^{(k)}, \frac{\alpha}{\gamma}) \\
d_y^{(k+1)} &\text{ = shrink}(\nabla_y t^{(t+1)} + b_y^{(k)}, \frac{\alpha}{\gamma}) \\
b_x^{(k+1)} &\text{ = } b_x^{(k)} + (\nabla_x t^{(k+1)} - d_x^{(k+1)}) \\
b_y^{(k+1)} &\text{ = } b_y^{(k)} + (\nabla_y t^{(k+1)} - d_y^{(k+1)})
\end{align*}
\]
with the shrinkage function:

\[
\text{shrink}(x, \eta) = \text{sign}(x) \times \max(|x| - \eta, 0)
\]  

(5.26)

Through element-wise division between \(Q_c\) and \(t\), the matrices of the dehazed RGB digital image \(J_c, c = 1, 2, 3\), are solved. It is further verified that the presented algorithm can be directly extended to \(K\)-level DHWT low-pass sub-band image model in (5.13).

### 5.5 Computational Results

A large set of hazed images, including synthetic images and real case images, were tested to evaluate the performance of our proposed regularized convex optimization algorithm, in comparison with the state-of-the-art algorithms, including that of He et al. [38], Tarel and Hautiere [87], Fattal [29], Meng et al. [60], Wang and Fan [91], Nishino et al. [71], Zhu et al. [105] and Berman et al. [5]. Among these, the computations of the algorithms of Tarel et al., He et al., Meng et al., Zhu et al., and Berman et al. used the Matlab codes provided by the authors. Performance comparisons with the other algorithms used the results presented in the corresponding publications and the authors’ websites. The proposed regularized convex optimization algorithms CO, CO-DHWT\(_K\) were coded with Matlab. The Matlab code of the proposed algorithms and other state of the art algorithms were executed on a HP-Z420 workstation with a 3.30 GHz Intel E5-1660 CPU without using parallel processing in computations.
5.5 Computational Results

5.5.1 Synthetic Images

Unlike other computer graphics problems, e.g. denoising, there is no direct quantitative assessment technique for the image dehazing results for the lack of reference images for benchmarking. It has been suggested [36] that dehazing performance can be compared between images of a scene with and without haze. Following this suggestion, synthetic images with uniformly distributed haze, together with their synthetic ground true images, from the database FRIDA2 [88] were adopted to test our proposed CO.

Fig.5.4 presents the dehazing results of CO for synthetic images, compared with those of the histogram equalization approach, Tarel et al.’s algorithm [87], He et al.’s algorithm [39] and Zhu et al.’s algorithm [105]. Dehazing performance was assessed by how far the visible area extends in the restored images. Since the haze was very dense in the synthetic hazed images, the weighting parameter $\alpha$ of our proposed dehazing algorithm CO in (5.13) was set relatively small, $\alpha = 0.5$, so as to comparatively increase the weighting of the regularization $\|t\|_F^2$. Furthermore, to prevent over-saturation in the near field, we added a constraint $J_c > S \cdot 1$, that is $Q_c \geq S \cdot t$, to the CO (5.13), where, $C = 30$ is a threshold value relevant to the brightness of the input synthetic images.

It is observed from Fig.5.4 that Tarel et al.’s algorithm [87], He et al.’s algorithm [39], and Zhu et al.’s algorithm [105] cannot recover the hidden items in remote areas. Although the results of the histogram equalization approach exhibit certain remote scenes, they fail to present vivid colours. In contrast, our proposed algorithm outperforms all the other algorithms and offers better visibility and colour enhancement in remote areas.
Fig. 5.4 Comparative results for synthetic hazed image.
Table 5.1 Comparison of MAEs of different algorithms for synthetic images

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>MAEs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nothing</td>
<td>68.28</td>
</tr>
<tr>
<td>Histogram equalization</td>
<td>41.67</td>
</tr>
<tr>
<td>Tarel et al.'s algorithm [87]</td>
<td>49.01</td>
</tr>
<tr>
<td>He et al.'s algorithm [39]</td>
<td>40.12</td>
</tr>
<tr>
<td>Zhu et al.'s algorithm [105]</td>
<td>45.63</td>
</tr>
<tr>
<td>The proposed algorithm [87]</td>
<td>32.08</td>
</tr>
</tbody>
</table>

To further demonstrate the performance of the proposed dehazing algorithm CO, the mean absolute errors (MAEs) between the synthetic ground true image and the dehazed images of our algorithm, the histogram equalization approach and the algorithms in [87, 39, 105] are listed in Table 5.1. It is easily observed that our algorithm yields the least MAE.

### 5.5.2 Real Images

To verify the practical feasibility and effectiveness, the proposed regularized convex optimization algorithms at different levels of DHWT, i.e. CO, CO-DHWT\(_1\), CO-DHWT\(_2\), were applied to different real captured hazed images. The hazed images used for the dehazing objective were the "Canyon", "Manor", "House", "Desk", "Mountain", "River", and "Hill" images. These images were processed by the proposed CO, CO-DHWT\(_1\) and CO-DHWT\(_2\), algorithms and the dehazing results are presented, respectively, in Fig.5.5-Fig.5.13.

For comparison purposes, the original hazed images and the images dehazed by the state-of-the-art algorithms of He et al. [38], Tarel and Hautiere [87], Meng et al. [60], Wang and Fan [91], Nishino et al. [71], Zhu et al. [105] and Berman et al. [5] are also shown in Fig.5.5 -Fig.5.13. Discussion of the dehazing results and performance comparison of these algorithms are provided in the following section.
5.5 Comparative results for the "Canyon" image.

(a) Hazy input

(b) Tarel et al. [87]

(c) He et al. [38]

(d) Meng et al. [60]

(e) Berman et al. [5]

(f) CO, $\lambda_1 = 1, \lambda_2 = 1$

(g) CO-DHWT$_1$, $\lambda_1 = 1, \lambda_2 = 1$

(h) CO-DHWT$_2$, $\lambda_1 = 1, \lambda_2 = 1$
Fig. 5.5 presents the dehazing results of the CO, CO-DHWT\textsubscript{1} and CO-DHWT\textsubscript{2} algorithms for the "Canyon" image, compared with those of the algorithms presented in [87, 38, 60, 5]. The dehazing results produced by the CO, CO-DHWT\textsubscript{1} and CO-DHWT\textsubscript{2} algorithms present more faithful and balanced contrast and colour reconstruction over the whole image. In comparison, there are fringe artifacts in the result of Tarel et al.'s algorithm [87]. An over-saturated effect is produced in He et al.'s [38] result, because its statistical observation tends to underestimate the transmission function. Over-saturated artifacts are also shown in Meng et al.'s [60] result in the upper cloud part of the image, though it attempts to improve from He et al.'s algorithm by including a boundary constraint for restoring bright colour and limiting over-saturation. The dehazing result of Berman et al.'s algorithm [5] also shows over-correction similar to that of He et al. and Meng et al..

Fig. 5.6 shows the dehazing results of the CO, CO-DHWT\textsubscript{1} and CO-DHWT\textsubscript{2} algorithms compared with those of [38, 60, 105, 5] for the "Desk" image. The algorithms of He et al. [38] and Berman et al. [5] produce obvious colour over-correction and saturation in the dehazing results, especially on the objects at the lower left corner of the image. A halo effect around the pink hexagonal object can also be seen in the results of He et al.'s [38] and Berman et al.'s [5] algorithms. This effect may be caused by the DCP technique used in [38] and the hazy-line technique used in [5], which are pixel-wise based without taking into account continuities between adjacent pixels. The colour over-correction and halo effects, similar to the results of He et al.'s [38] and Berman et al.'s [5] algorithms, can also be found in the result of Meng et al.'s algorithm [60]. While the algorithm of Zhu et al. [105] presents improved results for the colour over-correction and halo effect problems, it only partially removes the haze, resulting in an image of relatively low contrast. In contrast, the results of the proposed CO, CO-DHWT\textsubscript{1} and CO-DHWT\textsubscript{2} algorithms present uniform colour and contrast reconstructions without showing the colour over-correction and halo effect problems.
5.5 Computational Results

Fig. 5.6 Comparative results for the "Desk" image.
5.5 Computational Results

Fig. 5.7 Comparative results for the "Hill" image.

(a) Hazy input  (b) He et al. [38]  (c) Wang et al. [91]

(d) Meng et al. [60]  (e) Zhu et al. [105]  (f) Berman et al. [5]

(g) CO, $\lambda_1 = 2.3, \lambda_2 = 1$
(h) CO-DHWT$^2, \lambda_1 = 2.6, \lambda_2 = 1$
(i) CO-DHWT$^2, \lambda_1 = 2.6, \lambda_2 = 1$
Fig. 5.7 displays the dehazing results of CO, CO-DHWT₁ and CO-DHWT₂ for the “Hill” image, compared with those of the algorithms presented in [87, 38, 91, 60, 105, 5]. Again, Taral et al. [87] introduces too many undesired artifacts. The dehazing results of He et al. [38] and Wang et al. [91] are over-saturated in the hill and forest part, leading to information loss. For the algorithm of Berman et al. [5], the over-sharpening and colour shift of the left hill make the result visually unpleasant. The algorithm of Meng et al. [60] fails in the white cloud part and the algorithm of Zhu et al. [105] is unable to thoroughly remove the haze. In contrast, our results appear natural in terms of both colour and the profile of the objects in the scene.

Fig. 5.8 shows and compares the performance of CO, CO-DHWT₁ and CO-DHWT₂ to the other algorithms of [38, 29, 71, 60, 105, 5] for the “House” image. Although all the algorithms provide enhanced colour contrast in their results, it is easily observed that the algorithms of [38, 60, 105, 5] generate halo effects at the left tree branch. The results of Fattal [29] and He et al. [38] exhibit colour saturation at the left tree branch and accordingly lose detailed information. The algorithm of Nishion et al. [71] presents a visually unpleasant result due to the colour distortion, which is the consequence of their physically deficient MRF model. In contrast, our proposed algorithms offer excellent results in both colour enhancement, edge sharpening and information preservation.

Fig. 5.9 presents the dehazing results of our proposed algorithms for the “Hill” image, compared with those of the algorithms presented in [38, 60, 105, 5]. The algorithm of He et al. [38] generates over-correction and dark colour over the entire image. The algorithm of Meng et al. [60, 5] performs poorly at the white flower and introduces colour distortion. The result of Zhu et al. [105] is acceptable, but is insufficiently dehazed. Our results clearly outperform the others.
Fig. 5.8 Comparative results for the "House" image.
5.5 Computational Results

(a) Hazy input

(b) He et al. [38]

(c) Meng et al. [60]

(d) Zhu et al. [105]

(e) Berman et al. [5]

(f) CO, $\lambda_1 = 0.6, \lambda_2 = 1$

(g) CO-DHWT$_1$, $\lambda_1 = 0.8, \lambda_2 = 1$

(h) CO-DHWT$_2$, $\lambda_1 = 1, \lambda_2 = 1$

Fig. 5.9 Comparative results for the "Lily" image.
Fig. 5.10 Comparing results for the "Manor" image.

Fig. 5.10 shows and compares the performance of our proposed algorithms to the algorithms presented in [38, 60, 105, 5]. The results of the algorithms in [38, 60, 5] exhibit colour shift and distortion at the most remote area, while the algorithm of [105] brings in halo effects around the trees. In contrast, our algorithms remove most hazes in the image and produce a clear image with vivid colour information.

Fig. 5.11 presents and compares the performance of our proposed algorithms and other algorithms including that in [87, 38, 29, 60, 105, 5] for the "Mountain" image. The results by Tarel et al. [87] and Fattal [29] have heavy colour saturation in the upper part of the image. There also exist contrast over-correction in the sky and cloud part of the dehazed images by He et al. [38], Meng et al. [60] and Berman et al. [5]. The result by Zhu et al. [105] does not produce sufficient dehazing and contains obvious halo effect in the lower part of the mountain. In comparison with these results, our proposed CO, CO-DHWT and

\begin{align*}
\text{(e) Berman et al. [5]} & \quad \text{(f) CO, } \lambda_1 = 1.2, \lambda_2 = 1 \\
\text{(g) CO-DHWT}_1, & \quad \lambda_1 = 1.2, \lambda_2 = 1 \\
\text{(h) CO-DHWT}_2, & \quad \lambda_1 = 1.2, \lambda_2 = 1
\end{align*}
Fig. 5.11 Comparative results for the "Mountain" image.
CO-DHWT₂ algorithms present an appropriate level of dehazing and more natural colour and contrast reconstruction.

Fig.5.12 presents and compares the performance of our proposed algorithms and the algorithms of [38, 60, 105, 5] for the "Tower" image. The results by He et al. [38], Meng et al. [60] and Berman et al. [5] exhibit severe Moire artifacts in the sky and over-saturation in the remote area. The result by Zhu et al. [105] shows halo artifacts around the fringe of bush. In comparison with these results, our proposed algorithms overcome the Moire artifact and halo effect and produce improved dehazing results.

The dehazing performance comparison presented in Fig.5.13 is for the "River", where the input photographs taken on haze-free and hazy days are shown. Referring to the haze-free and hazy input images, it is conspicuous that the dehazing results provided by our proposed algorithms produce the most natural and pleasing visual scenery. In comparison, the other dehazing results suffer from colour shift, distortion and over-saturation, which are caused by the false estimation of atmospheric air light and the under-estimation of the transmission function.

To sum up, based on observations of the above dehazing results, a problem with He et al.'s algorithm [38] is its under-estimation of the transmission function which leads to colour saturation and causes information loss. The deficiency in Meng et al.'s algorithm [60] is that the algorithm generates colour distortion when dealing with objects with similar or brighter colours compared with the atmospheric light value. While the algorithm of Berman et al. [5] can produce improvement in dealing with colour distortion, it generates halo artifacts around the edges, as a result of its pixel-wise-based algorithm. The algorithm of Zhu et al. [105] has a problem of insufficient dehazing and can only partially remove the haze effect. Different from these state-of-the-art algorithms, our proposed algorithms, based on the optical model and regularized optimization, provide more effective haze removal and
Fig. 5.12 Comparative results for the "Tower" image.
5.5 Computational Results

Fig. 5.13 Comparative results for the "River" image.

(a) Clear day  
(b) Hazy input  
(c) He et al. [38]

(d) Meng et al. [60]  
(e) Zhu et al. [105]  
(f) Berman et al. [5]

(g) $\text{CO, } \lambda_1 = 1, \lambda_2 = 1$  
(h) $\text{CO-DHWT}_1, \lambda_1 = 1, \lambda_2 = 1$  
(i) $\text{CO-DHWT}_2, \lambda_1 = 1, \lambda_2 = 1$
more accurate computation of the pixel values. As a result, the images reconstructed by our algorithms produce more natural images.

Furthermore, the dehazing results of our proposed algorithms CO, CO-DHWT$_1$, CO-DHWT$_2$ shown in Fig.5.5 - Fig.5.13 verify the validity of DHWT, in terms of its dehazing performance.

5.5.3 Quantitative Assessment

Hautiere et al. [36] propose a quantitative assessment approach for dehazing algorithms. Three quantitative descriptors $e$, $\bar{r}$ and $\Sigma$ are derived in [36]. The descriptor $e$ represents the rate of edges newly visible after restoration; the $\bar{r}$ describes the mean ratio of the gradients at visible edges; and $\Sigma$ expresses the percentage of pixels which becomes completely black or completely white after restoration. In the assessment, the visible edges are assumed to be the pixels having a local contrast above 5%.

The average values of descriptors $e$, $\bar{r}$ and $\Sigma$ of our proposed algorithms CO, CO-DHWT$_1$ and CO-DHWT$_2$ as well as the algorithms of [38, 60, 105, 5] were computed based on the dehazing results from Fig.5.5 - Fig.5.13 and the values are listed in Table 5.2. As the Table indicates, the value $\Sigma$, which denotes the saturation degree, is comparatively high for the algorithms in [38, 5]. The value $\bar{r}$ of Zhu et al.’s algorithm [105] is relatively low, which indicates insufficient dehazing. The values $e$, $\bar{r}$ and $\Sigma$ of the proposed algorithms and Meng et al.’s algorithm [60] are moderate, which demonstrates the dehazing ability of the proposed algorithms and Meng et al.’s algorithm in contrast enhancement and information preservation.

Although these quantitative descriptions can be used to analyse the dehazing results, they cannot be regarded as the criteria for the dehazing algorithm for the following reasons. Foremost, the halo effects and the Moire artifacts can increase the value of descriptor $e$, which
### 5.5 Computational Results

#### Table 5.2 Quantitative descriptors $e$, $r$ and $\Sigma$

<table>
<thead>
<tr>
<th></th>
<th>He et al. [38]</th>
<th>Meng et al. [60]</th>
<th>Zhu et al. [105]</th>
<th>Berman et al. [5]</th>
<th>CO</th>
<th>CO-DHWT$_1$</th>
<th>CO-DHWT$_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>1.2</td>
<td>2</td>
<td>1.2</td>
<td>1.8</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
</tr>
<tr>
<td>$\bar{r}$ (%)</td>
<td>1.4</td>
<td>2</td>
<td>0.8</td>
<td>1.4</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>0.7</td>
<td>0.06</td>
<td>0</td>
<td>0.6</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

decreases the credibility of $e$. In addition, these three descriptors contain no information regarding the colour shift and colour distortion.

#### 5.5.4 Computational Performance

Based on the low frequency characteristics of haze, a novel technique in our proposed dehazing algorithms is the application of DHWT for reducing the problem dimension and, hence, reducing the computational workload and speeding up the dehazing process. To further demonstrate the computational efficiency of the proposed dehazing algorithms, the processing times of our algorithms and the algorithms in [38, 60, 105, 5] for the dehazing processing of different images from Fig.5.5 - Fig.5.13 are listed in Table 5.3. These operation times were recorded on the same processor under the same conditions. The table shows that our algorithms can perform faster dehazing processing than other algorithms. The table also shows the effectiveness of DHWT in speeding up the dehazing process as the CO-DHWT$_1$ and CO-DHWT$_2$ algorithms performed faster dehazing processing than the other algorithms. In addition to the above computational results, the author applied the proposed CO, CO-DHWT$_1$ and CO-DHWT$_2$ algorithms to a number of other hazed images of different dimensions. Their average computational time durations, in seconds, are listed in Table 5.4, in comparison with those of other algorithms for the same images under the same computational conditions.
### Table 5.3
Comparison of computational time (in seconds) of different algorithms for images of different sizes

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Image</th>
<th>Canyon</th>
<th>Desk</th>
<th>Hill</th>
<th>House</th>
<th>Lily</th>
<th>Manor</th>
<th>Mountain</th>
<th>River</th>
<th>Tower</th>
</tr>
</thead>
<tbody>
<tr>
<td>He et al. [39]</td>
<td>[107,556]</td>
<td>14.5</td>
<td>24.4</td>
<td>17.2</td>
<td>7.8</td>
<td>12.0</td>
<td>19.9</td>
<td>7.5</td>
<td>4.3</td>
<td>9.1</td>
</tr>
<tr>
<td>Meng et al. [60]</td>
<td>[1200,056]</td>
<td>3.8</td>
<td>9.4</td>
<td>3.9</td>
<td>2.8</td>
<td>3.5</td>
<td>4.6</td>
<td>2.6</td>
<td>2.2</td>
<td>27</td>
</tr>
<tr>
<td>Zhu et al. [105]</td>
<td>[576,768]</td>
<td>2.8</td>
<td>4.1</td>
<td>3.8</td>
<td>2.1</td>
<td>3.0</td>
<td>3.7</td>
<td>2.5</td>
<td>1.8</td>
<td>2.2</td>
</tr>
<tr>
<td>Berman et al. [5]</td>
<td>[600,480]</td>
<td>2.9</td>
<td>6.4</td>
<td>2.2</td>
<td>1.6</td>
<td>2.1</td>
<td>1.7</td>
<td>1.8</td>
<td>1.1</td>
<td>2.5</td>
</tr>
<tr>
<td>CO</td>
<td>[440,480]</td>
<td>2.9</td>
<td>6.4</td>
<td>2.2</td>
<td>1.6</td>
<td>2.1</td>
<td>1.7</td>
<td>1.8</td>
<td>1.1</td>
<td>2.5</td>
</tr>
<tr>
<td>CO-DHWT₁</td>
<td>[440,480]</td>
<td>2.3</td>
<td>6.7</td>
<td>2.2</td>
<td>1.6</td>
<td>2.1</td>
<td>1.8</td>
<td>1.7</td>
<td>1.1</td>
<td>2.5</td>
</tr>
<tr>
<td>CO-DHWT₂</td>
<td>[440,480]</td>
<td>1.1</td>
<td>2.9</td>
<td>1.0</td>
<td>0.6</td>
<td>1.0</td>
<td>0.9</td>
<td>1</td>
<td>0.6</td>
<td>0.53</td>
</tr>
<tr>
<td>CO-DHWT₃</td>
<td>[440,480]</td>
<td>0.8</td>
<td>1.6</td>
<td>0.8</td>
<td>0.52</td>
<td>0.73</td>
<td>0.78</td>
<td>0.58</td>
<td>0.49</td>
<td>0.58</td>
</tr>
</tbody>
</table>

**Notes:**
- CO-DHWT₁ and CO-DHWT₂ are two different versions of the CO-DHWT algorithm.
Table 5.4 Comparison of average computational time (in seconds) of different algorithms for images of different sizes

<table>
<thead>
<tr>
<th>Algorithm \ Size \</th>
<th>600 × 450</th>
<th>1024 × 768</th>
<th>1536 × 1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>He et al. [39]</td>
<td>10.04</td>
<td>29.46</td>
<td>65.4</td>
</tr>
<tr>
<td>Meng et al. [60]</td>
<td>4.782</td>
<td>5.60s</td>
<td>10.27</td>
</tr>
<tr>
<td>Zhu et al. [105]</td>
<td>2.68</td>
<td>4.08</td>
<td>8.08</td>
</tr>
<tr>
<td>Berman et al. [5]</td>
<td>1.4</td>
<td>4.7</td>
<td>9.1</td>
</tr>
<tr>
<td>CO</td>
<td>1.3</td>
<td>3.8</td>
<td>8.16</td>
</tr>
<tr>
<td>CO-DHWT₁</td>
<td>0.7</td>
<td>1.7</td>
<td>3.3</td>
</tr>
<tr>
<td>CO-DHWT₂</td>
<td>0.55</td>
<td>1.1</td>
<td>1.9</td>
</tr>
</tbody>
</table>

5.6 Conclusion

By reformulating the bilinearly coupled haze-free image and light transmission distribution terms in the hazed image model and using an assumption of the low-pass and smooth features of the light transmission distribution, this chapter has presented a convex optimization formulation for the image dehazing problem in the low-pass sub-band of the Haar wavelet transform domain. The convex optimization enables systematic and efficient computation of the image dehazing solution and the sub-band wavelet transform significantly reduces the problem dimensions and the computational workload. The proposed algorithms CO, CO-DHWT₁, CO-DHWT₂ have been extensively applied to a number of hazed images, in comparison with state-of-the-art algorithms. Experimental results for both synthetic images and real-world images have demonstrated that our proposed algorithm is capable of removing haze effectively and outperforms other state-of-the-art algorithms.
Chapter 6

Video Dehazing by Regularized Optimization

In this chapter, the video dehazing problem is investigated. A digital video comprises a series of digital images displayed in rapid succession at a constant rate. In the context of digital video, the images are called frames. A video is generally characterized by strong temporal coherence or temporal continuity between its adjacent frames. The presence of temporal coherence minimizes the sudden change of the content from frame to frame. Perceptual studies have shown that humans are very sensitive to abrupt temporal variations such as popping and flickering. Therefore, the visibility and colours of the video should vary smoothly to ensure temporal coherence and fluid animation [75].

The regularized optimization dehazing method proposed in the last chapter provides good results for static images. However, when applied to each frame of a hazed digital video sequence independently, it may break the temporal continuity and produce a restored video with severe flickering artifacts.
6.1 Temporal Coherence Guarantee

Theoretically, the temporal coherence of a dehazed digital video can be enforced by yielding highly consistent atmospheric light values $a_c, c = 1, 2, 3$ and highly correlated spatial distance maps between its adjacent frames [46, 102, 83]. Therefore, in this thesis, an adaptive technique for atmospheric light estimation is proposed to ensure consistency of atmospheric light values between adjacent frames. Furthermore, a temporal coherence cost function is formulated depending on the transmission function $t$ to enforce the correlation of distance maps between adjacent frames.

6.1.1 Atmospheric Light Estimation

Taking into account the consistency of the atmospheric light values between adjacent frames, an simple adaptive technique is proposed to estimate the atmospheric light value.

Adaptive Technique for Atmospheric Light Estimation: Initially, the atmospheric light values $a_c, c = 1, 2, 3$ for the frames in the video sequence are estimated independently. Then, the estimated atmospheric light values of the current frame $\hat{a}_c, c = 1, 2, 3$ are compared with that of the previous frame $\hat{a}_c^p, c = 1, 2, 3$ and their difference is computed:

$$\delta = \sum_{c=1,2,3} |\hat{a}_c - \hat{a}_c^p|, \quad c = 1, 2, 3.$$

Let $\delta_s$ denote a threshold value. If the error $\delta$ is smaller than $\delta_s$, then the atmospheric light value estimated from the previous frame is applied, otherwise, the current estimated
atmospheric light value is preserved for the calculation:

\[
\hat{a}_c = \begin{cases} 
\hat{a}_c^p, & \text{if } \delta \leq \delta_s \\
\hat{a}_c, & \text{if } \delta > \delta_s
\end{cases}
\]

As a result, if there is no abrupt change of the atmospheric light values \(a_c, c = 1, 2, 3\) between adjacent frames, the previous values of the atmospheric light \(\hat{a}_c^p, c = 1, 2, 3\) remained and are applied to achieve highly consistent atmospheric light values over the video sequence. Otherwise, if the atmospheric light values change dramatically between adjacent frames, for example, there is a sudden change of the scene content or a sudden change of the environment illumination, then the estimated atmospheric values are kept and used to prevent colour distortion.

### 6.1.2 Transmission Coherence

Given the assumption that the temporal coherence can be improved by yielding highly correlated spatial distance distributions between adjacent frames in the video sequence, and based on the continuous relationship between the distance map and the light transmission distribution (4.12), a temporal coherence cost function depending on the transmission function is formulated to enforce temporal coherence and alleviate flickering artifacts as below:

\[
\|t - t^p\|_F^2
\]

(6.1)

where \(t^p\) is the transmission distribution of the previous frame.
Together with the regularizations for static image dehazing (5.12), the regularization function \( R(t, Q_c, c = 1, 2, 3) \) for video dehazing is specified as:

\[
R(t) = \|t\|_2^2 + \alpha \|t\|_{TV} + \beta \|t - t^p\|_F^2
\]

(6.2)

where, \( \beta \) is the weighting parameter. As \( \beta \) becomes larger, the temporal coherence is emphasized more strongly and the flickering artifacts are depressed more effectively. However, an overly large \( \beta \) may fix the transmission distribution value for each frame over the whole video sequence, which leads to blurring artifacts and degrades the quality of the restored video. Therefore, \( \beta \) should be determined by considering the trade-off between the flickering artifacts and the dehazing quality of individual frames.

In consequence, the regularized convex optimization without DHWT for coherence-enhanced video dehazing is formulated as:

\[
\begin{array}{ll}
\min_t & \|t\|_F^2 + \alpha \|t\|_{TV} + \beta \|t - t^p\|_F^2 \\
\text{s.t.} & Y_c - Q_c + a_c t = 0, \\
& 0 \preceq t \preceq 1, \ 0 \preceq Q_c, \ c = 1, 2, 3,
\end{array}
\]

(6.3)

where, \( Q_c = J_c \odot t \) and \( Y_c = Q_c - a_c 1, \ c = 1, 2, 3 \). This convex optimization for video dehazing is called COV (convex optimization for video dehazing) in the remainder of this paper, and it has a similar form to the CO in (5.14).

**Remark:** With the assumption that the light transmission distribution \( t \) is a \( 2^K \)-patch piecewise constant for \( k \geq 1 \), the conditions for the proposed COV for the original hazed image model (2.31) are applicable to the \( K \)-level DHWT low-pass sub-band hazed image model (5.2), which results in the following convex optimization for coherence-enhanced...
video dehazing with reduced image dimension:

\[
\min_{\hat{t}_K} \|\hat{t}_K\|_F^2 + \alpha \|\hat{t}_K\|_{TV} + \beta \|\hat{t}_K - \hat{t}_p^K\|_F^2
\]

(6.4)

s.t. \[\hat{Y}^{a,c}_{c,K} - \hat{Q}^{a,c}_{c,K} + \hat{a}_{c,K} \hat{t}_K = 0,\]

\[0 \prec \hat{t}_K \leq 1, \ 0 \preceq \hat{Q}^{a,c}_{c,K}, \ c = 1, 2, 3,\]

where, \(\hat{t}_K^p\) is the \(K\)-level sub-band transmission function of the previous frame and \(\hat{Q}^{a,c}_{c,K} = \hat{J}^{a,c}_{c,K} \odot \hat{t}_K\). \(\hat{Y}^{a,c}_{c,K} = \hat{Q}^{a,c}_{c,K} - a_{c,K} \mathbf{1}, \ c = 1, 2, 3.\) Further, the \(K\)-level DHWT-based convex optimization formulation for coherence-enhanced video dehazing is called COV-DHWT\(K\) (convex optimization for video dehazing - \(K\)-level discrete Haar wavelet transform) in the remainder of this paper.

6.2 Iterative Algorithm for COV

The optimization algorithm for COV (6.3) is similar to the Iterative Algorithm proposed for static image dehazing in Section 5.4. It is further verified that, the only difference is the update of the transmission distribution \(t\), and the update of \(t\) for video dehazing is given as:

\[
t^{(k+1)}(m,n) = \frac{\gamma}{2 + 2\beta - 2\lambda^{(k)} + \mu^{(k)} - \gamma \Delta} (t^{(k)}(m+1,n)) + t^{(k)}(m-1,n) + t^{(k)}(m,n+1)
\]

\[+ t^{(k)}(m,n-1) + d^{(k)}_v(m-1,n) - d^{(k)}_v(m,n) + d^{(k)}_v(m,n-1) - d^{(k)}_v(m,n)
\]

\[-b^{(k)}_v(m-1,n) + b^{(k)}_v(m,n) - b^{(k)}_v(m,n-1) + b^{(k)}_v(m,n))
\]

\[+ \frac{1}{2 + 2\beta - 2\lambda^{(k)} + \mu^{(k)} - \gamma \Delta} (-\lambda^{(k)} \odot (Q^{(k)}_c/a_c - Y^{(k)}_c/a_c) + \mu(Q^{(k)}_c/a_c - Y^{(k)}_c/a_c) + 2\beta t^p)
\]
The other variables $Q_c, c = 1, 2, 3$ and parameters $\lambda, \mu, d_x, d_y, b_x, b_y$ are updated as in the Iterative Algorithm proposed for CO. Their initial values are also set the same.

## 6.3 Computational Results

To demonstrate its practical feasibility and effectiveness, the proposed video dehazing algorithm was applied to real captured hazed videos. The 240p hazed video "Road View" was the dehazing objective. Considering the real-time constraint for video dehazing, the COV-DHWT$_2$ was employed. Our proposed COV-DHWT$_2$ was coded in C and parallelly implemented using a CUDA-enabled graphics processing unit (GPU). Experiments showed that the algorithm COV-DHWT$_2$ achieved real-time dehazing for 240p video on an HP-Z420 workstation with a 3.30 GHZ Intel E5-1660 CPU.

Several frames from the dehazing result for COV-DHWT$_2$ on the video "Road View" are presented in Fig.6.1. For comparison purposes, the corresponding frames of the original hazed video as well as the frames from the dehazing results of Kim et al.’s algorithm [46] and the proposed image dehazing algorithm CO-DHWT$_2$, are also shown in Fig.6.1. The static image dehazing algorithm CO-DHWT$_2$ is applied to each frame of the sequence independently. The dehazing result of Kim et al. [46] is from the authors’ website. A comparison of frames No.101 and No.201 of the dehazing result of CO-DHWT$_2$ in Fig.6.1 shows that the CO-DHWT$_2$ generates flickering artifacts in the video sequence, which is caused by the variations in the estimated atmospheric values. Although Kim et al. [46] provide a dehazed result without flickering artifacts, their algorithm is unable to thoroughly remove the haze, as shown in the bottom parts of the frames. In contrast, our proposed coherence-enhanced video dehazing algorithm yields fluid results by suppressing flickering artifacts.
6.3 Computational Results

Fig. 6.1 Comparative results for the "Roadview" video.
6.4 Conclusion

In this chapter, a coherence-enhanced video dehazing algorithm has been proposed. An adaptive method was proposed for atmospheric light estimation. In addition, a coherence cost depending on the transmission function was formulated and introduced into our regularization functions. Experimental results have shown that the proposed method achieves better performance of video dehazing with fewer flickering artifacts than existing methods.
Chapter 7

Conclusion

7.1 Conclusion

This thesis has investigated the image and video dehazing problem and proposed a novel dehazing method using regularized optimization. Through physical optical analysis of the underlying formation of a digital hazed image, it has been shown that the formation of hazed image and video is a combination of two light scattering effects, direct attenuation and airlight. By studying the optical fundamentals of these two scattering effects, an optical model for digital hazed RGB colour images has been proposed.

The formulation of the proposed optimization problem is based on the optical hazed image model. However, the inherent bilinearity and non-convexity that arises in the optical model for hazed images make dehazing challenging. This thesis has resolved the non-convex difficulty by regarding the bilinearly coupled term as one single variable, which leads to a linear model and therefore provides an efficient means of optimization.
In order to find feasible and meaningful solutions for the optimization model, it is crucial to incorporate available knowledge and information about the image or video conditions into appropriate regularizations, in order to regulate their solutions to satisfactory values. For image dehazing, the regularizations have been formulated based on the image contrast dependency on the light transmission distribution and the piecewise smoothness of the light transmission function. Nevertheless, for video dehazing, the temporal coherence characteristics of digital video were considered, and a coherence cost function has been formulated and introduced into the regularization functions.

Graphical analysis has been concluded on the discrete Haar wavelet transforms (DHWTs) of hazed images, which shows that coefficients in the low frequency sub-band have a high value due to the concentration of haze in the low frequency, while the coefficients containing details in the high frequency sub-band are degraded and weakened by the effect of haze. Then, based on the piecewise constant assumption of the light transmission function and the application of the DHWT onto the optical image model, a low-pass sub-band hazed image model, with reduced dimensions, has been derived. It has been inductively verified that the application of multiple $K$-level DHWT on the optical image model can formulate a sub-band image model with further reduced model dimensions, provided that the transmission function is $2^K$-patch piecewise constant. Furthermore, the DHWT low-pass sub-band image model is of exactly the same form as the original optical image model. In addition, it has been demonstrated that solving the dehazing problem of the low-pass sub-band hazed image model is sufficient for the solution of the original dehazing problem. The DHWT low-pass sub-band model with considerable dimension reduction in the optimization formulation can lead to fast dehazing processing.

Extensive computational experiments have been carried out to test the dehazing performance of our proposed optimization algorithm in comparison with a number of state-of-the-
art dehazing algorithms. The results have demonstrated the advantages of our algorithm as well as its computational efficiency.

7.2 Future Work

Some possible directions for further research are as follows:

- The dehazing results of the proposed algorithms CO, CO-DHWT, COV, and COV-DHWT sometimes present over-saturation for near objects, which is the result of the exponential manner between the transmission distribution and the distance map. Therefore, some other regularizations can be formulated to alleviate over-saturation.

- Although we achieved real-time processing for videos of 240p, this complexity is still too high to be employed in applications with limited computing resources, such as driving assistance systems. Further complexity reduction is a future research issue.

- The deep learning technique can be utilized into our proposed dehazing method to automatically find the values of the weighting parameters $\alpha, \beta$ by incorporating the information on hazed images and videos.

- Quantitative measurement of the performance for video dehazing can be proposed and applied for objective assessment.
Publications Arising from this Thesis

Some work described in this thesis has been published in the following papers:


References


References


References


