Sliding Mode Control of Fuzzy Dynamic Systems

Khoo Suiyang, Man Zhihong, Zhao Shengkui
School of Computer Engineering, Nanyang Technological University
Nanyang Avenue, Singapore 639798.
Email: khoo0032@ntu.edu.sg, aszhman@ntu.edu.sg, zhao0024@ntu.edu.sg

Abstract—In this paper, a sliding mode control scheme is developed for a class of complex nonlinear systems with their T-S fuzzy models. It is shown that a set of extreme fuzzy subsystems are first derived, and a constructive sliding mode control law is then developed to guarantee the stability of the closed-loop fuzzy system. Simulation results are presented in support of the proposed scheme.

Keywords—Sliding mode control; Fuzzy dynamic systems; Extreme fuzzy systems; Fuzzy control

I. INTRODUCTION

Sliding mode control (SMC) systems have been studied extensively and received many applications [1-4]. For the design of SMC systems, a sliding mode surface is first defined to specify the desired system dynamics. A sliding mode controller is then designed to drive the sliding variable to converge to zero. On the sliding mode surface, not only the desired system dynamics can be achieved, but also, the closed-loop system exhibits invariance properties, for instance, the closed-loop dynamics is insensitive to parameter variations and external disturbances.

It is known that, in order to design a sliding mode control system, only a rough mathematical model of the physical system is needed, i.e., the upper and lower bounds of the unknown parameters are required. Because of this advantage, the SMC technique has been used to control complex nonlinear systems with their T-S fuzzy models [5, 6].

The characteristics of the T-S fuzzy modelling technique for complex nonlinear systems can be briefly described as follows: (i) The whole state-space is first divided into subspaces, (ii) a simple linear model is used to approximate the complex nonlinear system in each subspace, (iii) the fuzzy inference rules are then used to fuzzify all the subsystem parameter matrices to obtain a global dynamic fuzzy model. In order to use the SMC technique to design a global controller for a complex nonlinear system with its T-S fuzzy model in this paper, we borrow the concept about fuzzy extreme subsystems from [7] to determine the information of the uncertainty bounds.

As described in [7], the fuzzy extreme subsystems can be obtained by decomposing a global fuzzy state-space into \( m \) subspaces. In each subspace, a dominant membership function, together with its local fuzzy subsystem, dominates the global fuzzy system. An upper bound for the subsystem can be found by considering all the interactions among the local fuzzy subsystems in the worst stability case. This uncertain bound information can then be used in the sliding mode controller design to stabilize a complex nonlinear system with its T-S fuzzy model.

The paper is organized as follows. Section 2 introduces complex nonlinear system and the fuzzy dynamic model. Section 3 discusses the concept of extreme fuzzy subsystem. Section 4 presents a robust SMC design for the complex nonlinear systems with matched uncertainties. Section 5 presents a numerical simulation example in support of the developed SMC scheme.

II. PROBLEM FORMULATION

Consider a complex nonlinear dynamic system described by

\[
\dot{x} = A(x) + \Delta A(x) + [B(x) + \Delta B(x)]u,
\]

where \( x \in \mathbb{R}^n \) is a measurable state variable vector, \( u \in \mathbb{R} \) is a control input. \( A(x) \) and \( B(x) \) are known nonlinear functions. \( \Delta A(x) \) and \( \Delta B(x) \) are system uncertainties.

In this paper, we assume that \( \Delta A(x) \) and \( \Delta B(x) \) satisfy the following matched conditions:

\[
\Delta A(x) = B(x) \Delta f_m(x) \quad \text{and} \quad \Delta B(x) = B(x) \Delta B_m(x).
\]

Furthermore, the following assumption is made:

Assumption 1: The uncertainties, \( \Delta f_m(x) \) and \( \Delta B_m(x) \) are bounded in Euclidean norm with known upper bounds as

\[
\| \Delta f_m(x) \| \leq \rho_m \quad \text{and} \quad \| \Delta B_m(x) \| < e_m < 1,
\]

where positive numbers \( \rho_m \) and \( e_m \) are assumed to be known.

For further analysis, the nominal system model without uncertainties can be written as follows:

\[
\dot{x} = A(x) + B(x)u.
\]

T-S fuzzy dynamic model of the nominal system (4) is described by the following fuzzy inference rules [5, 7]:
\[ R^i : \text{IF} \quad z_1 \text{ is } F^i_1 \text{ AND } \ldots \quad z_n \text{ is } F^i_n \]

THEN

\[ \dot{x} = A^i x + B^i u, \]

for \( i = 1, \ldots, m, \) (5)

where \( R^i \) represents the \( i^{th} \) fuzzy inference rule, \( m \) the number of inference rules, \( F_j \) \((j = 1, \ldots, n)\) the fuzzy sets, \( x \) the system state variable vector, and \( u \) the system input. The matrices \( A^i \in \mathbb{R}^{n \times n} \) and \( B^i \in \mathbb{R}^{n \times d} \). \( z = (z_1, \ldots, z_n)^T \) represents the subsystem parameters.

Denote \( \mu^i \) as the normalized fuzzy membership function of the inferred fuzzy set \( F^i \) where

\[ F^i = \prod_{j=1}^{m} F^i_j, \] (6)

\[ \mu^i = \frac{F^i}{\sum_{i=1}^{m} F^i}, \] (7)

and

\[ \sum_{i=1}^{m} \mu^i = 1. \] (8)

Using the fuzzy inference method with a center-average defuzzifier, product inference and singleton fuzzifier, the global dynamic fuzzy model of the nominal system (4) can be expressed as:

\[ \dot{x} = A(\mu)x + B(\mu)u(t), \] (9)

where

\[ A(\mu) = \sum_{i=1}^{m} \mu^i A^i, \quad B(\mu) = \sum_{i=1}^{m} \mu^i B^i, \] (10)

\[ \mu = (\mu^1, \mu^2, \ldots, \mu^m). \] (11)

Before we proceed, we have the following assumptions:

Assumption 2: Each linear subsystem in (5) is controllable, i.e. the controllability matrices
\[ M^i = \begin{bmatrix} B^i, A^i B^i, (A^i)^2 B^i, \ldots, (A^i)^{n-1} B^i \end{bmatrix} \] for \( i = 1, \ldots, m \) have full rank.

Assumption 3: The global fuzzy model (9) is controllable in the state space, i.e. the controllability matrix
\[ M = \begin{bmatrix} B(\mu), A(\mu)B(\mu), \ldots, A^{n-1}(\mu)B(\mu) \end{bmatrix} \] has full rank in the state space.

For convenience, we let the reference model share the same fuzzy sets with the fuzzy system in (5) where

\[ R^i : \text{IF} \quad z_1 \text{ is } F^i_1 \text{ AND } \ldots \quad z_n \text{ is } F^i_n \]

THEN

\[ \dot{x}_r = A^i_r x + B^i_r r, \]

for \( i = 1, 2, \ldots, m, \) (12)

with \( A^i_r \in \mathbb{R}^{n \times n}, \Re(\lambda[A^i_r]) < 0, \) \( B^i_r \in \mathbb{R}^{n \times d}, \) and \( r \in \mathbb{R} \) is a known input of the reference model.

The output of the overall reference model is inferred as follows,

\[ \dot{x}_r = A^i_r(\mu)x + B^i_r(\mu)r. \] (13)

III. FUZZY EXTREME SUBSYSTEM

To obtain the fuzzy extreme subsystem, the global fuzzy state-space is first decomposed into \( m \) subspaces,

\[ S_i = \left\{ x | \mu^i \geq \mu^l, l = 1, 2, \ldots, m, l \neq i \right\}, \]

for \( i = 1, 2, \ldots, m. \) (14)

The characteristic function of \( S_i \) is defined by

\[ \eta^i = \begin{cases} 1(x \in S_i), \\ 0(x \not\in S_i). \end{cases} \] (15)

Let \( G \) be the set of membership functions satisfying (8). Then, on every subspace, the fuzzy model (9) can be denoted by

\[ \dot{x} = \ddot{A}^{i}(\mu)x + \ddot{B}^{i}(\mu)u, \]

\[ = \left[ A^i + \nabla A^i(\mu) \right] x + \left[ B^i + \nabla B^i(\mu) \right] u, \]

\[ \nabla A^i(\mu) = \sum_{l=1}^{m} \eta^l \nabla A^l, \quad \nabla B^i(\mu) = \sum_{l=1}^{m} \eta^l \nabla B^l, \]

\[ \nabla A^i = A^i - A^l, \quad \nabla B^i = B^i - B^l, \]

\[ \eta^l \in G, \quad l = 1, \ldots, m, \quad \eta^l \neq \mu^i, \]

\[ \eta^l \neq 0, \quad \forall x \in S_i, \quad \text{for } i = 1, 2, \ldots, m. \] (16)

Obviously, the interactions of the subsystems are represented by \( \{ \nabla A^i(\mu), \nabla B^i(\mu) \} \). Here the \( i^{th} \) subsystem is different from the fuzzy dynamical local model in (5), because it considers all the interactions among the local fuzzy models.

Using the characteristic function of \( S_i \), (16) can be denoted by

\[ \dot{x} = \left[ A^i + \nabla A^i(\mu) \right] x + \left[ B^i + \nabla B^i(\mu) \right] w^i, \] (17)

where \( i = 1, 2, \ldots, m, \quad \dot{x} = \eta^i x, \) and \( w^i = \eta^i u. \)

It can be seen that each subsystem in (16) is time-varying. For the design of a sliding mode controller, we define the following upper bounds as in [8]:

\[ R^i : \text{IF} \quad z_1 \text{ is } F^i_1 \text{ AND } \ldots \quad z_n \text{ is } F^i_n \]
\[
\begin{align*}
\begin{bmatrix} \nabla A(\mu) \end{bmatrix}^T \begin{bmatrix} \nabla A(\mu) \end{bmatrix} &\leq (E^i)^T (E^i) = \begin{bmatrix} E^{i1} & E^{i2} \end{bmatrix}^T \begin{bmatrix} E^{i1} & E^{i2} \end{bmatrix}, \\
\nabla A(\mu) &= \begin{bmatrix} \nabla A(\mu) & \nabla B(\mu) \end{bmatrix}.
\end{align*}
\]
Then, the following \( m \) subsystems can be defined as the extreme subsystems of (16):
\[
\dot{x}_i = \begin{bmatrix} A^i + E^{i3} \end{bmatrix} \dot{x}_i + \begin{bmatrix} B^i + E^{i2} \end{bmatrix} w_i, \quad i = 1, \ldots, m.
\]
It is noted that the upper bound in (18) must be chosen as the worst upper bounds in the stability sense; that is, the upper bound will reduce the stability margin of the system [7].

IV. SMC FOR SYSTEMS WITH MATCHED UNCERTAINTIES

In this section, we consider the SMC design for the complex nonlinear systems with matched uncertainties.

Using (1), (2), and (9), we can express the fuzzy model of the complex nonlinear system with matched uncertainties as follows:
\[
\dot{x} = A(\mu) x + B(\mu) \begin{bmatrix} I + \Delta B_m(x) \end{bmatrix} u + \Delta f_m(x).
\]
For further analysis, a sliding variable \( s(\tilde{x}) \) is defined as follows:
\[
s(\tilde{x}) = c^T \tilde{x},
\]
where \( \tilde{x} = x - x_r \) and \( c \in \mathbb{R}^{n_1} \) is selected such that the dynamics of \( s(\tilde{x}) = 0 \) is Hurwitz.

Differentiating the sliding variable (21) with respect to time and substituting (13), (20) into it, we have
\[
\dot{s}(\tilde{x}) = c^T \begin{bmatrix} A(\mu) x + B(\mu) \begin{bmatrix} I + \Delta B_m(x) \end{bmatrix} u + \Delta f_m(x) \\
-A(\mu) x_r - B(\mu) r \end{bmatrix}.
\]
For the \( i^{th} \) subsystem, the time derivative of the sliding variable gives
\[
\dot{s}(\tilde{x}^i) = c^T \begin{bmatrix} A^i + \nabla A^i(\mu) \end{bmatrix} \dot{x}_i + \begin{bmatrix} B^i + \nabla B^i(\mu) \end{bmatrix} \begin{bmatrix} I + \Delta B^i_m(x) \end{bmatrix} w_i + \Delta f^i_m(x) - \begin{bmatrix} A^i + \nabla A^i(\mu) \end{bmatrix} \dot{x}_i - \begin{bmatrix} B^i + \nabla B^i(\mu) \end{bmatrix} r_i.
\]

Theorem 1: Consider the fuzzy model with matched uncertainties (20). If the sliding mode surface is chosen as (21), and the control input for \( i^{th} \) subsystem is designed as
\[
w_i = k^i(x') - \frac{s(\tilde{x}^i)}{\|s(\tilde{x}^i)\|} \begin{bmatrix} \psi^i_1 + \psi^i_2 \end{bmatrix},
\]
where
\[
k^i(x') = -\left( c^T B^i \right)^{-1} \begin{bmatrix} c^T A^i x' - c^T A^i x_r - c^T B^i r_i \end{bmatrix},
\]
then the output tracking error \( \tilde{x} = x - x_r \) will asymptotically converge to zero.

Proof: Consider the following Lyapunov function for \( i^{th} \) subsystem
\[
V^i = 0.5 \tilde{x}^T (\tilde{x}^i) s(\tilde{x}^i).
\]
By differentiating \( V^i \) with respect to time, we have
\[
\dot{V}^i = s^T (\tilde{x}^i) \dot{s}(\tilde{x}^i)
\]
\[
= s^T (\tilde{x}^i) c^T \left[ \dot{\tilde{x}}^i - \dot{x}_r \right]
\]
\[
= s^T (\tilde{x}^i) c^T \left[ \begin{bmatrix} A^i + \nabla A^i(\mu) \end{bmatrix} \dot{x}_i + \begin{bmatrix} B^i + \nabla B^i(\mu) \end{bmatrix} \begin{bmatrix} I + \Delta B^i_m(x) \end{bmatrix} w_i + \Delta f^i_m(x) - \begin{bmatrix} A^i + \nabla A^i(\mu) \end{bmatrix} \dot{x}_i - \begin{bmatrix} B^i + \nabla B^i(\mu) \end{bmatrix} r_i \right]
\]
\[
= s^T (\tilde{x}^i) \left[ c^T \nabla A^i(\mu) \dot{x}_i - c^T B^i k^i(x') + c^T \left[ B^i + \nabla B^i(\mu) \right] \begin{bmatrix} I + \Delta B^i_m(x) \end{bmatrix} w_i \right.
\]
\[
+ c^T \left[ B^i + \nabla B^i(\mu) \right] \Delta f^i_m(x') - c^T \nabla A^i(\mu) \dot{x}_i
\]
\[
\left. - c^T \nabla B^i(\mu) r_i \right]
\]
\[
= s^T (\tilde{x}^i) \left[ c^T \nabla A^i(\mu) \dot{x}_i + s^T (\tilde{x}^i) c^T B^i \Delta B^i_m(x') k^i(x') \right.
\]
\[
+ s^T (\tilde{x}^i) c^T \nabla B^i(\mu) \left. \left[ 1 + \Delta B^i_m(x') \right] k^i(x') \right]
\]
\[
- \left. \left\| s(\tilde{x}^i) \right\| c^T \nabla B^i(\mu) \left[ 1 + \Delta B^i_m(x') \right] \left\| \psi^i_1 + \psi^i_2 \right\| \right.
\]
\[
+ s^T (\tilde{x}^i) c^T \nabla B^i(\mu) \left. \left[ 1 + \Delta B^i_m(x') \right] k^i(x') \right]
\]
\[
- \left. \left\| s(\tilde{x}^i) \right\| c^T \nabla B^i(\mu) \left[ 1 + \Delta B^i_m(x') \right] \left\| \psi^i_1 + \psi^i_2 \right\| \right]
\]
\[
\leq \frac{s^T (\tilde{x}^i) c^T B^i \left( \dot{x}^i - x_r \right) c^T B^i \left( \dot{x}^i - x_r \right)}{\| \psi^i_1 + \psi^i_2 \|}
\]
\[
+ s^T (\tilde{x}^i) e^T \nabla B^i(\mu) \left[ 1 + \Delta B^i_m(x') \right] \left\| \psi^i_1 + \psi^i_2 \right\|.
\]
\[
+s^T(\vec{x}^o)c^TB' + \nabla B' (\mu) \Delta f_m (\vec{x}^o) - s^T(\vec{x}^o)c^T\nabla A'_m (\mu) x'_m
- s^T(\vec{x}^o)c^T \nabla B'_m (\mu) r' \n\]
\[
< \|s(\vec{x}^o)\|c^T E^s \vec{x}^o + \|s(\vec{x}^o)\|c^T \varepsilon_m k' (\vec{x}^o) - \|s(\vec{x}^o)\|\|c^T \nabla A'_m (\mu) x'_m
+ \|s(\vec{x}^o)\|c^T \nabla B'_m (\mu) r' - \|s(\vec{x}^o)\|\{\phi_1, \phi_2\} \n\]
\[
\leq \|s(\vec{x}^o)\|(1 - \phi_1) \{c^T E^s \vec{x}^o + \|c^T \nabla B'_m (\mu) r'\}\]
\[
< 0 \text{ for } s(\vec{x}^o) \neq 0, \tag{29}\n\]

Expression (29) is the sufficient condition for the switching plane variable \(s(\vec{x}^o)\) to reach the sliding mode surface. On the sliding mode surface, using expression (21), we can conclude that the output tracking error will asymptotically converge to zero.

V. SIMULATION EXAMPLE

In this section, we consider an inverted pendulum via gear train [9, 10], described by the following state equations:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
\frac{-g}{l} \sin(x_1) + \frac{NK}{ml^2} x_3 \\
\frac{-K_bN}{L_a} x_2 - \frac{R_s}{L_a} x_3 \\
-10 x_2 - 10 x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} u, \tag{30-a}
\]

\[
y = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}, \tag{30-b}
\]

where \(K_m\) is the motor torque constant, \(K_b\) is the back emf constant, and \(N\) is the gear ratio.

In this simulation, \(g = 9.8 m/s^2\), \(l = 1 m\), \(m = 1 kg\), \(N = 10\), \(K_m = 0.1 Nm/A\), \(K_b = 0.1 \text{ Vs/rad}\), \(R_s = 1 \Omega\), and \(L_a = 100 mH\). Using the values of the system parameters in (30-a), we have

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
x_2 \\
9.8 \sin(x_1) + x_3 \\
-10 x_2 - 10 x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
10
\end{bmatrix} u. \tag{31}
\]

The following dynamic fuzzy model is used to represent the nonlinear system (31):

Plant Rule 1:

\[
\text{IF } x_1 \text{ is } F_1, \quad \text{THEN } \begin{bmatrix}
\dot{x} = A_1 x + B_1 u(t) \\
y = D_1 x
\end{bmatrix},
\]

Plant Rule 2:

\[
\text{IF } x_1 \text{ is } F_2, \quad \text{THEN } \begin{bmatrix}
\dot{x} = A_2 x + B_2 u(t) \\
y = D_2 x
\end{bmatrix},
\]

where,

\[
x = [x_1 \quad x_2 \quad x_3]^T,
\]

\[
A_1 = \begin{bmatrix}
0 & 1 & 0 \\
9.8 & 0 & 1 \\
0 & -10 & -10
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

\[
D_1 = \begin{bmatrix}
1 & 0
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -10 & -10
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

\[
D_2 = \begin{bmatrix}
1 & 0
\end{bmatrix},
\]

\(x_1\) is the angle of the pendulum, \(x_2 = x_1\), and \(x_3\) is the current of the motor.

We use the following membership functions, \(F_1\) and \(F_2\):

\[
F_1 (x_1) = \frac{1-1/(1 + \exp(-5(x_1 - 1)))}{1 + \exp(-5(x_1 + 1))},
\]

\[
F_2 (x_1) = 1 - F_1 (x_1).
\]

The following parameters for matched uncertainties are chosen for the simulation:

\[
\Delta f_m (x) = \sin(x_1) + 0.7 \cos^2 (x_2),
\]

\[
\Delta B_m (t) = 0.2 \sin(t).
\]

The uncertain bounds \(\varepsilon_m = 0.5\) and \(\rho_m = 2\) are assumed to be known. In this example, the initial states are selected as

\[
x(t_0) = [0 \quad 0 \quad 0]^T,
\]

and

\[
x_r(t_0) = [1 \quad 0 \quad 0]^T.
\]
The sliding mode variable is prescribed as \( s(t) = [81 \ 18 \ 1] \), \( \tilde{x} = 0 \). The extreme subsystems can be calculated as

\[
E^1 = E^{11} = E^{21} = \frac{1}{2} \begin{bmatrix}
0 & 1 & 0 \\
9.8 & 0 & 1 \\
0 & -10 & -10
\end{bmatrix},
\]

\[
E^2 = E^{12} = E^{22} = \begin{bmatrix}
0 \\
0 \\
5
\end{bmatrix}.
\]

The desired reference model is given as

\[
\dot{x}_r = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-72 & -57 & -14
\end{bmatrix} x_r + \begin{bmatrix}
0 \\
0 \\
35
\end{bmatrix} r.
\]

Fig.1 shows the good tracking performance though the inverted pendulum system is highly nonlinear and unstable. Fig.2 shows the sliding mode control input signal. The corresponding tracking error is shown in Fig. 3. Obviously, the effects of the system uncertainties are eliminated and good tracking performance is guaranteed by using the proposed SMC scheme.

To show the important of the extreme subsystem idea in robust SMC design, we do not consider the interaction between two subsystems by letting \( E^{11} = E^{21} = 0 \). As shown in Fig. 4, the system state trajectory fails to follow the reference trajectory. However, after the extreme subsystem idea is employed by letting \( E^{11} = E^{21} = \begin{bmatrix}
1 & 0 & 0 \\
9.8 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \), as shown in Fig. 5, good tracking performance is achieved.

It is obvious that by applying the extreme system idea, the design of the fuzzy control system can be decomposed into the design of a set of fuzzy subsystems. Each subsystem can be designed independently, and the individual control laws can be combined to get a global control law for the global fuzzy system.
V. CONCLUSION

A robust sliding mode tracking control scheme has been developed in this paper for a class of complex nonlinear systems with their T-S fuzzy models. The concept about the fuzzy extreme subsystem has been used to provide the uncertainty bound information. Several advantages of the proposed SMC scheme have been seen from the theoretical analysis and the simulation results. These include the strong robustness property with respect to the system uncertainties and the fast convergence of the output tracking error on the sliding mode surface. A simulation example has been given in support of the proposed control scheme.

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