An Extended Half-scan Feldkamp-type CT Reconstruction

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ABSTRACT

In this paper, a Feldkamp-type approximate algorithm is proposed for helical multislice Computed Tomography (CT) image reconstruction. For a planar transversal reconstruction slice under consideration, the algorithm adopts a set of scanning data samples such that all points of the planar plane satisfy Tuy’s exact reconstruction condition and, therefore, have potential to be exactly reconstructed. This can provide a practically feasible compromise between image quality and computation efficiency in the reconstruction. Simulation results can show advantages of this algorithm in reduction of artifacts and improvement of computational efficiency in comparison with the existing algorithms.

Keywords: Helical multislice CT, Feldkamp-type reconstruction, Half-scan, Tuy’s condition

1. INTRODUCTION

Helical multislice Computed Tomography (CT) has been recently investigated and developed for rapid and volumetric scanning with high axial image resolution for medical diagnosis. Popularly, the objective of CT reconstruction algorithms is to produce a sequence of planar transversal image slices representing the objective volume for clinical applications.

Helical multislice CT reconstruction algorithms can be classified into exact and approximate algorithms. While the exact algorithms can produce high quality images\textsuperscript{1–6}, they require extensive computational resource and power in practical implementation. Approximate algorithms, on the other hand, can provide feasible compromise between the image quality and computational efficiency so they are widely adopted in practical CT scanners\textsuperscript{8–15}. Among approximate algorithms, Feldkamp (FDK) algorithm and its extensions are well known and popularly employed\textsuperscript{11–15}.

Most approximate algorithms including FDK-type algorithms for helical multislice CT scanners exhibit considerable artifacts in the reconstructed image when the cone-beam angle or helix pitch value is relatively large. Efforts have been made for reducing the artifacts and improving the image quality for approximation algorithms, such as the helix tangential filtering\textsuperscript{14} and rotational filtering\textsuperscript{16}. Recently, Hu et al. proposed a FDK-type approximate algorithm which considered reconstruction of mutating curved surfaces using short scan data\textsuperscript{13}. It is shown that all points on the mutating curved surface satisfy Tuy’s exact reconstruction condition\textsuperscript{7} thus have potential to be exactly reconstructed. The conventional planar transversal image slice is then obtained by computing its points on the corresponding mutating curved surfaces or by interpolating the mutating curved surfaces.

The proposed algorithm in this paper is motivated by Hu et al.’s concept for approximate reconstruction of the image slice which satisfies Tuy’s exact reconstruction condition. Instead of reconstructing the mutating curved surface as considered in Hu et al.’s proposal, we consider direct reconstruction of transversal planes and derive a shortest scanning helical trajectory such that all points on the transversal plane satisfy Tuy’s condition and, therefore, have potential to be exactly reconstructed. A FDK-type approximate algorithm is then applied to directly reconstruct the planar image. It is shown that the proposed algorithm can reduce imaging artifacts.

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from that of existing FDK-type algorithms, and it utilizes minimum scanning data to meet Tuy’s condition for all points on the reconstructed transversal plane which results in enhancement of computational efficiency.

The rest of this paper is organized as follows. Section 2 presents the coordinates system for helical scanning and Tuy’s condition for the transversal reconstruction plane. Section 3 presents the proposed algorithm and reconstruction procedure, followed by computer simulations and discussions. The conclusion is presented in Section 4.

2. GEOMETRY AND RECONSTRUCTION CONDITION

2.1. Helical scanning geometry

The geometry of the helical multislice CT scanning is shown in Fig. 1 in a Cartesian coordinate system $x - y - z$. The X-ray source moves along a helical trajectory with the $z$ axis as the helix axis. Denote the radius of the helix as $R$, the radius of objective cylindrical support as $r$, the source rotation angle as $\theta$. The pitch value of the helix is $h$ and the detector collimation width is $w$, therefore we have the table translation speed as $p = hw$. The position of the source can be represented as $(x, y, z)^T = (R \cos \theta, R \sin \theta, p\theta)^T$.

Consider a helix segment between two source rotation angles $\theta_1$ and $\theta_2$ with $\theta_1 < \theta_2$. As shown in Fig. 1, the PI-lines emitting from the helix terminal point at $\theta_2$ and connecting the helix points between $(\theta_1, \theta_2)$ form a surface, denoted as $S_1$. Similarly, PI-lines emitting from the terminal point at $\theta_1$ form another surface, denoted as $S_2$. Recall the well-known Tuy’s exact reconstruction condition which can be stated as that an object can be exactly reconstructed if and only if all the planes intersecting the object also intersect the source scanning trajectory. Referring to Fig. 1 we can see that only the object being totally covered between surface $S_1$ and $S_2$ can satisfy the Tuy’s condition and, therefore, has potential to be exactly reconstructed.

It is noted that the half-scan scheme is popularly employed in existing approximate algorithms and it requires a $\pi + \alpha_d$ scanning, where $\alpha_d = 2 \arcsin \frac{r}{w}$, for reconstruction of a transversal planar image. While it uses reduced amount of data in order to save the radiation dose and computational workload, some part of the reconstruction slice may not satisfy Tuy’s condition for exact reconstruction. This can be geometrically illustrated as in Fig. 2, where some parts of the round reconstruction slice are not covered between the surfaces $S_1$ and $S_2$. As a result, these parts may not be satisfactorily reconstructed by exact or approximate algorithms.

Figure 1. Helical CT scanning geometry.
2.2. Tuy’s condition for transversal reconstruction plane

It is clear that Tuy’s condition for exact reconstruction of the transversal plane can be satisfied by extending the length of the scanning helix. And, for a given transversal planar slice, there exists a minimum helix length for such a condition to be satisfied. With respect to the helix segment within $\theta \in [\theta_1, \theta_2]$ and the object slice positioned at $z = \frac{p(\theta_2 - \theta_1)}{2}$ as shown Fig.3, we can express the minimum helix angular interval $\theta_{\text{min}} = \theta_2 - \theta_1$ for all points of the object slice to satisfy Tuy’s condition in the following formula.

$$\theta_{\text{min}} = \max \left\{ f(\theta) = \theta + \sqrt{\frac{2\pi^2 - \cos \theta - 1}{1 - \cos \theta}}, \ \theta \in [2 \arccos \frac{r}{R}, 2\pi - 2 \arccos \frac{r}{R}] \right\} \tag{1}$$

It can be verified that $f(\theta)$ in (1) is a convex function in $\theta \in [2 \arccos \frac{r}{R}, 2\pi - 2 \arccos \frac{r}{R}]$ and has a unique maximum. Thus the computing of $\theta_{\text{min}}$ is simple and straightforward.

To derive (1), we introduce Fig.4 (a) and consider that a PI-line starting from point $A$ of the source trajectory at angle $\theta_1$, passing through the object and meeting the point $C$ on the trajectory at angle $\theta \in (\theta_1, \theta_2)$. Let the point where the PI-line $AC$ passing out from the object be $(x, y, z)^T$. Without loss of generality, we assume
θ₁ = 0. Then θ₂ ∈ (π + α_d, 2π] and the position of point C is (R cos θ, R sin θ, pθ)T. We can project the helical scanning system in Fig.4 (a) onto a transversal plane where the helix starting point A locates, as shown in Fig.4 (b), where the projections of the helix AB and the PI-line AC are, respectively, AB’ and AC’. Referring to the triangle proportional relationship, we have:

\[
\frac{R - x}{R - R \cos \theta} = \frac{y}{R \sin \theta} = \frac{z}{p\theta}
\]

(2)

Since the point \((x, y, z)\) is at the peripheral of the object, \(x^2 + y^2 = r^2\) is satisfied. It then follows from (2) that

\[
\left( \frac{Rx(1 - \cos \theta)}{p\theta} - R \right)^2 + \left( \frac{Rx \sin \theta}{p\theta} \right)^2 = r^2
\]

(3)

\[
2R^2 \frac{(1 - \cos \theta)}{p\theta^2} z^2 - 2R^2 \frac{(1 - \cos \theta)}{p\theta} z + (R^2 - r^2) = 0
\]

(4)

The solution for \(z\) of the above quadratic equation (after removing the irrelevant solution) as a function in \(θ\) is

\[
z(θ) = \frac{p\theta}{2} \left( 1 + \frac{2 \frac{p\theta}{2} - \cos \theta - 1}{1 - \cos \theta} \right)
\]

(5)

It can be verified that, for \(θ \in [2 \arccos \frac{p\theta}{2}, 2\pi - 2 \arccos \frac{p\theta}{2}]\) which means the PI-line AC passing through the object, the function \(z(θ)\) in (5) is a convex function and has a unique maximum, named as \(z_m\).

Considering Fig.4 (a), if the transversal object slice with radius \(r\) is positioned at \(z_m\), the PI-lines emitting from point A will no longer pass through the slice and there is one and only one PI-line touching the slice’s circular rim. If this \(z\)-position is the middle \(z\)-position of helix, i.e. \(z_m = \frac{p\theta}{2}\), it can be seen according to the geometrical symmetry that PI-lines emitting from point B will not pass through the slice either and there is also one and only one PI-line touching the slice’s circular rim. Once the slice’s \(z\)-position is lower (higher) than \(z_m\), there will be PI-lines of surface \(S_2\) (surface \(S_1\)) passing through the slice. Therefore the object slice is completely covered between surface \(S_1\) and \(S_2\) which satisfies the Tuy’s condition. Also it can be seen that \(\frac{2z_m}{p\theta}\) is the minimal angular interval of the helix for the objective slice with radius \(r\) to satisfy this condition. This immediately yields (1).
### 3. RECONSTRUCTION AND SIMULATION

#### 3.1. Reconstruction using extended scanning helix

We now reconstruct the transversal planar image slice using data from the extended half-scanning helix with FDK filtered back-projection scheme. The reconstruction procedure is presented as following:

1. Collect the data set denoted as $D(m, n, \theta)$ along the helix segment, where $(m, n)$ is the index of detector cells with $(0, 0)$ being the detector center, and $\theta$ is source rotation angle, $\theta \in [\theta_1, \theta_2]$ and $[\theta_1, \theta_2]$ is the angular interval of extended half-scanning helix. Denote the distance from source to detector plane is $d_{sd}$, and define $\tau = R/d_{sd}$. A standard cone-beam weighting is applied to $D(m, n, \theta)$ before filtering, as follows:

$$D_w(m, n, \theta) = \sqrt{R D(m, n, \theta)}.$$  \hspace{1cm} (6)

2. Weight $D_w(m, n, \theta)$ with Parker weighting factors, as follows:

$$D_{pw}(m, n, \theta) = D_w(m, n, \theta) w(m, \theta),$$  \hspace{1cm} (7)

where $w(m, \theta)$ is the Parker weighting factor for short scanning reconstruction.\(^{17}\)

3. Filter the data set using the helix-tangential filtering technique, which is for improving the image quality and longitudinal resolution. Denote the angle between the helix-tangential unit vector and the $x - y$ plane as $\beta$ and the ramp filter as $h(t)$. The filtering process is:

$$D_{f_{pw}}(m, n, \theta) = D_{pw}(m_t, n_t, \theta) \ast \frac{1}{2} h(\tau m_t),$$  \hspace{1cm} (8)

where $(\ast)$ denotes the convolution, and

$$\begin{pmatrix} m_t \\ n_t \\ \theta \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ \theta_0 \end{pmatrix}.$$  \hspace{1cm} (9)

4. Implement the FDK back-projection to obtain the reconstructed image. Denote the objective slice as $f(x, y, z_0)$ where $z_0$ is the $z$-position. The back-projection is:

$$f(\vec{r}) = \int_{\theta_1}^{\theta_2} \frac{R^3 d\theta}{(R - s)^2} D_{f_{pw}}(R t / (R - s), R (z_0 - s \theta) / (R - s), \theta),$$  \hspace{1cm} (10)

where $\vec{r} = (t, s, z_0)^T$ denotes a point of objective slice:

$$\begin{pmatrix} t \\ s \\ z_0 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z_0 \end{pmatrix}.$$  \hspace{1cm} (11)

In the following section the proposed algorithm is simulated with different phantoms, and compared with Hu et al.’s nutating curved reconstruction algorithm\(^{13}\) and transversal reconstruction algorithm with conventional half-scanning.\(^{11}\)

#### 3.2. Computer simulation

In this section, the proposed FDK-type algorithm using extended scanning helix (abbreviated as ExFDK) is tested by Shepp-Logan phantom, Clock phantom and Dish phantom. The algorithm is compared with Hu et al.’s nutating curved half-scan FDK reconstruction (abbreviated as NcFDK) and the conventional $\pi + \alpha_d$ half-scan FDK reconstruction (abbreviated as HsFDK\(_1\)). The data sample collected from above three algorithms are all filtered along the helix-tangential direction. For further comparison, the half-scan FDK reconstruction with
normal filtering (i.e. filtering along detector row) is also simulated (abbreviated as HsFDK). Parameters of the simulated CT scanning system are listed in Table 1.

First we test these algorithms with a 256 × 256 × 256 voxel (one voxel is 1 × 1 × 1mm) low-contrast Shepp-Logan Phantom. Its grey value ranges in [0, 0.2]. The display window is [0.095, 0.105]. The objective square slice is 256 × 256mm. So we have $r = 256 \times 0.707 = 181.0 mm$, and $\alpha_d = 2 \arcsin \frac{r}{R} = 0.1951\pi$. The extended half-scan helix of ExFDK is calculated following (1), and the result is 1.3872$\pi$. With NcFDK algorithm, the transversal image slice is reconstructed point by point. Each point is reconstructed with data sample collected from its corresponding PI-helix segment (we denote the helix segment between two tips of PI-line as PI-helix. Note that a point inside a helix belongs to one and only one PI-line.\(^{1-3}\) The position of PI-helix segment are pre-calculated and stored in a looking-up table. In this simulation, we position the expected square slice on the $x$ – $y$ plane (refer to Fig.3), with its four tip points positioned on $x$ and $y$ axis respectively. The union of all involved helix segments for reconstructing a transversal slice is 1.61$\pi$.

The reconstruction images using four different FDK-type algorithms are shown in Fig.5. It is shown that the result of the ExFDK can provide good image quality in terms of sharp edge and less stripe artifacts, and is as good as that of NcFDK. However, the results of HsFDK\(_1\) and HsFDK\(_2\) show much more stripe artifacts.

Next a Clock phantom, which has the same size as the above Shepp-Logan phantom, is adopted for simulation. Its grey value also ranges in [0, 0.1], and the display window is [0.09, 0.105]. As shown in Fig.6, the reconstruction results of proposed ExFDK algorithm and NcFDK algorithm are much better than the result of HsFDK\(_1\) and HsFDK\(_2\), in terms of less shading artifacts around small white balls.

A Dish phantom of the same size is also adopted for numerical comparison of these four FDK-type algorithms. The dish is positioned at the center of the cubical phantom, its grey value is 0.1, while the ambient is 0. The reconstruction values at the central straight line of dish are displayed in Fig.7, compared with the original values. We can see that the HsFDK\(_1\) and HsFDK\(_2\) produce rugged curves, while the ExFDK reconstruction curve is much closer to the original values which shows much better image reconstruction.

The involved helix lengths of ExFDK, NcFDK and conventional half-scanning schemes are compared at several $r/R$ values, as shown in Table 2. Without finding out all the PI-helix involved in NcFDK, we only pick up four PI-helix which belong to four tips of square slice (two points at $x$ axis and two at $y$ axis respectively) and compare their union with the helix involved in ExFDK and conventional half-scanning. It can be seen from Table 2 that the union length of only four PI-helices of NcFDK algorithm are longer than the helix involved in

<table>
<thead>
<tr>
<th>$r/R$</th>
<th>HsFDK(_1), (_2)</th>
<th>NcFDK</th>
<th>ExFDK</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.1282$\pi$</td>
<td>1.2978$\pi$</td>
<td>1.2502$\pi$</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1940$\pi$</td>
<td>1.6086$\pi$</td>
<td>1.3873$\pi$</td>
</tr>
<tr>
<td>0.5</td>
<td>1.3333$\pi$</td>
<td>1.8696$\pi$</td>
<td>1.6972$\pi$</td>
</tr>
</tbody>
</table>

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Table 1. Scanning system’s parameters

<table>
<thead>
<tr>
<th>Source-to-axis distance ($R$)</th>
<th>600mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axis-to-detector distance</td>
<td>400mm</td>
</tr>
<tr>
<td>Detector cell</td>
<td>$1 \times 1$mm</td>
</tr>
<tr>
<td>Detector dimension</td>
<td>$600 \times 64$ cells</td>
</tr>
<tr>
<td>Projections per rotation</td>
<td>512</td>
</tr>
<tr>
<td>Helical pitch ($h$)</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 2. Helix length against ratio $r/R$.
Figure 5. Helical reconstruction of Shepp-Logan phantom. (a) is by ExFDK, (b) is by NcFDK, (c) is by HsFDK₁, and (d) is by HsFDK₂.

ExFDK. This indicates that for reconstruction of a transversal planar slice, NcFDK uses longer helix segment than that of ExFDK.

3.3. Discussion

The simulation results show that the proposed FDK-type algorithm using extended scanning helix have the following advantages in comparison with existing algorithms:

- **Data Range**: In medical practice, transversal planar image slice is conventional and commonly adopted. Although NcFDK only requires a minimum data set to reconstruct a single nutating curved surface, it needs reconstructing a group of nutating curved surfaces for interpolation or, alternatively, computing pointwisely using each corresponding nutating curved surface to obtain a planar slice. Since different nutating curved surface or different point requires different PI-helix segment, the union of all helices involved can be considerably longer than the that required for a single nutating curved surface. Differently, our proposed algorithm directly reconstructs the planar transversal slice using a data set from only one helix segment.
Our simulation results have shown that the helix segment involved in the proposed ExFDK is shorter than the helix union required by NcFDK for construction of a planar slice.

- **Computation Load**: With NcFDK, reconstruction of a planar transversal slice will involve a large amount of PI-helix segments which require a considerable amount of computation for building up the looking-up table for reconstruction. Our proposed ExFDK only requires one single helix segment for reconstruction of a planar transversal slice. It is straight-forward to obtain the precise length and position of this helix segment.

  The short-scanning schemes adopt weighted conjugated projection values to reduce the signal-noise-ratio (SNR). It is well known that some $\theta$-dependent weighting methods\textsuperscript{17,18} can provide better results in comparison with the conventional $\theta$-independent weighting methods.\textsuperscript{19} With our proposed ExFDK it only needs the data set from one helix segment for reconstruction of a transversal planar slice, and it needs only one weighting procedure applied to the whole data set if a $\theta$-dependent weighting method is adopted. With NcFDK the reconstruction of each nutating curved surface requires data set from its unique helix segment and the corresponding unique weighting procedure if a $\theta$-dependent weighting method is adopted.
As a result, a great deal of computational workload is inevitable when a transversal planar slice is to be reconstructed by NcFDK, which involves a significant number of different helix segments. Moreover, in cerebral CT imaging, it often needs to reconstruct a series of transversal slices that are continuous at axial direction. Since every point at every slice has its unique PI-helix segment, to reconstruct a series of z-continuous transversal planar slices with NcFDK, it is computationally expensive to determine all these different helices. In contrast the length of the extended scanning helix is independent of the z-position as shown in (1). So with our proposed algorithm, there is no need to repeatedly computing the helix segment length slice by slice.

4. CONCLUSIONS

This paper has presented a FDK-type approximate algorithm for helical multislice CT using extended scanning helix such that all points of the objective slice satisfy Tuy’s condition for exact reconstruction. We have derived a formula for obtaining the minimum helix segment for transversal planar slices to satisfy Tuy’s condition, applied it in the proposed algorithm, and conducted simulations to evaluate the proposed algorithm in comparison with some existing algorithms. It is shown that the proposed algorithm can improve the image quality and reduce artifacts exhibited in earlier FDK-type algorithms. It can also provide more efficient computation in reconstructing transversal planar slices in comparison with the algorithm proposed recently which reconstructed Tuy-satisfying nutating curved surface with FDK-type method.

REFERENCES